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The Connectivity and Reliability of A Class of Extremal Graphs

by

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ABSTRACT

The topology of computer-communication networks has been modelled by Generalized Moore Graphs, an extension of the Moore Graphs first studied by Hoffman and Singleton. The construction of these graphs has been studied in a number of previous papers and a large number of such graphs have been exhibited. This paper contains a proof that the connectivity of the class of trivalent Generalized Moore Graphs is three. This result is important for the design of computer networks and is closely related to the reliability of such networks.

THE CONNECTIVITY AND RELIABILITY OF A CLASS OF EXTREMAL GRAPHS

D.D. Cowan

1. *Introduction*

In recent studies of the topological design of computer communication networks [1,2,3,4,5,6] a class of extremal graphs was analyzed. Each graph in this class was regular and had a minimum average path length. This paper discusses the connectivity of such graphs and proves that members of this class with valence 3 have a connectivity of 3. Previous papers have presented a census of these graphs and explored methods for constructing them.

The class of graphs possessing the property of minimum average path-length is interesting in itself, and includes the Petersen [8], Heawood, McGee [10] and Tutte 8-Cage [12]. Also all graphs which have been found with the exception of the Petersen graph and a graph on 28 nodes [7,11] have Hamiltonian circuits.

These graphs have been called Generalized Moore Graphs since they were a natural extension of the graphs defined by E.F. Moore and first studied by Hoffman and Singleton [9].

It has also been shown in a previous paper [2] that some graphs satisfying admissible parameters for this class do not exist.

2. *Generalized Moore Graphs*

The study of cost, delay and reliability in computer communication networks led one to consider networks or graphs in which the distance travelled by a message was minimized. Such considerations would lead to a complete graph which certainly solved the delay and reliability problem but would present the designer with a network whose costs were prohibitive. In order to construct an optimum network design which attempted to produce reasonable values for all three variables one was led to an analysis of regular graphs with minimum average path length. The minimum average path length P(N,V) is

$$\frac{V \sum_{j=0}^{m-1} (v-1)^j (j+1-m)+(M-1)m}{N-1}, \tag{1}$$

where N is the number of nodes in the graph, V is the valence of a node, and

$$m = \lceil \log_{V-1} \frac{N(V-2)+2}{V} \rceil \quad (V>2). \tag{2}$$

Networks which could satisfy this expression were a reasonable compromise [1]. They minimized delay and maximized reliability for a fixed valence and they could be constructed for a reasonable cost.

The development of this expression led to a number of questions:

(i) Are there many graphs which satisfy this minimum average path length constraint?

(ii) If such graphs exist, can one exhibit them?

(iii) Do these graphs have maximum reliability; is the valence of such a graph equal to its connectivity?

A short presentation of the derivation of expression (1) is appropriate at this point, as it provides much of the background for the discussion of questions (i), (ii) and (iii).

Consider a tree with N nodes in which each node has either valence V or valence 1. The nodes of valence 1 are called leaves. A tree with N = 10 and V = 3 is illustrated in Figure 1. One node will be chosen as the root node, and it is labelled R in Figure 1. The root node R will be considered to be at level zero in the tree, the V nodes adjacent to R will be at level one, the V(V-1) nodes adjacent to those at level one will be at level 2, and so forth. The levels are shown on the right side of Figure 1.

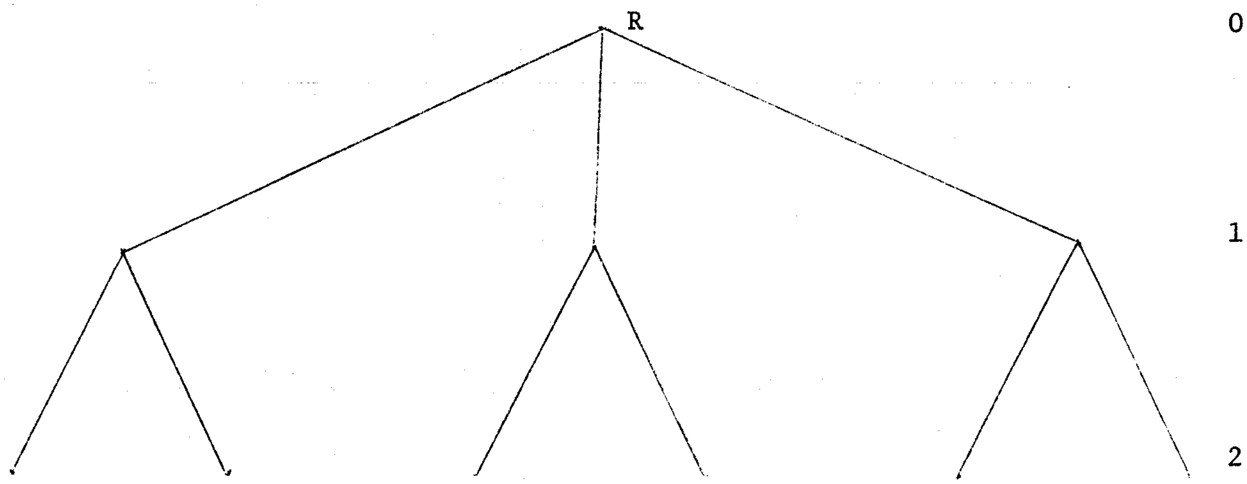

Figure 1

A tree such as this with

$$1 + V \sum_{j=0}^{m-1} (V-1)^j \qquad (3)$$

nodes, where m is the maximum level, will be called a complete tree, since new nodes can only be added by starting a new level. The graph formed by joining the nodes at the top level of the tree so that all nodes have valence V has been called a Moore Graph [9] and the tree has been called a Moore Tree.

In order to find the average path length from R to all nodes in the tree, it is necessary to sum all paths and then divide by N-1. Since there are V paths of length 1, $V(V-1)$ paths of length 2, $V(V-1)^2$ paths of length 3, and so on, then the average path length from R in a full tree with m levels is

$$\frac{V \sum_{j=0}^{m-1} (V-1)^j (j+1)}{N-1} , \qquad (4)$$

By removing nodes and their corresponding edges from the highest-numbered level of the full tree, it is possible to arrive at the general formula for average path length in any tree of this type. If the number of vertices in the tree is N then the number of vertices removed is

$$1 + V \sum_{j=0}^{m-1} (V-1)^j - N . \qquad (5)$$

Then the average path length P(N,V) in this "pruned" tree is

$$P(N,V) = \frac{V \sum_{j=0}^{m-1} (V-1)^j (j+1) - \{1 + V \sum_{j=0}^{m-1} (V-1)^j - N\} m}{N-1} , \qquad (6)$$

which can be rewritten as

$$P(N,V) = \frac{V \sum_{j=0}^{m-1} (V-1)^j (j+1-m) + (N-1)m}{N-1} . \qquad (7)$$

The level m can easily be computed by noting that

$$1 + V \sum_{j=0}^{m-1} (V-1)^j - N \geq 0 , \qquad (8)$$

and that m is the smallest integer which satisfies this inequality.

Since

$$\sum_{j=0}^{m-1} (V-1)^j = \frac{(V-1)^m - 1}{V-2} , \qquad (9)$$

then

$$1 + \frac{V\{(V-1)^m - 1\}}{V-2} - N \geq 0, \qquad (10)$$

and

$$m = \lceil \log_{V-1} \frac{N(V-2) + 2}{V} \rceil \text{ for } V \geq 2. \qquad (11)$$

Here $\lceil x \rceil$ denotes the least integer y such that $y \geq x$.

If the leaves of this tree can be joined by edges to make them V-valent so that each node can be treated as a root, then expression (7) represents the minimum average path length for a graph of this type. Graphs of this type have been called Generalized Moore Graphs, and are denoted by the symbol M(N,V).

We shall denote the trivalent case by M(2N,3) since all of these graphs must have an even number of nodes.

3. *Some Preliminary Results*

In the next section the connectivity of M(2N,3) - the trivalent Generalized Moore Graph will be shown to have a value of 3. Before proceeding to discuss the connectivity of M(2N,3) some results of graph theory will be stated without proof. These results can be found in the literature and references are given.

We denote a linear graph by G, its connectivity or point connectivity by C(G), its line connectivity by L(G), its minimum valence by V(G), and its diameter by D(G). Definitions of these quantities can be found in [8].

Lemma 1 [8]

The connectivity, line-connectivity, and minimum valence are related by the following inequality

$$C(G) \leq L(G) \leq V(G). \tag{12}$$

Lemma 2 [8]

The connectivity and line-connectivity are equal in every cubic graph, hence

$$C(G) = L(G). \tag{13}$$

Lemma 3 [2,3]

The diameter D(M(2N,3)) of M(2N,3) is equal to m, the number of levels.

4. The Connectivity of M(2N,3)

This section concludes with a theorem which states that M(2N,3) has a connectivity or point-connectivity of 3. Since for a cubic graph connectivity and line-connectivity are equal, this means that three edges or three nodes must be removed before the graph is divided into two disjoint components.

The proof proceeds in stages. First we prove that the connectivity must be at least two and then we prove a number of lemmas which lead to the result that the connectivity is three.

Lemma 4

The connectivity of G = M(2N,3) is at least two, hence

$$C(G) \geq 2.$$

Proof:

We assume that C(G) = 1; hence there is a cutpoint in G. Since G satisfies the minimum average path length condition, any node could be chosen as the root node R. We choose R as the cutpoint and then G can be represented as the two graphs G_1 and G_2 connected by the node R as in Figure 2.

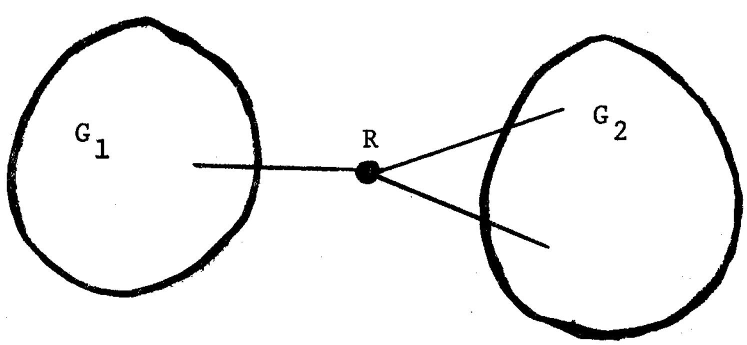

Figure 2.

If we assume that G has m levels with level m possibly incomplete then the distance from R to level m-1 in both G_1 and G_2 is m-1.

Hence the diameter

$$D(G) \geq 2m-1,$$

since the nodes in the incomplete level may all be in G_1 or G_2.

Since there are m levels in G

$$D(G) = m,$$

and

$$m \geq 2m-1,$$

which is a contradiction for $m > 1$.

The complete graph on four nodes which is the only graph with one level has no cutpoint and so there is a contradiction for all m.

Hence

$$C(G) \geq 2.$$

From lemma 1 it is obvious that C(G) is either two or three since G is trivalent. We assume that C(G) = 2 which implies that L(G) = 2.

This assumption means that G can be partitioned into two graphs G_1 and G_2 which are joined by exactly two edges as in Figure 3. We observe that both a and b and c and d are distinct nodes otherwise there would be a cutpoint in G.

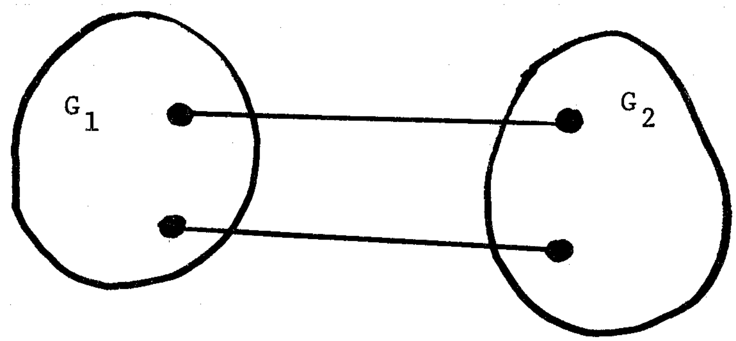

Figure 3

As part of our proof we need to know whether the number of nodes in G_1 and G_2 is odd or even. We prove the following lemma.

Lemma 5

Both G_1 and G_2 have an even number of nodes.

Proof:

Suppose the number of nodes in G_1 is odd and equal to 2p+1. Then the number of edges in G_1, not counting the edges connecting it to G_2, is

$$\frac{3(2p+1)}{2} - 2.$$

This expression is not an integer and so the number of nodes in G_1 must be even.

The number of nodes in G_1 will be denoted by 2p. Similarly G_2 has an even number of nodes which can be denoted by 2q. Since G is a cubic or trivalent graph the number of nodes in G will be 2N and

$$N = p + q.$$

This leads to a lemma which contradicts our assumption that L(G) = 2 and shows that most of the Generalized Moore Graphs are three-connected; in particular those which have more than 6 nodes. The proof of the lemma is obtained by replacing the graphs G_1 and G_2 by two trivalent graphs H and K which have minimum diameter. This new graph is illustrated in Figure 4. We then calculate the diameter of this new graph and show that this diameter is larger than the diameter of the original graph G and hence we obtain a contradiction.

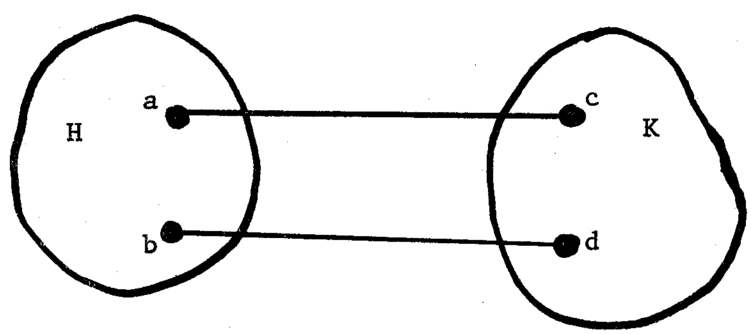

Figure 4

Lemma 6

Generalized Moore Graphs M(2N,3) are three-connected for 2N > 6.

Proof:

It has been assumed that L(G)=2 and hence the graph $M(2N,3)$ can be represented as shown in Figure 3. We now replace the graph G_1 by a trivalent graph H of minimum diameter on 2p nodes and then have

$$D(G_1) \geq D(H),$$

where

$$D(H) = \lceil \log_2 \frac{2p+2}{3} \rceil .$$

Similarly G_2 is replaced by a trivalent graph K on 2q nodes with

$$D(G_2) \geq D(K),$$

where

$$D(K) = \lceil \log_2 \frac{2q+2}{3} \rceil .$$

The new graph showing H and K replacing G_1 and G_2 is illustrated in Figure 4. The connecting nodes between the two graphs are still labelled, a,b,c, and d.

Since H is a minimum diameter trivalent graph, then either a or b may be designated as a root node of H. Hence there is at least one node in H which is distance D(H) from a and b. Similarly there is at least one node in K which is distance D(K) from c and d.

Then the diameter of the combined graph in Figure 4 is given by

$$\begin{aligned} D(H) + D(K) + 1 > D(H) + D(K) &= \lceil \log_2 \frac{2p+2}{3} \rceil + \lceil \log_2 \frac{2q+2}{3} \rceil \\ &\geq \lceil \log_2 (\frac{2p+2}{3})(\frac{2q+2}{3}) \rceil \\ &\geq \lceil \log_2 (\frac{2p+2q+2}{3}) \rceil \text{ for } p\geq 2, q\geq 2 \\ &= \lceil \log_2 (\frac{2N+2}{3}) \rceil \\ &= m. \end{aligned}$$

Hence the diameter of the graph in Figure 4 where H and K each have 4 or more nodes is larger than the diameter of the original graph G. Since this graph was constructed by replacing G_1 and G_2 by H and K we have a contradiction. Hence the line connectivity of M(2N,3) is three for 2N>6.

It is now necessary to examine the cases for 2N = 4 and 6, and to complete the proof using the following lemma.

Lemma 7

The graphs M(4,3) and M(6,3) are three-connected.

Proof:

At least one of the graphs G_1 and G_2 must contain two nodes in order to construct the graph of Figure 3 since each one must have an even number of nodes. Choose G_1 to contain the two nodes. Hence G_1 must appear as in Figure 5. This means that there are at least four edges joining G_1 to G_2 which violates our assumption. Hence M(4,3) and M(6,3) are three-connected. The results of these lemmas lead to the

Theorem:

All trivalent Generalized Moore Graphs M(2N,3) are three-connected.

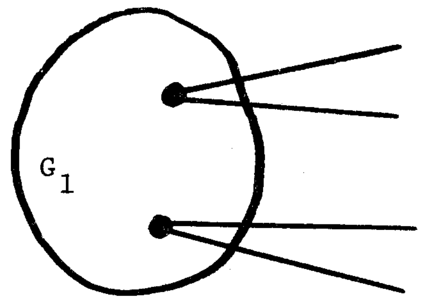

Figure 5

Reliability of M(2N,3)

Since the graphs M(2N,3) were originally chosen as models for the topology of a computer-communications network we should examine their reliability. Reliability is a generalization of connectivity in that one wishes to find a mixed set of nodes and edges which separates a graph into two components. Specifically, one wishes to find a connectivity pair of a graph G. A connectivity pair [8] is an ordered pair (x,y) of non-negative integers such that there is some set of a nodes and b edges whose removal separates the graph, and there is not a set of a-1 nodes and b edges, or a nodes and b-1 edges with this property. For example (C(G),0) and (0,L(G)) are two such pairs. The following results [8] have been shown for connectivity pairs:

(i) There are C(G) + 1 connectivity pairs.

(ii) If (x,y) is a connectivity pair with y > 0, then (x+1, y-1) is also a connectivity pair.

These two facts allow us to construct the connectivity pairs for M(2N,3). Obviously (0,3) is a connectivity pair and using (ii) so is (1,2) and (2,1). A final connectivity pair is (3,0) and according to (i) this computes all the connectivity pairs for M(2N,3).

This simple analysis reveals that this model for a computer network has a high degree of reliability. Its connectivity and its valence are the same, a situation which is highly desirable in any network which must remain operational under failure.

Conclusions

We have presented in this paper a discussion of the connectivity and reliability of a class of extremal graphs. These are the trivalent Generalized Moore Graphs M(2N,3). These graphs have been used as models for computer-communication networks, and hence it is quite important to study connectivity and reliability, since such a network must be able to function in spite of the failure of some components.

There are a number of other problems which are associated with these graphs. For example, it would be interesting to enumerate graphs for other values of valence up to 4 or 5. One would also wish to investigate the connectivity and reliability of these configurations as well. Finally an algorithm or heuristic should be found which will allow the extension of M(2N,3) to a graph which is close to if not exactly M(2N+2,3). Hopefully, this would be done without changing the connectivity or reliability properties. This extension method is required since one must be able to expand networks without extensive reconfiguration.

Some of these questions have been studied and will be presented in subsequent papers.

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