SUBSEQUENCES IN STACK SORTABLE PERMUTATIONS
AND THEIR RELATION TO BINARY TREES

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Research Report CS-78-42
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ABSTRACT

The class $SS_n$ of stack sortable permutations is known to be in 1-1 correspondence with n-noded binary trees. Expressions are derived for the average length of several types of monotonic subsequences in members of $SS_n$. The relations between these subsequences and properties of the corresponding tree are demonstrated. It is also shown that the permutation graph of a member of $SS_n$ is an interval graph of a special type.
1. Introduction

Given a permutation $\Pi = < p_1, p_2, \ldots, p_n >$ and an empty stack, the elements of $\Pi$ can be passed through the stack using two elementary operations coded 'S' and 'X'. The operation 'S' denotes 'put the next element of $\Pi$ on top of stack' and 'X' stands for 'transfer the element on top of stack to the output'. A sequence $L$ of the above mentioned operations, is called a valid operation sequence (or simply an operation sequence) if and only if (1) all elements of $\Pi$ are transferred to the output and (2) the operation 'X' is never specified when the stack is empty. Conditions (1) and (2) imply that an operation sequence must consist of $2n$ operations, $n$ of each kind, the number of 'X' operations may never exceed the number of 'S' operations when $L$ is scanned from left to right.

We denote by $L(\Pi)$ the output permutation which results from passing $\Pi$ through a stack. For example if $\Pi = < 1, 3, 2, 4 >$ and $L = < S, X, S, X, S, S, S, X >$ then $L(\Pi) = < 1, 3, 4, 2 >$. A permutation $\Pi$ is sortable with a stack if and only if there exists an operation sequence $\overline{\Pi}$ such that $L(\Pi) = < 1, 2, \ldots, n >$, it is realizable with a stack if and only if an operation sequence $\overline{\Pi}$ exists such that $L(1, 2, \ldots, n) = \Pi$.

Given a permutation $\Pi$, let $\overline{\Pi}$ be the sequence of operations which sorts $\Pi$ with a stack. Scanning $\overline{\Pi}$ from left to right, we call each sequence of consecutive 'S' operations an $S$-group and such a sequence of 'X' operations an $X$-group. Clearly, the number of $X$-groups is equal to the number of $S$-groups, two $S$-groups are separated by an $X$-group and vice versa. The $S$-specification and the $X$-specification of $\overline{\Pi}$ are
vectors \( <s_1, s_2, \ldots, s_\lambda> \) and \( <x_1, x_2, \ldots, x_\lambda> \) respectively, where for \( 1 \leq i \leq \lambda \), \( s_i \) denotes the size of the \( i \)th \( S \)-group and \( x_i \) the size of the \( i \)th \( X \)-group.

We denote by \( SS_n \) the class of permutations of order \( n \) which are sortable with a stack, and by \( SR_n \) the class of permutations of the same order which are realizable with a stack. Those two classes are related as follows,

\[
\Pi \in SS_n \text{ if and only if } \Pi^{-1} \in SR_n. \tag{1}
\]

The class \( SR_n \) is characterized by Knuth [3, p. 239] by the following theorem,

**Theorem 1:** The permutation \( \Pi = <p_1, p_2, \ldots, p_n> \) is a member of \( SR_n \) if and only if it does not contain a subsequence

\[
< p_i, p_j, p_k > \text{ such that } p_i > p_k > p_j. \tag{2}
\]

From this theorem and the relation (1) we obtain a characterization of \( SS_n \) as follows,

**Theorem 1**: \( \Pi \in SS_n \) if and only if it does not contain a subsequence

\[
< p_i, p_j, p_k > \text{ such that } p_j > p_i > p_k. \tag{3}
\]

Two binary trees \( T \) and \( T' \) are similar (\( T = T' \)) if they have the same 'shape', formally, they both have the same number of nodes, with the left subtree of \( T \) similar to the left subtree of \( T' \) and the same holds true for right subtrees. For a node \( j \) in \( T \), we denote by \( L_T(j) \) and \( R_T(j) \) the left and right subtrees of \( j \) respectively.
A permutation can be mapped into a labelled binary tree $T_\pi$ using the following well-known construction.

**Construction - $T$**

Given $\pi = \langle p_1, p_2, \ldots, p_n \rangle$ and an empty tree $T$, assign $p_1$ to the root of the tree; for each $p_k$, $k = 2, 3, \ldots, n$ apply the rule

-- if $p_k$ is inserted into a non-empty subtree rooted by $p_i$, it is inserted into $L_T(p_i)$ if $p_k < p_i$ otherwise $p_k$ is inserted into $R_T(p_i)$ --

until an empty subtree is reached and then a root labeled $p_k$ is created to that subtree.

Construction-$T$ establishes a 1-1 correspondence between the set $SS_n$ and the set of n-noded binary trees [3;6.2.2]. Given a labeled tree $T$, its corresponding member of $SS_n$ can be obtained by reading the labels of $T$ in symmetric order (root, left subtree and right subtree).

The class $SS_n$ was studied in Knuth [3] and its relation to the classical ballot problem is shown in [4] and [7]. The correspondence between $SS_n$ and the set of binary trees is used in [8] to generate and rank all 'shapes' of n-noded binary trees. The cardinality of $SS_n$ is $C_n = (n+1)^{-1} \binom{2n}{n}$ (the n'th catalan number).

In this paper we study in detail some of the combinatorial properties of the class $SS_n$. In sections 2 and 3 expressions are derived for the expected length of some types of monotonic subsequences and the average number of inversions. The set $SS_n \cap SR_n$ is characterized and enumerated in section 4. In the last section the permutation graph associated with $\pi \in SS_n$ is shown to be an interval graph of a special type.
2. Monotonic subsequences in $SS_n$ and their relation to binary trees.

Let $\Pi = <p_1, p_2, \ldots, p_n>$ be a permutation on the set $N = \{1, 2, \ldots, n\}$. A descending subsequence of length $k$ in $\Pi$ satisfies,

$$p_{i_1} > p_{i_2} > \ldots > p_{i_k} \quad \text{and} \quad i_1 < i_2 < \ldots < i_k.$$ 

A descending subsequence is maximal in $\Pi$ if no element of $\Pi$ can be added to it without violating its monotonicity. A longest descending subsequence in $\Pi$ (LDS) contains the maximum number of elements among all descending subsequences in $\Pi$. We get the corresponding definitions for ascending subsequences by replacing $'>'$ with $'<'$ in the above, where LAS stands for 'longest ascending subsequence'. For $j \in N$, we denote by $R_\Pi(j)$ the set of elements to the right of $j$ in $\Pi$, and by $L_\Pi(j)$ the set of elements to the left of $j$ in $\Pi$. Two elements $p_i$ and $p_j$ form an inversion in $\Pi$ if $(p_i - p_j)(i-j) < 0$.

A descending run in $\Pi$ is a sequence of successive elements $p_{i_1}, p_{i_1+1}, \ldots, p_{i_1+k}$ such that

(a) $p_{i-1} < p_i$
(b) $p_{i+k} < p_{i+k+1}$
(c) $p_i > p_{i+1} > \ldots > p_{i+k}$

(We assume that $p_1 > p_0$ and $p_n < p_{n+1}$.

The inversion-table of $\Pi$, is a vector $<b_1, b_2, \ldots, b_n>$ such that for $1 \leq i \leq n$ $b_i$ counts the number of elements in $R_\Pi(i)$ which are smaller than $i$. It is well-known, that an inversion-table uniquely determines its corresponding permutation. We denote by $\Pi^{-1}$ the inverse permutation of $\Pi$, if $\Pi = \Pi^{-1}$ it is called an involution.
Example:

Let $\pi = <3,6,4,5,2,1>$. Then $<3,2,1>$ is a maximal descending subsequence in $\pi$, $<6,4,2,1>$ and $<3,4,5>$ are an LDS and an LAS respectively in $\pi$, $R_\pi(4) = <5,2,1>$ and $L_\pi(6) = <3>$. The inversion-table of $\pi$ is $<0,1,2,2,2,4>$. The runs of $\pi$ are $<3>$, $<6,4>$ and $<5,2,1>$. □

Theorem 3: The expected length of an LDS in a random permutation in $SS_n$ is asymptotically

$$\sqrt{\ln n} - 1.5 + \frac{11}{24}\sqrt{\frac{n}{\ln n}} + O(n^{-3/2}) \quad (4)$$

Proof: We show that the length of an LDS in $\pi \in SS_n$ is equal to the depth of stack which is needed to traverse $T_\pi$ in symmetric order.

Equation (4) is Knuth's result for the average depth of stack [3, Ex. 2.3.11].

We observe that the sequence of insertions and removals from stack made during the symmetric traversal of $T_\pi$ is equivalent to the sequence of operations required to sort $\pi$ with a stack.

Let $D = <d_{i_1}, d_{i_2}, \ldots, d_{i_\lambda}>$ be an LDS in $\pi$. While sorting $\pi$, no member of $D$ can leave the stack before $d_{i_\lambda}$ so the stack must have at least $\lambda$ entries.

Conversely, assume that the stack contains $m$ elements during the sorting process and $m > \lambda$. Let $B = <b_{i_1}, b_{i_2}, \ldots, b_{i_m}>$ be the elements in the stack, then $B$ must be a descending subsequence in $\pi$, a contradiction to the definition of $D$. □
Remark: The problem of finding the expected length of an LDS (or an LAS) in a random permutation is still unsolved analytically. Experimental results show good agreement with $2\sqrt{n}$ [2].

We need the following definitions to prove the corresponding result on the LAS.

A composition of a whole number $n$ into $m$ parts is a vector $C = <c_1, c_2, \ldots, c_m>$ such that $c_i > 0$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} c_i = n$. A composition $C$ of $n$ can be represented as a zig-zag graph, this graph contains $m$ rows with $c_i$ dots in the $i$-th row, for $i > 1$ the first dot in the $i$-th row is written under the last dot in row $i-1$. Given a composition $C$, we obtain its conjugate composition $\overline{C} = <\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_{n+1-m}>$ such that for $1 \leq i \leq n+1-m$, $\overline{c}_i$ is equal to the number of dots in the $i$-th column (from left) of the zig-zag graph of $C$. For example let $C = <3, 2, 4, 1>$ be a composition of the integer 10. The zig-zag graph of $C$ is

$$
\begin{array}{ccc}
& & \\
& & \\
& & \\
\vdots & \vdots & \\
& & \\
& & \\
\end{array}
$$

therefore $\overline{C} = <1, 1, 2, 2, 1, 1, 2>$.  

Let $\Pi$ and $\Pi_{RF}$ be two members of $SS_n$ (not necessarily distinct) such that their corresponding trees $T_\Pi$ and $T_{\Pi_{RF}}$ are reflections of each other about the vertical axis.
Lemma 1: Let $X = \langle x_1, x_2, \ldots, x_k \rangle$ and $X_{RF} = \langle x_1', x_2', \ldots, x_m' \rangle$ be the $X$-specifications of $\Gamma$ and $\Gamma_{RF}$ respectively. Then the vectors $X^R = \langle x_k, x_{k-1}, \ldots, x_1 \rangle$ (the reverse of $X$) and $X_{RF}$ are conjugate compositions of $n$.

Illustration: Consider the permutations $\Pi = \langle 6, 3, 2, 1, 4, 5, 8, 7 \rangle$ and $\Pi_{RF} = \langle 3, 1, 2, 6, 5, 4, 7, 8 \rangle$. The corresponding binary trees are shown in Figure 1 (a) and (b) respectively.

The $X$-specification of $\Gamma$ is $X = \langle 3, 1, 2, 2 \rangle$ and $X^R = \langle 2, 2, 1, 3 \rangle$. The zig-zag graph of $X^R$ is \ldots and therefore its conjugate is $X^R = \langle 1, 2, 3, 1, 1 \rangle$, we then have $X^R = X_{RF}$.

![Figure 1](image-url)

Proof: A binary tree is traversed in reverse symmetric order if a root and its two subtrees are visited in the order (1) right subtree (2) root (3) left subtree. We observe that the operations which are required in order to traverse $T_{\Pi}$ in reverse symmetric order are equivalent to those necessary for traversing $T_{\Pi_{RF}}$ in symmetric order. Therefore $\Gamma$ and $\Gamma_{RF}$ specify the stack operations for traversing $T_{\Pi}$ in symmetric and reverse symmetric order respectively. For two consecutive labels $i$ and $i-1$ we can have
(a) \( i \in R_\pi (i-1) \) or (b) \( i-1 \in L_\pi (i) \). While traversing \( T_\pi \) in symmetric order, (a) implies that \( i \) must be stacked after \( i-1 \) is written on output and therefore \( X(i-1) \) and \( X(i) \) are in different \( X \)-groups, (b) implies that \( i \) is present in stack when \( i-1 \) is written, hence \( X(i) \) and \( X(i-1) \) are in the same \( X \)-group. It is easy to see that in the reverse symmetric order traversal of \( T_\pi \), we have exactly the converse, i.e. the labels \( i \) and \( i-1 \) are written on output by the same \( X \)-group in \( \Gamma_{RF} \) in case (a) and by different \( X \)-groups in case (b).

We can represent \( X \) as a zig-zag graph in which the \( i \)-th row contains the elements written by the \( i \)-th \( X \)-group in \( \Gamma \). By the above argument, it follows that the \( i \)-th \( X \)-group in \( \Gamma_{RF} \) will write out elements of the \( i \)-th column in this graph, where counting starts from the rightmost column. For example in the above illustration the graph is  \[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
\end{array}
\]
\( \Gamma_{RF} \) write out \( <8>, <7,6>, <5,4,3>, <2>, <1> \), where brackets enclose elements of the same \( X \)-group. Therefore \( X^R \) and \( X_{RF} \) are conjugate compositions and \( k = n+1-m \).

\[\square\]

**Lemma 2:** The length of the LAS in \( \Pi \in SS_n \) is equal to the number of components in the \( S \)-specification (\( X \)-specification) of its sorting sequence.

**Proof:** Let \( \Gamma \) be a sorting sequence for \( \Pi \) with \( S \)-specification
<s_1, s_2, ..., s_k>. Then clearly II must have exactly k descending runs where the size of the i-th run is s_i. Let an LAS in II be of length k. Then k ≤ n since no two elements in an LAS are in the same descending run.

To show that n ≤ k, we construct a sequence D = <d_1, d_2, ..., d_k> where d_i is the last element in the i-th descending run in II. We show that D is an ascending subsequence in II by deriving a contradiction. Suppose that for some i, d_i > d_{i+1}, then there must be an element d in the i+1-th run such that d > d_i and d > d_{i+1} and II must contain a forbidden subsequence <d_i, d, d_{i+1}>

Theorem 4: The expected length of the LAS in a random permutation is SS_n is \( \frac{n+1}{2} \).

Proof: We define a mapping RF:SS_n → SS_n such that II ∈ SS_n is mapped into II_{RF} by RF. Suppose that the length of the LAS in II is equal to k. By Lemma 2 this is also the number of components in the S-specification and X-specification of the sorting sequence \( \bar{I} \). From Lemma 1, the length of the LAS in II_{RF} is n+1-k. Since RF is a one-to-one correspondence our result follows. □

Another subsequence, which was studied in permutations is the sequence of left to right maxima which is also called the distinguished subsequence by Brock & Baer [1]. For example the distinguished subsequence in <1,3,2,5,4,6> is <1,3,5,6>. It is shown by Knuth [3], [4] and in [1] that the expected length of this subsequence in a random permutation is \( H_n \) (the n-th harmonic number). The next theorem gives the corresponding
result for a random permutation in $SS_n$.

**Theorem 5:** The expected length of the distinguished subsequence in a random permutation of $SS_n$ is $3 - \frac{6}{n+2}$.

**Proof:** Given $\Pi = <p_1, p_2, \ldots, p_n> \in SS_n$ let $<p_{i_1}, p_{i_2}, \ldots, p_{i_n}>$ be the distinguished subsequence in $\Pi$. We can form $k+1$ permutations $\Pi_1, \Pi_2, \ldots, \Pi_{k+1}$ of length $n+1$ from $\Pi$ by inserting the number $n+1$ in each of the positions immediately to the left of $p_{i_j}$ in $\Pi$ for $1 \leq j \leq k$ or placing $n+1$ as the last element in $\Pi$. For example, if $\Pi = <1,3,2,4>$ then $<5,1,3,2,4> <1,5,3,2,4> <1,3,2,5,4> \text{ and } <1,3,2,4,5>$ are formed in this way. We now show that for $1 \leq i \leq k+1$ $\Pi_i \in SS_{n+1}$. If not, then for some $2 \leq j \leq k$ $\Pi_j$ must contain a forbidden subsequence of the form $<p_i, n+1, p_\ell>$ and $p_i < p_\ell$.

But this implies that $\Pi$ must contain a subsequence

$$<p_i, p_{i_j}, p_\ell>$$

Now $p_{i_j}$ is in the distinguished subsequence, and therefore sequence (6) is of type (3) thus contradicting $\Pi \in SS_n$. It is easy to see that inserting $n+1$ in any other position of $\Pi$ will create a permutation $\Pi'$ such that $\Pi' \notin SS_{n+1}$. On the other hand all the members of $SS_{n+1}$ can be generated from the members of $SS_n$ in this way. Let $a_\Pi$ be the length of the distinguished subsequence in $\Pi$. Then

$$|SS_{n+1}| = C_{n+1} = \sum_{\Pi \in SS_n} (a_\Pi + 1).$$
\[ C_{n+1} = \Sigma a_{\pi} + C_n \] (8)

\[
\frac{\Sigma a_{\pi}}{C_n} = \frac{C_{n+1}}{C_n} - 1 = \frac{1}{n+2} \left( \frac{2n + 2}{n+1} \right) - 1
\] (9)

which gives

\[
\frac{\Sigma a_{\pi}}{C_n} = 3 - \frac{6}{n+2}
\] (10)

Remark: This result is directly related to a theorem by Munro [5] which shows that the average length of a random walk on a binary tree is

\[ 2 - \frac{6}{n+2} \]

Corollary: The expected length of the first descending run in a random permutation of \( SS_n \) is \( 3 - \frac{6}{n+2} \).

Proof: Given \( \pi \in SS_n \), the elements of the LAS in \( \pi \) form the rightmost path in \( T_\pi \). By symmetry, the average length of the leftmost path over all n-noded binary trees is also \( 3 - \frac{6}{n+2} \). This path in \( T_\pi \) is formed by the members of the first descending run in \( \pi \), since under Construction -T this path is completed before any other part of the tree is constructed. \( \square \)
3. The Number of Involution in $SS_n$

It is well known [4] that a permutation is an involution if and only if it does not contain a cycle with more than two elements. Using this fact, we prove in Lemma 3 that the set of involutions in $SS_n$ is equal to $SS_n \cap SR_n$. A simple expression for the cardinality of this set is then calculated in Theorem 6.

Lemma 3: Let $\pi \in SS_n$, then $\pi$ is an involution if and only if $\pi \in SS_n \cap SR_n$.

Proof: The 'only if' part follows directly from the definitions. We prove the 'if' part by showing that a permutation which is a non-involution must contain at least one of the subsequences (2) or (3), therefore it is not a member of $SS_n \cap SR_n$.

Let $\pi$ be a non-involution, then $\pi$ contains a cycle of length $k \geq 3$. Let this cycle be $[a_1, a_2, \ldots, a_k]$ where $a_1$ is the smallest element in this cycle. We can arrange the elements of the cycle according to their original order in $\pi$ in the following way. First we sort the cycle into ascending order, then write under each element its right successor in the cycle, the second line thus obtained forms a subsequence of $\pi$. For example, if $\pi$ contains the cycle $[1, 4, 3, 6, 5]$ then the above operations will give $\begin{bmatrix} 1 & 3 & 4 & 5 & 6 \end{bmatrix}$ and $\begin{bmatrix} 4 & 6 & 3 & 1 & 5 \end{bmatrix}$ is a subsequence of $\pi$ ($a_1$ is considered to be the right successor of $a_k$). We distinguish between two cases:
**Case 1:** \( a_2 < a_3 \). Let \( k = 3 \), then after sorting the cycle we get

\[
\begin{bmatrix}
a_1 & a_2 & a_3 \\
a_2 & a_3 & a_1
\end{bmatrix}
\]

and \( <a_2,a_3,a_1> \) forms a subsequence (3) in \( \Pi \). Assume that \( k > 3 \). We sort the cycle by placing \( a_2 \) on the right of \( a_1 \), then inserting the elements \( a_k,a_{k-1},...,a_3 \) one by one into their correct positions. We write under each element its right successor when it is inserted. If \( a_k > a_2 \), then \( a_k \) is inserted on the right of \( a_2 \) and we get the same result as in the case \( k = 3 \), \( a_k \) playing the role of \( a_3 \).

Assume that \( a_k < a_2 \). We insert \( a_{k-1},a_{k-2},...,a_2 \) into their positions until an element \( a_{k-i} \) is found such that \( a_{k-i} > a_2 \), the existence of such an element is guaranteed since \( a_3 > a_2 \). The element \( a_{k-i+1} \) is smaller than \( a_2 \), hence after inserting \( a_{k-i} \) we have the following configuration

\[
\begin{bmatrix}
a_1 & \cdots & a_{k-i+1} & \cdots & a_2 & a_{k-i} \\
a_2 & \cdots & a_{k-i+2} & \cdots & a_3 & a_{k-i+1}
\end{bmatrix}
\]

(11)

and \( <a_2,a_3,a_{k-i+1}> \) forms a subsequence (3) in \( \Pi \).

**Case 2:** \( a_2 > a_3 \). If \( k = 3 \) we have the configuration

\[
\begin{bmatrix}
a_1 & a_3 & a_2 \\
a_2 & a_1 & a_3
\end{bmatrix}
\]

after sorting the cycle, and \( <a_2,a_1,a_3> \) forms a subsequence (2) in \( \Pi \). Assume \( k > 3 \). If \( a_k < a_2 \) we obtain the same subsequence, we therefore consider the case \( a_k > a_2 \). We use the same procedure as in Case 1, this time we search for the first element \( a_{k-i} \) such that \( a_{k-i+1} > a_2 \) and
$a_{k-i} < a_2$. We then have the configuration:

$$\begin{bmatrix}
a_1 & a_{k-i} & a_2 & \cdots & a_{k-1} & \cdots \\
a_2 & a_{k-i+1} & a_3 & \cdots & a_{k-1+2} & \cdots
\end{bmatrix}$$

(12)

and $< a_2, a_{k-i+1}, a_3 >$ is a subsequence (3) in $\Pi$. □

**Theorem 6:** The number of involutions in $SS_n$ is equal to $2^{n-1}$.

**Proof:** By Lemma 3, we have to show that there are $2^{n-1}$ permutations of length $n$ which do not contain subsequences (2) or (3). A permutation $\Pi \in SS_n \cap SR_n$ can be characterized by the following property of its maximal descending subsequences.

Let $D = < d_1, d_2, \ldots, d_k >$ be a maximal descending subsequence in a permutation $\Pi$ of order $n$, then $\Pi \in SS_n \cap SR_n$ if and only if for $1 \leq j \leq k-1$,

$$d_{i_j} = d_{i_{j+1}} + 1 \text{ (elements of } D \text{ appear in reverse natural order).}$$

(13)

**Proof:** Clearly every permutation which satisfies condition (13) is a member of $SS_n \cap SR_n$, since each of the forbidden subsequences (2) and (3) have at least one pair of elements which belong to a descending subsequence and are not in reverse natural order. We now show that if any violations of condition (13) occur in $\Pi$ then $\Pi \not\in SS_n \cap SR_n$.

Suppose that for some index $m$, $(1 \leq m \leq k-1)$ $d_{i_m} \neq d_{i_{m+1}} + 1$. Let $d_{i_{m+1}} = \lambda$. Then $\lambda$ cannot appear between $d_{i_m}$ and $d_{i_{m+1}}$ in $\Pi$,
since it is not a member of $D$. Therefore one of the two subsequences 
$<e, d_i, d_{i+1}, \ldots, x>$ or $<d_i, d_{i+1}, x>$ must appear in $\Pi$, thus 
contradicting $\Pi \in SS_n \cap SR_n$.

For each permutation $\Pi \in SS_n \cap SR_n$, we can generate two permutations 
$\Pi_1$ and $\Pi_2$ of order $n+1$ as follows;
(a) generate $\Pi_1$ by inserting $n+1$ one position to the left of $n$ in $\Pi$,
(b) generate $\Pi_2$ by putting $n+1$ after the rightmost element in $\Pi$.

Clearly, condition (13) is not violated in $\Pi_1$ and $\Pi_2$ thus generated.
Furthermore, inserting $n+1$ in any other position of $\Pi$, generates a 
maximal descending subsequence (with $n+1$ as its first element) which does 
not satisfy condition (13). Therefore $\Pi_1$ and $\Pi_2$ belong to $SS_{n+1} \cap SR_{n+1}$.
Since all the elements of $SS_{n+1} \cap SR_{n+1}$ are generated in this way, we have

$$|SS_{n+1} \cap SR_{n+1}| = 2|SS_n \cap SR_n|.$$  \hspace{1cm} (14)

Our result follows from the fact that $SS_3 \cap SR_3$ contains 4 elements, 
namely, $<1,2,3>$, $<1,3,2>$, $<2,1,3>$, $<3,2,1>$.  \hspace{1cm} $\Box$
4. The Average Number of Inversions in $SS_n$

**Lemma 4:** Let $<b_1, b_2, \ldots, b_n>$ be the inversion-table of $\pi \in SS_n$, then for node labelled $k$ in $T_\pi$, $|L_T(k)| = b_k$.

**Proof:** We show that the elements which are counted by $b_k$ are exactly the ones which are inserted into $L_T(k)$ by Construction-T. Clearly, only an element $j$ such that $j < k$ and $j \in L_\pi(k)$ can be inserted into $L_T(k)$. If no such element exists in $\pi$ then $b_k = 0$ and the subtree $L_T(k)$ is empty. Assume $b_k > 0$. Since $\pi \in SS_n$, all elements in $L_\pi(k)$ are either bigger or smaller than both $k$ and $j$, any other possibility will create a subsequence (3) in $\pi$. Therefore, application of the rule of Construction-T will force $j$ to be inserted into the same subtrees as $k$, finally $j$ must be compared with $k$ and since $j < k$ it follows that $j \in L_T(k)$. \[\square\]

**Theorem 7:** The average number of inversions in a random permutation of $SS_n$ is

$$\frac{1}{2} \left( \frac{4^n}{c_n} - 3n - 1 \right).$$ (15)

**Proof:** Let $i(\pi)$ denote the number of inversions in a permutation $\pi$ and $\text{int}(T)$ the internal path length of the tree $T$. The sum of sizes of all subtrees in a binary tree (or any other tree) is equal to $\text{int}(T)$. This follows from the fact that in a tree $T$, the distance of vertex $i$
from the root is equal to the number of subtrees in which \( i \) participates.

Let \( \langle b_1, b_2, \ldots, b_n \rangle \) be the inversion-table of a permutation \( \Pi \in SS_n \), then by definition

\[
\sum_{i=1}^{n} b_i = i(\Pi). \tag{16}
\]

By lemma 4, \( i(\Pi) \) is the sum of sizes of all left subtrees in \( T_\pi \). Hence, by the symmetry of left and right subtrees

\[
\sum_{\Pi \in SS_n} \text{int}(T_\pi) = 2 \sum_{\Pi \in SS_n} i(\Pi). \tag{17}
\]

The value of the left member of (17) is given in [3,p. 404] as

\[
\sum_{\Pi \in SS_n} \text{int}(T_\pi) = 4^n - (3n+1)C_n, \tag{18}
\]

from which (15) follows.

It is interesting to note that on the average a random permutation of \( SS_n \) contains \( O(n^{1.5}) \) inversions, where as the corresponding value for a random permutation of order \( n \) is \( O(n^2) \).
5. Graphs Associated with $SS_n$

We give some definitions and notations from graph theory which are required in this section.

A graph $G(V, E)$, consists of a vertex set $V$ and an edge set $E$, such that each edge in $E$ is associated with two vertices in $V$ called its end points. We consider here only graphs which have no two edges with the same two end points (parallel edges), and no edge for which its two end points are the same (self loop). Two vertices are adjacent if they are the end points of the same edge, this is denoted by $v_i \overset{G}{\longrightarrow} v_j$, otherwise they are non-adjacent denoted by $v_i \overset{G}{\not\longrightarrow} v_j$. The complement of $G$, denoted by $G^c$, has the same vertex set as $G$, two vertices are adjacent in $G^c$ if and only if they are non-adjacent in $G$.

A direction can be assigned to the edge $v_i \overset{G}{\longrightarrow} v_j$, this is denoted by $v_i \rightarrow v_j$. If all edges of $G$ are assigned a direction, it is called a digraph (directed graph). A digraph is transitive if for $v_i, v_j, v_k \in V$, the existence of $v_i \rightarrow v_j$ and $v_j \rightarrow v_k$ implies $v_i \rightarrow v_k$. A graph $G$ is transitively orientable (TRO), if it is possible to orient all its edges such that its directed image is transitive.

Let $G(N)$ be a graph which has its vertices labeled by the set $N = \{1, 2, \ldots, n\}$. Then $G(N)$ has a defining permutation with respect to its labeling, if there is a permutation $\Pi$ on $N$ such that;

$$i \overset{G(N)}{\longrightarrow} j \quad (\text{vertices are called by their labels}) \text{ if and only if } \quad i \text{ and } j \text{ form an inversion in } \Pi.$$

A graph $G$ is a permutation graph, if at least one of the possible
labelings of its vertices with \( N \), gives rise to a defining permutation.

**Example:** A permutation graph \( G \), with two labelings and their respective defining permutations, is shown in Figure 2

\[
\begin{align*}
\pi_1 &= <3,2,4,5,1> \\
\pi_2 &= <2,4,3,5,1>
\end{align*}
\]

*Figure 2*

The next theorem of [6] demonstrates the connection between permutation graphs and transitive graphs.

**Theorem:** A graph \( G \) is a permutation graph if and only if both \( G \) and \( G^C \) are TRO graphs.

A graph \( G \) with vertex set \( V(\{V\} = n) \), is an **interval graph** if there exists a family of intervals on the line \( I = (I_1, I_2, \ldots, I_n) \) such that
$v_i \in V$ corresponds to an interval $I_i$, and $v_i \overset{G}{\longrightarrow} v_j$ if and only if $I_i \cap I_j \neq \phi$. A nested interval graph is an interval graph which has a representing family $I$ such that for each pair of intervals $I_i$ and $I_j$, if $I_i \cap I_j \neq \phi$ then either $I_i \subset I_j$ or $I_j \subset I_i$ holds.

**Theorem 8:** The following conditions are equivalent:

1. $G$ is a permutation graph, with a defining permutation $\Pi \in SS_n$.
2. $G$ is a nested interval graph.

**Proof:** (2) Consider the sorting sequence of $\Pi$, where a line is drawn from each $S$ operation to its corresponding $X$ operation which removes from stack the element stacked by $S$. For example, for $\Pi = \langle 3, 1, 2 \rangle$ the following sorting sequence and lines are drawn $S \underline{S} \underline{X} S \underline{X} X$. Let $I_i$ be the line drawn between the $S$ and $X$ which stack and unstack $i$ in $\Pi$. For a pair of intervals $I_i$ and $I_j$ assume that $I_i$ has its left end to the left of $I_j$ ($i \in L_{\Pi}(j)$). Then two cases are possible:

(a) $i < j$, $i$ leaves the stack before $j$ is stacked and $I_i \cap I_j = \phi$

(b) $i > j$, $i$ leaves the stack only after $j$ is unstacked and $I_i \supset I_j$.

In the permutation graph $G$ labeled with $\Pi$, vertices labeled $i$ and $j$ are adjacent only in case (b) where $i$ and $j$ form an inversion in $\Pi$ hence $G$ is a nested interval graph. Conversely, let $I$ be a family of $n$ intervals which is represented by a nested interval graph $G$. Then, $I$ can be mapped into a sequence of $S$'s and $X$'s by reversing the above procedure. By reading this sequence of $S$'s and $X$'s from left to right we obtain a sorting sequence of some $\Pi \in SS_n$ and $\Pi$ is a defining permutation for $G$. □
Conclusions

In this paper we studied some of the combinatorial properties of members of $SS_n$, and the relations of these properties to the corresponding binary tree. It was observed that members of $SS_n$ tend to be more 'ordered' than ordinary permutations in the sense that on the average they contain less inversions, longer maximum ascending subsequences and shorter maximum descending subsequences.
References


