SUBSEQUENCES IN STACK SORTABLE PERMUTATIONS AND THEIR RELATION TO BINARY TREES

by

D. Rotem

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Department of Computer Science
University of Waterloo
Waterloo, Ontario, Canada

ABSTRACT

The class SS_n of stack sortable permutations is known to be in 1-1 correspondence with n-noded binary trees. Expressions are derived for the average length of several types of monotonic subsequences in members of SS_n. The relations between these subsequences and properties of the corresponding tree are demonstrated. It is also shown that the permutation graph of a member of SS_n is an interval graph of a special type.

1. Introduction

Given a permutation $\Pi = \langle p_1, p_2, \ldots, p_n \rangle$ and an empty stack, the elements of Π can be passed through the stack using two elementary operations coded 'S' and 'X'. The operation 'S' denotes 'put the next element of Π on top of stack' and 'X' stands for 'transfer the element on top of stack to the output'. A sequence L of the above mentioned operations, is called a valid operation sequence (or simply an operation sequence) if and only if (1) all elements of Π are transferred to the output and (2) the operation 'X' is never specified when the stack is empty. Conditions (1) and (2) imply that an operation sequence must consist of 2n operations, n of each kind, the number of 'X' operations may never exceed the number of 'S' operations when L is scanned from left to right.

We denote by $L(\Pi)$ the output permutation which results from passing Π through a stack. For example if $\Pi = \langle 1,3,2,4 \rangle$ and $L = \langle S,X,S,X,S,S,X,X \rangle$ then $L(\Pi) = \langle 1,3,4,2 \rangle$. A permutation Π is sortable with a stack if and only if there exists an operation sequence $\bar{L}$ such that $\bar{L}(\Pi) = \langle 1,2,\ldots,n \rangle$, it is realizable with a stack if and only if an operation sequence $\bar{R}$ exists such that $\bar{R}(\langle 1,2,\ldots,n \rangle) = \Pi$.

Given a permutation Π, let $\bar{L}$ be the sequence of operations which sorts Π with a stack. Scanning $\bar{L}$ from left to right, we call each sequence of consecutive 'S' operations an <u>S-group</u> and such a sequence of 'X' operations an <u>X-group.</u> Clearly, the number of X-groups is equal to the number of S-groups, two S-groups are seperated by an X-group and vice versa. The <u>S-specification</u> and the <u>X-specification</u> of $\bar{L}$ are vectors $< s_1, s_2, \ldots, s_\ell >$ and $< x_1, x_2, \ldots, x_\ell >$ respectively, where for $1 \le i \le \ell$ s_i denotes the size of the ith S-group and x_i the size of the ith X-group.

We denote by SS_n the class of permutations of order n which are sortable with a stack, and by SR_n the class of permutations of the same order which are realizable with a stack. Those two classes are related as follows,

$$\Pi \in SS_n \quad \text{if and only if} \quad \Pi^{-1} \in SR_n. \qquad (1)$$

The class SR_n is characterized by Knuth [3, p. 239] by the following theorem,

Theorem 1: The permutation $\Pi = < p_1, p_2, \ldots, p_n >$ is a member of SR_n if and only if it does not contain a subsequence

$$< p_i, p_j, p_k > \text{ such that } p_i > p_k > p_j. \qquad (2)$$

From this theorem and the relation (1) we obtain a characterization of SS_n as follows,

Theorem 1*: $\Pi \in SS_n$ if and only if it does not contain a subsequence

$$< p_i, p_j, p_k > \text{ such that } p_j > p_i > p_k. \qquad (3)$$

Two binary trees T and T' are similar $(T = T')$ if they have the same 'shape', formally, they both have the same number of nodes, with the left subtree of T similar to the left subtree of T' and the same holds true for right subtrees. For a node j in T, we denote by $L_T(j)$ and $R_T(j)$ the left and right subtrees of j respectively.

A permutation can be mapped into a labelled binary tree T_π using the following well-known construction.

Construction - T

Given $\Pi = \langle p_1, p_2, \ldots, p_n \rangle$ and an empty tree T, assign p_1 to the root of the tree; for each p_k, $k = 2,3,\ldots,n$ apply the rule

-- if p_k is inserted into a non-empty subtree rooted by p_i, it is inserted into $L_T(p_i)$ if $p_k < p_i$ otherwise p_k is inserted into $R_T(p_i)$ --

until an empty subtree is reached and then a root labeled p_k is created to that subtree.

Construction-T establishes a 1-1 correspondence between the set SS_n and the set of n-noded binary trees [3;6.2.2]. Given a labeled tree T, its corresponding member of SS_n can be obtained by reading the labels of T in symmetric order (root, left subtree and right subtree).

The class SS_n was studied in Knuth [3] and its relation to the classical ballot problem is shown in [4] and [7]. The correspondence between SS_n and the set of binary trees is used in [8] to generate and rank all 'shapes' of n-noded binary trees. The cardinality of SS_n is $C_n = (n+1)^{-1}\binom{2n}{n}$ (the n'th catalan number).

In this paper we study in detail some of the combinatorial properties of the class SS_n. In sections 2 and 3 expressions are derived for the expected length of some types of monotonic subsequences and the average number of inversions. The set $SS_n \cap SR_n$ is characterized and enumerated in section 4. In the last section the permutation graph associated with $\Pi \in SS_n$ is shown to be an interval graph of a special type.

2. Monotonic subsequences in SS_n and their relation to binary trees.

Let $\Pi = \langle p_1, p_2, \ldots, p_n \rangle$ be a permutation on the set $N = \{1, 2, \ldots, n\}$. A descending subsequence of length k in Π satisfies,

$$p_{i_1} > p_{i_2} > \ldots > p_{i_k} \quad \text{and} \quad i_1 < i_2 < \ldots < i_k.$$

A descending subsequence is maximal in Π if no element of Π can be added to it without violating its monotonicity. A longest descending subsequence in Π (LDS) contains the maximum number of elements among all descending subsequences in Π. We get the corresponding definitions for ascending subsequences by replacing '>' with '<' in the above, where LAS stands for 'longest ascending subsequence'. For $j \in N$, we denote by $R_\pi(j)$ the set of elements to the right of j in Π, and by $L_\pi(j)$ the set of elements to the left of j in Π. Two elements p_i and p_j form an inversion in Π if $(p_i - p_j)(i-j) < 0$.

A descending run in Π is a sequence of successive elements $p_i, p_{i+1}, \ldots, p_{i+k}$ such that

(a) $p_{i-1} < p_i$

(b) $p_{i+k} < p_{i+k+1}$

(c) $p_i > p_{i+1} > \ldots > p_{i+k}$

(We assume that $p_1 > p_0$ and $p_n < p_{n+1}$.

The inversion-table of Π, is a vector $\langle b_1, b_2, \ldots, b_n \rangle$ such that for $1 \le i \le n$ b_i counts the number of elements in $R_\pi(i)$ which are smaller than i. It is well-known, that an inversion-table uniquely determines its corresponding permutation. We denote by Π^{-1} the inverse permutation of Π, if $\Pi = \Pi^{-1}$ it is called an involution.

Example:

Let $\Pi = <3,6,4,5,2,1>$. Then $<3,2,1>$ is a maximal descending subsequence in Π, $<6,4,2,1>$ and $<3,4,5>$ are an LDS and an LAS respectively in Π, $R_\pi(4) = <5,2,1>$ and $L_\pi(6) = <3>$. The inversion-table of Π is $<0,1,2,2,2,4>$. The runs of Π are $<3>$, $<6,4>$ and $<5,2,1>$. □

Theorem 3: The expected length of an LDS in a random permutation in SS_n is asymptotically

$$\sqrt{\Pi n} - 1.5 + \frac{11}{24}\cdot\sqrt{\frac{\Pi}{n}} + O(n^{-3/2}) \qquad (4)$$

Proof: We show that the length of an LDS in $\Pi \in SS_n$ is equal to the depth of stack which is needed to traverse T_Π in symmetric order. Equation (4) is Knuth's result for the average depth of stack [3,Ex. 2.3,11].

We observe that the sequence of insertions and removals from stack made during the symmetric traversal of T_Π is equivalent to the sequence of operations required to sort Π with a stack.

Let $D = <d_{i_1}, d_{i_2}, \ldots, d_{i_\ell}>$ be an LDS in Π. While sorting Π, no member of D can leave the stack before d_{i_ℓ} so the stack must have at least ℓ entries.

Conversely, assume that the stack contains m elements during the sorting process and $m > \ell$. Let $B = <b_{i_1}, b_{i_2}, \ldots, b_{i_m}>$ be the elements in the stack, then B must be a descending subsequence in Π, a contradiction to the definition of D. □

Remark: The problem of finding the expected length of an LDS (or an LAS) in a random permutation is still unsolved analytically. Experimental results show good agreement with $2\sqrt{n}$ [2].

We need the following definitions to prove the corresponding result on the LAS.

A composition of a whole number n into m parts is a vector $C = \langle c_1, c_2, \ldots, c_m \rangle$ such that $c_i > 0$ for $1 \le i \le m$ and $\sum_{i=1}^{m} c_i = n$. A composition C of n can be represented as a zig-zag graph, this graph contains m rows with c_i dots in the i-th row, for $i > 1$ the first dot in the i-th row is written under the last dot in row $i-1$. Given a composition C, we obtain its conjugate composition $\overline{C} = \langle \overline{c}_1, \overline{c}_2, \ldots, \overline{c}_{n+1-m} \rangle$ such that for $1 \le i \le n+1-m$, $\overline{c}_i$ is equal to the number of dots in the i-th column (from left) of the zig-zag graph of C. For example let $C = \langle 3,2,4,1 \rangle$ be a composition of the integer 10. The zig-zag graph of C is

```
. . .
    . .
      . . . .
            .
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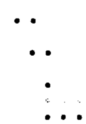
therefore $\overline{C} = \langle 1,1,2,2,1,1,2 \rangle$.

Let Π and Π_{RF} be two members of SS_n (not necessarily distinct) such that their corresponding trees T_π and $T_{\pi_{RF}}$ are reflections of each other about the vertical axis.

<u>Lemma 1</u>: Let $X = \langle x_1, x_2, \ldots, x_k \rangle$ and $X_{RF} = \langle x_1', x_2', \ldots, x_m' \rangle$ be the X-specifications of $\bar{L}$ and $\bar{L}_{RF}$ respectively. Then the vectors $X^R = \langle x_k, x_{k-1}, \ldots, x_1 \rangle$ (the reverse of X) and X_{RF} are conjugate compositions of n.

<u>Illustration</u>: Consider the permutations $\Pi = \langle 6,3,2,1,4,5,8,7 \rangle$ and $\Pi_{RF} = \langle 3,1,2,6,5,4,7,8 \rangle$. The corresponding binary trees are shown in Figure 1 (a) and (b) respectively.

The X-specification of $\bar{L}$ is $X = \langle 3,1,2,2 \rangle$ and $X^R = \langle 2,2,1,3 \rangle$. The zig-zag graph of X^R is



and therefore its conjugate **is $X^R = \langle 1,2,3,1,1 \rangle$, we then have $X^R = X_{RF}$.**

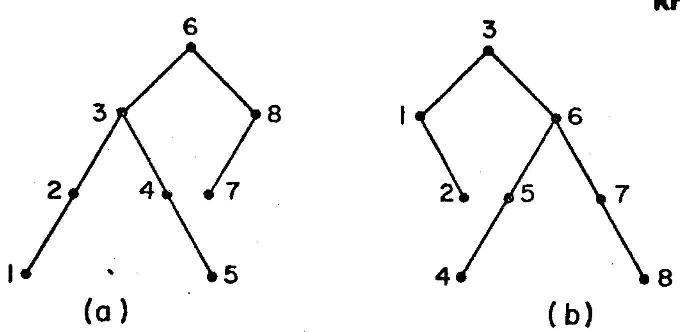


Figure 1

<u>Proof</u>: **A binary tree is traversed in reverse sy**mmetric order if a root and its two subtrees are visited in the order (1) right subtree (2) root (3) left subtree. We observe that the operations which are required in order to traverse T_π in reverse symmetric order are equivalent to those necessary for traversing $T_{\pi_{RF}}$ in symmetric order. Therefore $\bar{L}$ and $\bar{L}_{RF}$ specify the stack operations for traversing T_π in symmetric and reverse symmetric order respectively. For two consecutive labels i and $i-1$ we can have

(a) $i \in R_{T_\pi}(i-1)$ or (b) $i-1 \in L_{T_\pi}(i)$. While traversing T_π in symmetric order, (a) implies that i must be stacked after i-1 is written on output and therefore X(i-1) and X(i) are in different X-groups, (b) implies that i is present in stack when i-1 is written, hence X(i) and X(i-1) are in the same X-group. It is easy to see that in the reverse symmetric order traversal of T_π we have exactly the converse, i.e. the labels i and i-1 are written on output by the same X-group in $\overline{L}_{RF}$ in case (a) and by different X-groups in case (b).

We can represent X as a zig-zag graph in which the i-th row contains the elements written by the i-th X-group in $\overline{L}$. By the above argument, it follows that the i-th X-group in $\overline{L}_{RF}$ will write out elements of the i-th column in this graph, where counting starts from the rightmost column. For example in the above illustration the graph is

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123
  4
  56
   78
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and the X-group of $\overline{L}_{RF}$ write out < 8 > , < 7,6 > , < 5,4,3 > , < 2 > , < 1 > , where brackets enclose elements of the same X-group. Therefore X^R and X_{RF} are conjugate compositions and k = n+1-m. □

<u>Lemma 2</u>: The length of the LAS in $\Pi \in SS_n$ is equal to the number of components in the S-specification (X-specification) of its sorting sequence.

<u>Proof</u>: Let $\overline{L}$ be a sorting sequence for Π with S-specification

$<s_1, s_2, \ldots, s_\ell>$. Then clearly Π must have exactly ℓ descending runs where the size of the i-th run is s_i. Let an LAS in Π be of length k. Then $k \leq \ell$ since no two elements in an LAS are in the same descending run. To show that $\ell \leq k$, we construct a sequence $D = <d_1, d_2, \ldots, d_\ell>$ where d_i is the last element in the ith descending run in Π. We show that D is an ascending subsequence in Π by deriving a contradiction. Suppose that for some i $d_i > d_{i+1}$, then there must be an element d in the $i+1^{st}$ run such that $d > d_i$ and $d > d_{i+1}$ and Π must contain a forbidden subsequence $<d_i, d, d_{i+1}>$. □

Theorem 4: The expected length of the LAS in a random permutation is SS_n is $\frac{n+1}{2}$.

Proof: We define a mapping $RF: SS_n \to SS_n$ such that $\Pi \in SS_n$ is mapped into Π_{RF} by RF. Suppose that the length of the LAS in Π is equal to k. By Lemma 2 this is also the number of components in the S-specification and X-specification of the sorting sequence $\bar{C}$. From Lemma 1, the length of the LAS in Π_{RF} is $n+1-k$. Since RF is a one-to-one correspondence our result follows. □

Another subsequence, which was studied in permutations is the sequence of left to right maxima which is also called the distinguished subsequence by Brock & Baer [1]. For example the distinguished subsequence in $<1,3,2,5,4,6>$ is $<1,3,5,6>$. It is shown by Knuth [3], [4] and in [1] that the expected length of this subsequence in a random permutation is H_n (the n-th harmonic number). The next theorem gives the corresponding result for a random permutation in SS_n.

Theorem 5: The expected length of the distinguished subsequence in a random permutation of SS_n is $3 - \frac{6}{n+2}$.

Proof: Given $\Pi = \langle p_1, p_2, \ldots, p_n \rangle \in SS_n$ let $\langle p_{i_1}, p_{i_2}, \ldots, p_{i_n} \rangle$ be the distinguished subsequence in Π. We can form $k+1$ permutations $\Pi_1, \Pi_2, \ldots, \Pi_{k+1}$ of length $n+1$ from Π by inserting the number $n+1$ in each of the positions immediately to the left of p_{i_j} in Π for $1 \le j \le k$ or placing $n+1$ as the last element in Π. For example, if $\Pi = \langle 1,3,2,4 \rangle$ then $\langle 5,1,3,2,4 \rangle$ $\langle 1,5,3,2,4 \rangle$ $\langle 1,3,2,5,4 \rangle$ and $\langle 1,3,2,4,5 \rangle$ are formed in this way. We now show that for $1 \le i \le k+1$ $\Pi_i \in SS_{n+1}$. If not, then for some $2 \le j \le k$ Π_j must contain a forbidden subsequence of the form $\langle p_i, n+1, p_\ell \rangle$ and

$$\begin{array}{l} p_i > p_\ell \\ i < \ell \end{array} \tag{5}$$

But this implies that Π must contain a subsequence

$$\langle p_i, p_{i_j}, p_\ell \rangle \tag{6}$$

Now p_{i_j} is in the distinguished subsequence, and therefore sequence (6) is of type (3) thus contradicting $\Pi \in SS_n$. It is easy to see that inserting $n+1$ in any other position of Π will create a permutation Π' such that $\Pi' \notin SS_{n+1}$. On the other hand all the members of SS_{n+1} can be generated from the members of SS_n in this way. Let a_π be the length of the distinguished subsequence in Π. Then

$$|SS_{n+1}| = C_{n+1} = \sum_{\Pi \in SS_n} (a_\pi + 1). \tag{7}$$

$$C_{n+1} = \Sigma a_\pi + C_n \tag{8}$$

$$\frac{\Sigma a_\pi}{C_n} = \frac{C_{n+1}}{C_n} - 1 = \frac{\frac{1}{n+2}\binom{2n+2}{n+1}}{\frac{1}{n+1}\binom{2n}{n}} - 1 \tag{9}$$

which gives

$$\frac{\Sigma a_\pi}{C_n} = 3 - \frac{6}{n+2} \qquad \square \tag{10}$$

Remark: This result is directly related to a theorem by Munro [5] which shows that the average length of a random walk on a binary tree is $2 - \frac{6}{n+2}$.

Corollary: The expected length of the first descending run in a random permutation of SS_n is $3 - \frac{6}{n+2}$.

Proof: Given $\Pi \in SS_n$, the elements of the LAS in Π form the rightmost path in T_π. By symmetry, the average length of the leftmost path over all n-noded binary trees is also $3 - \frac{6}{n+2}$. This path in T_π is formed by the members of the first descending run in Π, since under Construction -T this path is completed before any other part of the tree is constructed. $\square$

3. The Number of Involutions in SS_n

It is well known [4] that a permutation is an involution if and only if it does not contain a cycle with more than two elements. Using this fact, we prove in Lemma 3 that the set of involutions in SS_n is equal to $SS_n \cap SR_n$. A simple expression for the cardinality of this set is then calculated in Theorem 6.

Lemma 3: Let $\Pi \in SS_n$, then Π is an involution if and only if $\Pi \in SS_n \cap SR_n$.

Proof: The 'only if' part follows directly from the definitions. We prove the 'if' part by showing that a permutation which is a non-involution must contain at least one of the subsequences (2) or (3), therefore it is not a member of $SS_n \cap SR_n$.

Let Π be a non-involution, then Π contains a cycle of length $k \geq 3$. Let this cycle be $[a_1, a_2, \ldots, a_k]$ where a_1 is the smallest element in this cycle. We can arrange the elements of the cycle according to their original order in Π in the following way. First we sort the cycle into ascending order, then write under each element its right successor in the cycle, the second line thus obtained forms a subsequence of Π. For example, if Π contains the cycle $[1,4,3,6,5]$ then the above operations will give $\begin{bmatrix} 1 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 1 & 5 \end{bmatrix}$ and $\langle 4,6,3,1,5 \rangle$ is a subsequence of Π (a_1 is considered to be the right successor of a_k). We distinguish between two cases:

Case 1: $a_2 < a_3$. Let $k = 3$, then after sorting the cycle we get $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \end{bmatrix}$ and $<a_2,a_3,a_1>$ forms a subsequence (3) in Π. Assume that $k > 3$. We sort the cycle by placing a_2 on the right of a_1, then inserting the elements $a_k,a_{k-1},\dots,a_3$ one by one into their correct positions. We write under each element its right successor when it is inserted. If $a_k > a_2$ then a_k is inserted on the right of a_2 and we get the same result as in the case $k = 3$, a_k playing the role of a_3. Assume that $a_k < a_2$. We insert $a_{k-1},a_{k-2},\dots$ into their positions until an element a_{k-i} is found such that $a_{k-i} > a_2$, the existence of such an element is guaranteed since $a_3 > a_2$. The element a_{k-i+1} is smaller than a_2, hence after inserting a_{k-i} we have the following configuration

$$\begin{bmatrix} a_1 \cdots\cdots a_{k-i+1} \cdots\cdots a_2 \; a_{k-i} \\ a_2 \cdots\cdots a_{k-i+2} \cdots\cdots a_3 \; a_{k-i+1} \end{bmatrix} \qquad (11)$$

and $<a_2,a_3,a_{k-i+1}>$ forms a subsequence (3) in Π.

Case 2: $a_2 > a_3$. If $k = 3$ we have the configuration $\begin{matrix} a_1 & a_3 & a_2 \\ a_2 & a_1 & a_3 \end{matrix}$ after sorting the cycle, and $<a_2,a_1,a_3>$ forms a subsequence (2) in Π. Assume $k > 3$. If $a_k < a_2$ we obtain the same subsequence, we therefore consider the case $a_k > a_2$. We use the same procedure as in Case 1, this time we search for the first element a_{k-i} such that $a_{k-i+1} > a_2$ and

$a_{k-i} < a_2$. We then have the configuration

$$\begin{bmatrix} a_1 & a_{k-i} & a_2 & \cdots & a_{k-i+1} & \cdots \\ a_2 & a_{k-i+1} & a_3 & \cdots & a_{k-i+2} & \cdots \end{bmatrix} \tag{12}$$

and $< a_2, a_{k-i+1}, a_3 >$ is a subsequence (3) in Π. □

Theorem 6: The number of involutions in SS_n is equal to 2^{n-1}.

Proof: By Lemma 3, we have to show that there are 2^{n-1} permutations of length n which do not contain subsequences (2) or (3). A permutation $\Pi \in SS_n \cap SR_n$ can be characterized by the following property of its maximal descending subsequences.

Let $D = < d_{i_1}, d_{i_2}, \ldots, d_{i_k} >$ be a maximal descending subsequence in a permutation Π of order n, then $\Pi \in SS_n \cap SR_n$ if and only if for $i \leq j \leq k-1$,

$d_{i_j} = d_{i_{j+1}} + 1$ (elements of D appear in reverse natural order). (13)

Proof: Clearly every permutation which satisfies condition (13) is a member of $SS_n \cap SR_n$, since each of the forbidden subsequences (2) and (3) have at least one pair of elements which belong to a descending subsequence and are not in reverse natural order. We now show that if any violations of condition (13) occur in Π then $\Pi \notin SS_n \cap SR_n$.

Suppose that for some index m, $(1 \leq m \leq k-1)$ $d_{i_m} \neq d_{i_{m+1}} + 1$. Let $d_{i_{m+1}} + 1 = \ell$. Then ℓ cannot appear between d_{i_m} and $d_{i_{m+1}}$ in Π,

since it is not a member of D. Therefore one of the two subsequences $<\ell, d_{i_m}, d_{i_{m+1}}>$ or $<d_{i_m}, d_{i_{m+1}}, \ell>$ must appear in Π, thus contradicting $\Pi \in SS_n \cap SR_n$.

For each permutation $\Pi \in SS_n \cap SR_n$, we can generate two permutations Π_1 and Π_2 of order n+1 as follows;

(a) generate Π_1 by inserting n+1 one position to the left of n in Π,

(b) generate Π_2 by putting n+1 after the rightmost element in Π.

Clearly, condition (13) is not violated in Π_1 and Π_2 thus generated. Furthermore, inserting n+1 in any other position of Π, generates a maximal descending subsequence (with n+1 as its first element) which does not satisfy condition (13). Therefore Π_1 and Π_2 belong to $SS_{n+1} \cap SR_{n+1}$. Since all the elements of $SS_{n+1} \cap SR_{n+1}$ are generated in this way, we have

$$|SS_{n+1} \cap SR_{n+1}| = 2|SS_n \cap SR_n|. \quad (14)$$

Our result follows from the fact that $SS_3 \cap SR_3$ contains 4 elements, namely, $<1,2,3>$, $<1,3,2>$, $<2,1,3>$, $<3,2,1>$. □

4. The Average Number of Inversions in SS_n

Lemma 4: Let $< b_1, b_2, \ldots, b_n >$ be the inversion-table of $\Pi \in SS_n$, then for node labelled k in T_π, $|L_{T_\pi}(k)| = b_k$.

Proof: We show that the elements which are counted by b_k are exactly the ones which are inserted into $L_{T_\pi}(k)$ by Construction-T. Clearly, only an element j such that $j < k$ and $j \in L_\pi(k)$ can be inserted into $L_{T_\pi}(k)$. If no such element exists in Π then $b_k = 0$ and the subtree $L_{T_\pi}(k)$ is empty. Assume $b_k > 0$. Since $\Pi \in SS_n$, all elements in $L_\pi(k)$ are either bigger or smaller than both k and j, any other possibility will create a subsequence (3) in Π. Therefore, application of the rule of Construction-T will force j to be inserted into the same subtrees as k, finally j must be compared with k and since $j < k$ it follows that $j \in L_{T_\pi}(k)$. □

Theorem 7: The average number of inversions in a random permutation of SS_n is

$$\frac{1}{2}\left(\frac{4^n}{C_n} - 3n - 1\right). \tag{15}$$

Proof: Let $i(\Pi)$ denote the number of inversions in a permutation Π and $int(T)$ the internal path length of the tree T. The sum of sizes of all subtrees in a binary tree (or any other tree) is equal to $int(T)$. This follows from the fact that in a tree T, the distance of vertex i

from the root is equal to the number of subtrees in which i participates.

Let $\langle b_1, b_2, \ldots, b_n \rangle$ be the inversion-table of a permutation $\Pi \in SS_n$, then by definition

$$\sum_{i=1}^{n} b_i = i(\Pi). \tag{16}$$

By lemma 4, $i(\Pi)$ is the sum of sizes of all left subtrees in T_π. Hence, by the symmetry of left and right subtrees

$$\sum_{\Pi \in SS_n} \mathrm{int}(T_\pi) = 2\sum_{\Pi \in SS_n} i(\Pi). \tag{17}$$

The value of the left member of (17) is given in [3,p. 404] as

$$\sum_{\Pi \in SS_n} \mathrm{int}(T_\pi) = 4^n - (3n+1)C_n, \tag{18}$$

from which (15) follows.

It is interesting to note that on the average a random permutation of SS_n contains $O(n^{1.5})$ inversions, where as the corresponding value for a random permutation of order n is $O(n^2)$.

5. Graphs Associated with SS_n

We give some definitions and notations from graph theory which are required in this section.

A graph $G(V,E)$, consists of a vertex set V and an edge set E, such that each edge in E is associated with two vertices in V called its end points. We consider here only graphs which have no two edges with the same two end points (parallel edges), and no edge for which its two end points are the same (self loop). Two vertices are adjacent if they are the end points of the same edge, this is denoted by $v_i \underset{G}{—} v_j$, otherwise they are non-adjacent denoted by $v_i \underset{G}{\not—} v_j$. The complement of G, denoted by G^C, has the same vertex set as G, two vertices are adjacent in G^C if and only if they are non-adjacent in G.

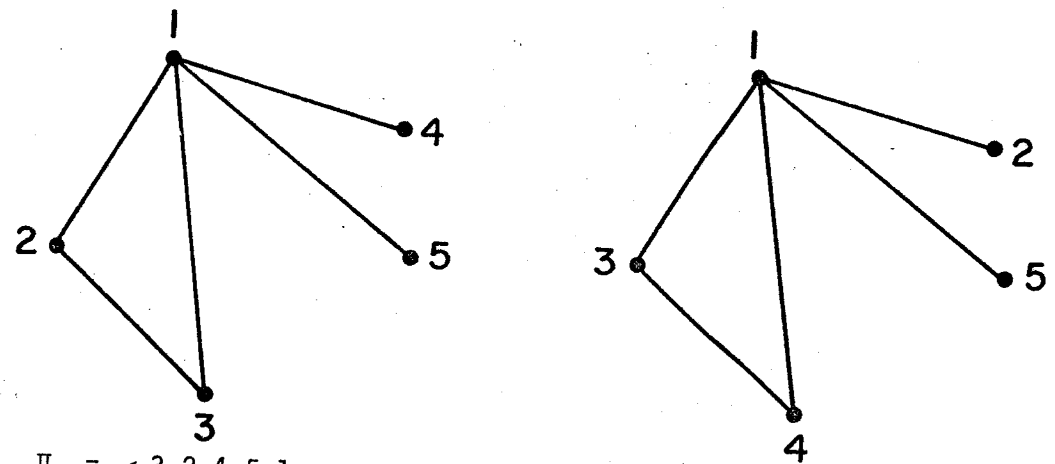
A direction can be assigned to the edge $v_i \underset{G}{—} v_j$, this is denoted by $v_i \rightarrow v_j$. If all edges of G are assigned a direction, it is called a digraph (directed graph). A digraph is transitive if for $v_i, v_j, v_k \in V$, the existence of $v_i \rightarrow v_j$ and $v_j \rightarrow v_k$ implies $v_i \rightarrow v_k$. A graph G is transitively orientable (TRO), if it is possible to orient all its edges such that its directed image is transitive.

Let $G(N)$ be a graph which has its vertices labeled by the set $N = \{1,2,\dots,n\}$. Then $G(N)$ has a defining permutation with respect to its labeling, if there is a permutation Π on N such that;

$i \underset{G(N)}{—} j$ (vertices are called by their labels) if and only if i and j form an inversion in Π.

A graph G is a permutation graph, if at least one of the possible labelings of its vertices with N, gives rise to a defining permutation.

Example: A permutation graph G, with two labelings and their respective defining permutations, is shown in Figure 2



$\Pi_1 = < 3,2,4,5,1 >$ $\Pi_2 = < 2,4,3,5,1 >$

Figure 2

The next theorem of [6] demonstrates the connection between permutation graphs and transitive graphs.

Theorem : A graph G is a permutation graph if and only if both G and G^C are TRO graphs.

A graph G with vertex set $V(|V| = n)$, is an interval graph if there exists a family of intervals on the line $I = (I_1, I_2, \ldots, I_n)$ such that

$v_i \in V$ corresponds to an interval I_i, and $v_i \underset{G}{—} v_j$ if and only if $I_i \cap I_j \neq \phi$. A <u>nested interval graph</u> is an interval graph which has a representing family I such that for each pair of intervals I_i and I_j, if $I_i \cap I_j \neq \phi$ then either $I_i \subset I_j$ or $I_j \subset I_i$ holds.

<u>Theorem 8</u>: The following conditions are equivalent:

(1) G is a permutation graph, with a defining permutation $\Pi \in SS_n$.

(2) G is a nested interval graph.

<u>Proof</u>: (2) Consider the sorting sequence of Π, where a line is drawn from each S operation to its corresponding X operation which removes from stack the element stacked by S. For example, for $\Pi = \langle 3,1,2 \rangle$ the following sorting sequence and lines are drawn $\underline{S\ \underline{S\ X}\ \underline{S\ X}\ X}$. Let I_i be the line drawn between the S and X which stack and unstack i in Π. For a pair of intervals I_i and I_j assume that I_i has its left end to the left of I $(i \in L_\pi(j))$. Then two cases are possible:

(a) $i < j$, i leaves the stack before j is stacked and $I_i \cap I_j = \phi$

(b) $i > j$, i leaves the stack only after j is unstacked and $I_i \supset I_j$.

In the permutation graph G labeled with Π, vertices labeled i and j are adjacent only in case (b) where i and j form an inversion in Π hence G is a nested interval graph. Conversely, let I be a family of n intervals which is represented by a nested interval graph G. Then, I can be mapped into a sequence of S's and X's by reversing the above procedure. By reading this sequence of S's and X's from left to right we obtain a sorting sequence of some $\Pi \in SS_n$ and Π is a defining permutation for G. $\square$

Conclusions

In this paper we studied some of the combinatorial properties of members of SS_n, and the relations of these properties to the corresponding binary tree. It was observed that members of SS_n tend to be more 'ordered' than ordinary permutations in the sense that on the average they contain less inversions, longer maximum ascending subsequences and shorter maximum descending subsequences.

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