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TWO ITERATION THEOREMS FOR THE LL(k) LANGUAGES*

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Abstract. The structure of derivation trees over an LL(k) grammar is explored and a property of these trees obtained which is shown to characterize the LL(k) grammars. This characterization, called the LL(k) Left Part Theorem, makes it possible to establish a pair of iteration theorems for the LL(k) languages. These theorems provide a general and powerful method of showing that a language is not LL(k) when that is the case. They thus provide for the first time a flexible tool with which to explore the structure of the LL(k) languages and with which to discriminate between the LL(k) and LR(k) language classes.

Examples are given of LR(k) languages which, for various reasons, fail to be LL(k). Easy and rigorous proofs to this effect are given using our LL(k) iteration theorems. In particular, it is proven that the dangling-ELSE construct allowed in PL/I and Pascal cannot be generated by any LL(k) grammar. We also give a new and straightforward proof based on the LL(k) Left Part Theorem that every LL(k) grammar is LR(k).

1. Introduction

The classical pumping lemma [3] and Ogden's lemma [19] are among the most powerful tools we possess for proving that languages are not context-free. Hence one goal of recent research has been to obtain analogous theorems for subclasses of the context-free languages. Thus Ogden [18] gives an iteration theorem for the deterministic context-free languages, Harrison and Havel have established an iteration theorem for the family of strict deterministic languages [11] which is also extendible to the context-free languages, and Boasson has established an iteration theorem for the one-counter languages [6]. More recently King has

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then the derivation is *leftmost*. By $\Rightarrow^n$ we mean a derivation of exactly n steps, for any $n \geq 0$, while $\Rightarrow_L^n$ denotes a leftmost derivation of exactly n steps. The relations $\Rightarrow_R$, $\Rightarrow_R^+$ and $\Rightarrow_R^*$, *etc.*, are similarly defined. If we use a Greek letter such as π (for example: $\Rightarrow_L^\pi$) which is constrained to belong to P^* then π represents the sequence of rules (possibly null) by which the derivation proceeds.

We will say that an occurrence of the symbol $X \in \Sigma$ is *exposed* at the $(n+1)^{\underline{st}}$ step of the leftmost derivation

$$S \Rightarrow_L^n wA\gamma \Rightarrow_L w\beta\gamma$$

if X appears somewhere in $\beta\gamma$ and there are no variables anywhere to the left of X in $\beta\gamma$.

The *context-free language* (*cfl*) $\mathcal{L}(G)$ generated by G is exactly the set of terminal strings which can be derived from the start symbol S. Similarly, if $\alpha \in V^*$ then $\mathcal{L}(\alpha)$ is the set of terminal strings which can be derived from α. The *left sentential forms* of G are exactly those strings of terminals and nonterminals which can be generated from S by a leftmost derivation.

G is said to be *unambiguous* if no string in $\mathcal{L}(G)$ has more than one distinct leftmost derivation. Otherwise G is said to be *ambiguous*.

The null string is written Λ. The length of a string x is written $|x|$. Thus $|\Lambda| = 0$.

A variable A of G is said to be *reduced* iff A derives at least one terminal string and itself appears in some string of terminals and nonterminals which can be derived from S. G is said to be *reduced* iff either the variables of G are all reduced or $P = \emptyset$.

A variable A of G is said to be *left recursive* iff $A \Rightarrow^+ A\beta$ for some string $\beta \in V^*$. G is left recursive iff some variable A of G is left recursive.

If w is a string and k a non-negative integer then w/k is the first k symbols of w if $|w| > k$ and is w itself if $|w| \leq k$. More generally, for a cfg $G = (N,\Sigma,P,S)$ we define

$$first_k(\beta) = \{ w \in \Sigma^* \mid (|w| \leq k \text{ and } \beta \Rightarrow^* w) \text{ or } (|w| = k \text{ and } \beta \Rightarrow^* wy \text{ for some } y \in \Sigma^+) \}$$

for any $\beta \in V^*$. $first_k$ is extended to sets in the usual way.

Next we review pertinent facts about LL(k) grammars.

Definition 1.1. A cfg $G = (N,\Sigma,P,S)$ is *LL(k)* iff for any $A \in N$; $w, x, y \in \Sigma^*$; $\beta, \beta', \gamma \in V^*$; and any two derivations

$$S \Rightarrow_L^* wA\gamma \Rightarrow_L w\beta\gamma \Rightarrow_L^* wx$$
$$S \Rightarrow_L^* wA\gamma \Rightarrow_L w\beta'\gamma \Rightarrow_L^* wy$$

for which $x/k = y/k$ we necessarily have $\beta = \beta'$. A language is LL(k) iff it is generated by an LL(k) grammar.

The following results are well-known or easily proven [5]. They will be used subsequently and are stated here for convenience.

Theorem 1.2. [21] No LL(k) grammar is ambiguous.

Theorem 1.3. [21] No LL(k) grammar is left recursive.

Theorem 1.4. Let $G = (N,\Sigma,P,S)$ be a cfg. G is an LL(k) grammar iff for any $A \in N$; $w, x, y \in \Sigma^*$; $\beta, \beta', \gamma, \gamma' \in V^*$; and any two derivations

$$S \Rightarrow_L^* wA\gamma \Rightarrow_L w\beta\gamma \Rightarrow_L^* wx$$
$$S \Rightarrow_L^* wA\gamma' \Rightarrow_L w\beta'\gamma' \Rightarrow_L^* wy$$

for which $x/k = y/k$ we necessarily have $\beta = \beta'$.

Theorem 1.4 allows the right context γ of A in the two derivations of definition 1.1 to differ. Definition 1.1 is taken from Aho and Ullman [2]; theorem 1.4 is actually the LL(k) definition used by Rosenkrantz and Stearns [21].

Theorem 1.5. [2] Let $G = (N,\Sigma,P,S)$ be a cfg. G is an LL(k) grammar iff given any $A \in N$, $w \in \Sigma^*$, and $\gamma \in V^*$ such that $S \Rightarrow_L^* wA\gamma$, we have

$$first_k(\beta\gamma) \cap first_k(\beta'\gamma) = \emptyset$$

for every distinct pair of rules $A \rightarrow \beta$ and $A \rightarrow \beta'$ in P.

Theorem 1.6. Let $G = (N,\Sigma,P,S)$ be a cfg. G is an LL(k) grammar iff given

(1) $w \in first_k(\Sigma^*)$
(2) $x \in \Sigma^*$
(3) $A \in N$

then there exists at most one rule $A \rightarrow \beta$ in P such that

(4) $S \Rightarrow^* xAw_2$

(5) $A \Rightarrow \beta \Rightarrow^* w_1$

(6) $(w_1w_2)/k = w$

for any w_1, $w_2 \in \Sigma^*$.

This was the definition of LL(k) grammars used by Lewis and Stearns [15].

The following special version of the LL(k) definition will be useful in section 3.

Theorem 1.7. Let $G = (N,\Sigma,P,S)$ be a reduced cfg. G is an LL(k) grammar iff for any $A \in N$; $w, x, y \in \Sigma^*$; $\beta, \beta', \gamma \in V^*$; and any two derivations

$$S \Rightarrow_L^n wA\gamma \Rightarrow_L w\beta\gamma \Rightarrow_L^* wx$$

$$S \Rightarrow_L^n wA\gamma \Rightarrow_L w\beta'\gamma \Rightarrow_L^* wy$$

for which $x/k = y/k$ we necessarily have $\beta = \beta'$. (Notice that $wA\gamma$ is derived in n steps in both derivations.)

Proof: A proof in the forward direction is trivial. To establish the reverse direction, suppose that G is not LL(k), but that the existence of two such derivations necessarily forces $\beta = \beta'$. Since G is not LL(k) it follows from theorem 1.5 that there exist strings $A \in N$; $w, x, y \in \Sigma^*$; $\beta, \beta', \gamma \in V^*$; such that $S \Rightarrow_L^* wA\gamma$ and

$$first_k(\beta\gamma) \cap first_k(\beta'\gamma) \neq \emptyset \qquad (1)$$

for some distinct pair of rules $A \to \beta$ and $A \to \beta'$ in P. Let x and y be strings in $\mathcal{L}(\beta\gamma)$ and $\mathcal{L}(\beta'\gamma)$, respectively, such that $x/k = y/k$ and suppose that S derives $wA\gamma$ leftmost in n steps. Then

$$S \Rightarrow_L^n wA\gamma \Rightarrow_L w\beta\gamma \Rightarrow_L^* wx$$

$$S \Rightarrow_L^n wA\gamma \Rightarrow_L w\beta'\gamma \Rightarrow_L^* wy$$

where $x/k = y/k$. By hypothesis we must have $\beta = \beta'$, which is a contradiction. Hence G must be LL(k). ■

Theorem 1.8. Let $G = (N,\Sigma,P,S)$ be a reduced LL(k) grammar. Let $G_A = (N,\Sigma,P,A)$ be the grammar formed from G by changing the start symbol from S to A, for any variable A of G. Then G_A is also an LL(k) grammar.

Proof: Suppose that G_A were not LL(k). Then for some $x, y_1, y_2 \in \Sigma^*$; $\beta, \beta', \gamma \in V^*$; $B \in N$; there must exist two derivations

$$A \Rightarrow_L^* xB\gamma \Rightarrow_L x\beta\gamma \Rightarrow_L^* xy_1$$
$$A \Rightarrow_L^* xB\gamma \Rightarrow_L x\beta'\gamma \Rightarrow_L^* xy_2$$

in G_A with $y_1/k = y_2/k$ and $\beta \neq \beta'$. But this is also a derivation in G. Since G is reduced, there also exists in G a derivation sequence $S \Rightarrow_L^* wA\delta$ for some $w \in \Sigma^*$ and $\delta \in V^*$. We obtain the following derivations in G:

$$S \Rightarrow_L^* wA\delta \Rightarrow_L^* wxB\gamma\delta \Rightarrow_L wx\beta\gamma\delta \Rightarrow_L^* wxy_1z$$
$$S \Rightarrow_L^* wA\delta \Rightarrow_L^* wxB\gamma\delta \Rightarrow_L wx\beta'\gamma\delta \Rightarrow_L^* wxy_2z$$

where z is any string derived from δ. Recall that $y_1/k = y_2/k$. If $|y_1| < k$ or $|y_2| < k$ then we must have $y_1 = y_2$, in which case $(y_1z)/k = (y_2z)/k$. If both y_1 and y_2 are of length k or greater then again $(y_1z)/k = (y_2z)/k$. Since G is LL(k), we must therefore have $\beta = \beta'$, which is a contradiction. Therefore G_A must also be LL(k). ■

We also need to introduce LR(k) grammars. We use the definition suggested by Geller and Harrison [10].

Definition 1.9. A cfg $G = (N,\Sigma,P,S)$ is *LR(k)* for some $k \geq 0$ iff $S \Rightarrow_R^+ S$ is impossible in G and for any w, w', $x \in \Sigma^*$; α, α', $\beta' \in V^*$; A, A' $\in$ N; and derivations

$$S \Rightarrow_R^* \alpha Aw \Rightarrow_R \alpha\beta w$$
$$S \Rightarrow_R^* \alpha'A'x \Rightarrow_R \alpha'\beta'x = \alpha\beta w'$$

if $w/k = w'/k$ then $(A \rightarrow \beta, |\alpha\beta|) = (A' \rightarrow \beta', |\alpha'\beta'|)$.

2. Trees

Following Harrison and Havel [11] we semi-formally develop the notion of trees, particularly derivation trees, and their properties. Our presentation is a compromise between the demands of rigor and a desire not to entirely sacrifice comprehensibility and intuition. To this end we will occasionally make informal use of pictures.

For our purposes a *tree* $\mathcal{T}$ is an acyclic, connected graph defined by a pair of sets $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ is a set of *nodes* and $\mathcal{E}$ is a set of *edges* $(x,y) \in \mathcal{V}\times\mathcal{V}$, in which all nodes save one (the *root node* of $\mathcal{T}$, written $rtn(\mathcal{T})$) have exactly one entering edge; the root node has no entering edges. For example, the tree in figure 1 is defined by

$$\left(\{x_0,x_1,x_2,x_3\}, \{ (x_0,x_1), (x_0,x_2), (x_2,x_3) \} \right)$$

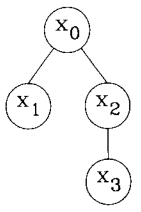


Fig. 1.

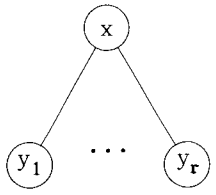


Fig. 2.

The edges (x,y) in $\mathcal{E}$ define the *immediate descendency* relation Γ; x is a *parent* of y and y is a *child* of x. In figure 1 we have $x_0 \Gamma x_1$ but not $x_1 \Gamma x_2$. The reflexive transitive closure Γ^* of Γ is called the *descendancy* relation. There is a *path* from node x to node y iff $x \Gamma^* y$. Thus in figure 1 there is a path from x_0 to x_3 since $x_0 \Gamma^* x_3$, but no path from x_3 to x_1. If $rtn(\mathcal{T})\, \Gamma^i\, y$ then y is said to be at *depth* i in $\mathcal{T}$. The *height* of $\mathcal{T}$ is the length of a longest path in $\mathcal{T}$; it is thus equal to the depth of a deepest node.

A node x is *internal* iff there exists a node y such that $x \Gamma y$. Otherwise x is a *leaf*, and has no children.

We will need a left to right ordering of the nodes in a tree. For this reason we assume that $\mathcal{E}$ is actually a sequence of edges so that we may define an additional relation $\sqcap$ on the nodes of a tree in the following way. If the r edges leaving an arbitrary node x are listed in $\mathcal{E}$ in the order $(x,y_1), \cdots, (x,y_r)$ then $y_1 \sqcap y_2 \sqcap \cdots \sqcap y_r$ and the edges will be drawn left to right according to this ordering, as in figure 2. Furthermore, if $p \Gamma y$ and there does not exist any node x such that $x \sqcap y$ then $p \Gamma_L y$ (y is a *leftmost child* of p). The relation Γ_R is defined similarly. Finally, we write $x \mathrel{L} y$ iff $(x,y) \in (\Gamma_R^{-1})^* \sqcap (\Gamma_L)^*$, so that $x \mathrel{L} y$ iff there are no nodes between x and y. The reflexive transitive closure L^* of L then defines the notion of *left to right order* in $\mathcal{T}$. (The relations Γ and L are identical to the relations represented by these symbols in Harrison and Havel [11].) If we list the leaves $l_1, \cdots, l_r$ of $\mathcal{T}$ in left to right order, which is to say that

$$l_1 \mathrel{L} l_2 \mathrel{L} \cdots \mathrel{L} l_r$$

then we obtain the left to right sequence of nodes

$$leaves(\mathcal{T}) = (l_1, l_2, \cdots, l_r)$$

Let us adopt the convention that if we list the nodes in a subtree $\mathcal{T}'$ of $\mathcal{T}$ then edges between those nodes in $\mathcal{T}$ are implicitly the edges of $\mathcal{T}'$ (the *induced subtree*). Then for any internal node x of the tree $\mathcal{T}$ the set $\{ y \mid x = y \text{ or } x \Gamma y \}$ defines the *elementary subtree of* $\mathcal{T}$ *with root* x. Also, if x is a node of $\mathcal{T}$ then we define $\mathcal{T}_x$ to be the largest induced subtree of $\mathcal{T}$ whose root is x. More precisely,

$$\mathcal{T}_x = \{ y \mid y \text{ is a node of } \mathcal{T} \text{ and } x \Gamma^* y \}$$

Since our trees represent context-free derivations we will want each node to represent a grammar symbol or, perhaps, Λ. Furthermore, it is often desirable to distinguish between a node and the symbol it represents since several nodes may represent the same grammar symbol. Hence we define a *labeled tree* to be a tree $\mathcal{T} = (\mathcal{V},\mathcal{E})$ together with a labeling function λ from $\mathcal{V}$ into a finite set $\mathcal{L}$ of labels such that $\mathcal{V} \cap \mathcal{L} = \emptyset$. The labeling function λ is then extended to sequences of nodes in the obvious way. Our labels will always be drawn from some set $V_\Lambda = V \cup \{\Lambda\}$, where V is the vocabulary of some cfg. Of particular interest are the *root label* and *frontier* of $\mathcal{T}$:

$$rtl(\mathcal{T}) = \lambda(rtn(\mathcal{T}))$$
$$fr(\mathcal{T}) = \lambda(leaves(\mathcal{T}))$$

Let $G = (N,\Sigma,P,S)$ be a context-free grammar, and let $\mathcal{T}$ be a labeled tree for which the labels are symbols from V_Λ. $\mathcal{T}$ is said to be a *grammatical tree* iff $fr(\mathcal{T}) \in \Sigma^*$ and either

$\mathcal{T}$ is a trivial tree consisting of a single labeled node

or

for every internal node x in $\mathcal{T}$, if $y_1, \cdots, y_r$ are all of x's children in left to right order then $\lambda(x) \rightarrow \lambda(y_1) \cdots \lambda(y_r)$ is a rule of G and $\lambda(y_i) = \Lambda$ iff $1 = i = r$.

Leaves which are labeled with terminals are referred to as *terminal nodes*. Leaves which are labeled with Λ are called Λ-*nodes*. Observe that a node x is internal iff $\lambda(x) \in N$. A grammatical tree $\mathcal{T}$ is said to be a *derivation tree* iff $rtl(\mathcal{T}) = S$.

Figure 3, for example, displays a grammatical tree over the context-free grammar $S \rightarrow aSbS \mid \Lambda$. Occasionally we will omit the names of nodes in a grammatical tree, leaving only the labels, in which case the tree of figure 3 would appear as in figure 4.

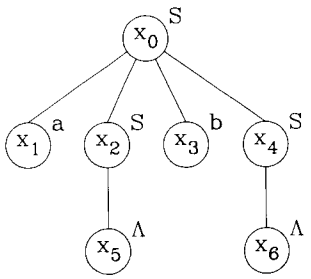


Fig. 3. A grammatical tree in which we distinguish nodes and labels.

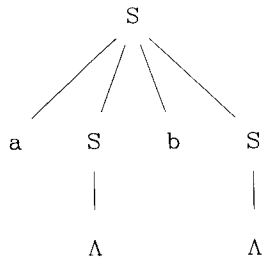


Fig. 4. A grammatical tree in which nodes and labels are not distinguished.

The sentential forms which appear in a derivation are embedded in a natural way in the grammatical tree representing that derivation. We represent this embedding by means of *cross sections* (CS's) and *canonical cross sections*, which we define inductively for a tree $\mathcal{T}$ by the following:

(1) $\eta = (x_0)$, where $x_0 = rtn(\mathcal{T})$, is a cross section at level 0.

(2) Let $\eta = (x_1, \cdots, x_k, \cdots, x_m)$ be a cross section of level l and let x_k be an internal node of $\mathcal{T}$. If $y_1, \cdots, y_r$ are all the children of x_k in left to right order then

$$\eta' = (x_1, \cdots, x_{k-1}, y_1, \cdots, y_r, x_{k+1}, \cdots, x_m)$$

is a cross section of level $l+1$.

(x_0) is also said to be a *left canonical cross section* (*LCCS*) of $\mathcal{T}$. If η is a LCCS of $\mathcal{T}$ and x_k, the node which is replaced, is the leftmost internal node of η, then η' is also a left canonical cross section of $\mathcal{T}$. *Right canonical cross sections* (*RCCS's*) are defined analogously. For readability we may sometimes write $(x_1\ x_2\ \cdots\ x_m)$ instead of $(x_1, x_2, \cdots, x_m)$.

For example, in the grammatical tree of figure 3 $(x_1\ x_5\ x_3\ x_4)$ is a LCCS, $(x_1\ x_2\ x_3\ x_6)$ is a CS but not a LCCS and $(x_1\ x_2\ x_0\ x_4)$ is neither a LCCS nor a CS.

The following properties of cross sections are intuitive. Consequently we state them without proof, though in an order convenient for rigorous development. More detail may be found in [5].

Fact 2.1. Let $\eta = (x_0, \cdots, x_m)$ be a cross section of some tree $\mathcal{T}$. Then $x_i \sqsubset x_{i+1}$, $1 \leq i < m$.

Fact 2.2. No node of any tree $\mathcal{T}$ appears more than once in any one cross section of $\mathcal{T}$.

Fact 2.3. [11] No two distinct LCCS's of a grammatical tree can be of the same level.

Fact 2.4. The level associated with any cross section is unique.

Fact 2.5. Let $\mathcal{T}$ be a tree and let $\mathfrak{n}$ be a node in $\mathcal{T}$. Then $\mathfrak{n}$ appears in at least one LCCS (respectively CS) of $\mathcal{T}$. Moreover, we may assume that there are no internal nodes to the left (respectively to the left and right) of $\mathfrak{n}$ in this cross section.

Fact 2.6. Let $\mathcal{T}$ be a tree. Then $\mathit{leaves}(\mathcal{T})$ is a LCCS of $\mathcal{T}$.

Next we delineate the relationship between cross sections and sentential forms. First we describe how to pass from cross sections to derivations.

Fact 2.7. Let $G = (N,\Sigma,P,S)$ be a cfg and let $\mathcal{T}$ be a grammatical tree over G. If η is a cross section of $\mathcal{T}$ at level l then $\mathit{frontier}(\mathcal{T}) \Rightarrow \lambda(\eta)$.

We have a stronger result for canonical cross sections.

Fact 2.8. Let $G = (N,\Sigma,P,S)$ be a cfg and let $\mathcal{T}$ be a grammatical tree over G. If η and η' are LCCS's of level l and $l+i$, for any l and $i \geq 0$, then $\lambda(\eta) \Rightarrow_L^i \lambda(\eta')$. If η and η' are instead RCCS's then $\lambda(\eta) \Rightarrow_R^i \lambda(\eta')$.

This result does <u>not</u> hold for cross sections in general. In figure 5 the cross section

$$\eta = (x_1\ x_5\ x_{13}\ x_7\ x_8\ x_3\ x_4)$$

is at level 3 and the cross section

$$\eta' = (x_1\ x_5\ x_6\ x_7\ x_{14}\ x_3\ x_9\ x_{15}\ x_{11}\ x_{16})$$

is at level 6, but $\lambda(\eta)$ = aabSbS cannot possibly derive $\lambda(\eta')$ = aaSbbab, the S in aaSbbab already having been erased in aabSbS.

Fact 2.9. [11] Let $\mathcal{T}$ be a derivation tree over some unambiguous cfg and let η and θ be two LCCS's (or RCCS's) in $\mathcal{T}$. If $\lambda(\eta) = \lambda(\theta)$ then $\eta = \theta$.

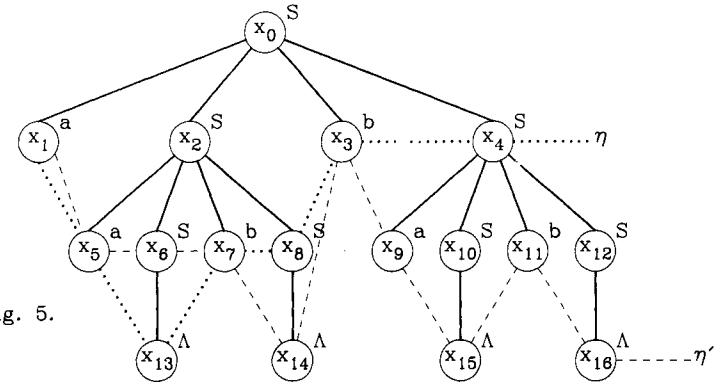


Fig. 5.

We pass from derivations to cross sections *via* the next two results.

Fact 2.10. Let $G = (N,\Sigma,P,S)$ be a cfg and let $A \Rightarrow^{i} \alpha \Rightarrow^{*} w$, where A is a variable, $\alpha \in V^{*}$ and w is a string of terminals. Then there exists a grammatical tree Υ containing a cross section η of level i such that $rtl(\Upsilon) = A$, $fr(\Upsilon) = w$ and $\lambda(\eta) = \alpha$. Moreover, if the derivation is leftmost or rightmost then η is respectively a left or right canonical cross section of Υ.

If we are dealing with an unambiguous grammar then we can prove a stronger result.

Fact 2.11. Let $G = (N,\Sigma,P,S)$ be an unambiguous cfg and Υ a grammatical tree over G. If $rtl(\Upsilon) \Rightarrow^{i} \alpha \Rightarrow^{*} fr(\Upsilon)$, where $\alpha \in V^{*}$, then there exists a cross section η at level i in Υ such that $\lambda(\eta) = \alpha$. Moreover, if the derivation is leftmost or rightmost then η is respectively a left or right canonical cross section of Υ.

In developing our arguments we will need to disassemble and reassemble derivation trees and cross sections in a highly specialized manner. Hence we next define the tree fragments about which we will be speaking.

Definition 2.12. Let Υ be a grammatical tree such that $|fr(\Upsilon)| = m$. Let $y_1, \cdots, y_m$ be a complete left to right sequence of the terminal nodes of Υ. If n lies in the range $1 \leq n \leq m$ then

$[n]\mathcal{T} = \{ x \mid x L^* \Gamma^* y_n \}$

$\{n\}\mathcal{T} = [n]\mathcal{T} \cup \{ x \mid \exists\, b \in \mathcal{T} \text{ s.t. } rtn(\mathcal{T})\, \Gamma^* b\, \Gamma^* y_n \text{ and } b \sqcap^+ x \}$

$[0]\mathcal{T} = \{0\}\mathcal{T} = (\emptyset,\emptyset)$ and for $n > m$, $[n]\mathcal{T} = \{n\}\mathcal{T} = \mathcal{T}$. $[n]\mathcal{T}$ is called a *left* $[n]$*-part* of $\mathcal{T}$ and $\{n\}\mathcal{T}$ is called a *left* $\{n\}$*-part* of $\mathcal{T}$. Thus if $\mathfrak{p}$ is the root-leaf path to the $n^{\underline{th}}$ terminal node (counting from the left), then $[n]\mathcal{T}$ consists of those nodes which are on or left of $\mathfrak{p}$, while $\{n\}\mathcal{T}$ consists of those nodes of $\mathcal{T}$ which are left of $\mathfrak{p}$, or on $\mathfrak{p}$, or are right of $\mathfrak{p}$ and have a parent on $\mathfrak{p}$. For example, in figures 7 and 8 we see in bold the left [4]-part and left {4}-part of the tree in figure 6. (Our left []-parts correspond to the left parts defined by Harrison and Havel [11].)

Next we establish those properties of left parts which will be needed later.

Theorem 2.13. [11] Let η be a RCCS of the grammatical tree $\mathcal{T}$ and let n be a positive integer. The restriction of η to $[n]\mathcal{T}$ is a RCCS of $[n]\mathcal{T}$.

Theorem 2.14. Let η be a LCCS of the grammatical tree $\mathcal{T}$ at level ℓ and let n be a positive integer. If the restriction η' of η to $\{n\}\mathcal{T}$ contains an internal node of $\mathcal{T}$ then $\eta' = \eta$ and η' is a LCCS of level ℓ in $\{n\}\mathcal{T}$. (Refer to figures 9 and 10.)

Proof: The proof proceeds by means of an induction on ℓ.

Basis ($\ell = 0$): Let $x_0 = rtn(\mathcal{T})$. We must have $\eta = (x_0)$, since this is the only LCCS of $\mathcal{T}$ having level 0. If the restriction of η to $\{n\}\mathcal{T}$ contains an internal node then it must contain x_0, in which case the restriction is exactly η, which is by definition a LCCS of $\{n\}\mathcal{T}$ for every $n \geq 1$.

Induction Step: We assume that the theorem is true for LCCS's having level ℓ or less and extend the theorem to LCCS's having level $\ell+1$. Let η be a LCCS of level $\ell+1$ and let θ be the LCCS of level ℓ from which it is obtained. Let

$\theta = (z_1 \cdots z_{g-1}\ z_g\ z_{g+1} \cdots z_r)$

$\eta = (z_1 \cdots z_{g-1}\ x_1 \cdots x_s\ z_{g+1} \cdots z_r)$

so that z_g is the leftmost internal node of θ. The leftmost internal node of η belongs to $\{n\}\mathcal{T}$ since our hypothesis is that the restriction of η contains at least one internal node. It follows that if one of $x_1, \cdots, x_s$ is

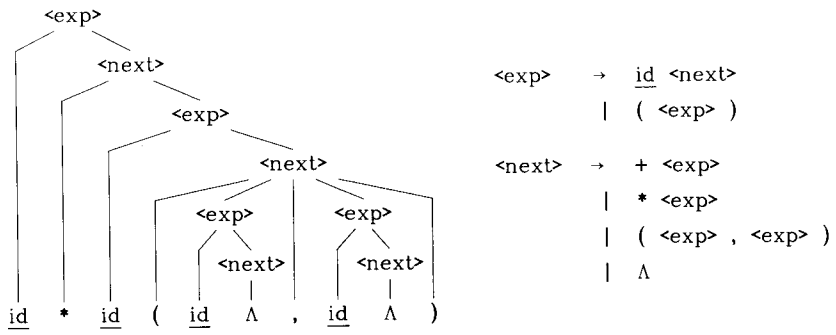


Fig. 6. The derivation tree $\mathcal{T}$ for id*id(id,id), over the indicated grammar.

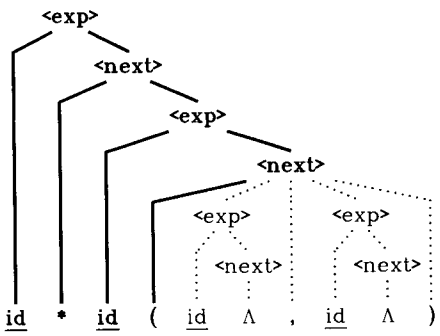


Fig. 7. The left part $^{[4]}\mathcal{T}$ of $\mathcal{T}$.

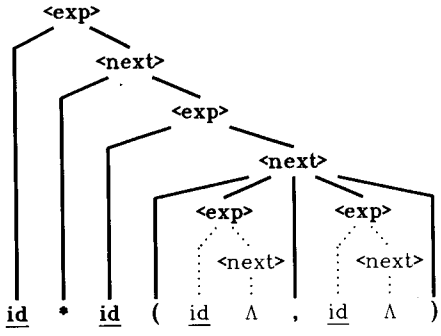


Fig. 8. The left part $^{\{4\}}\mathcal{T}$ of $\mathcal{T}$.

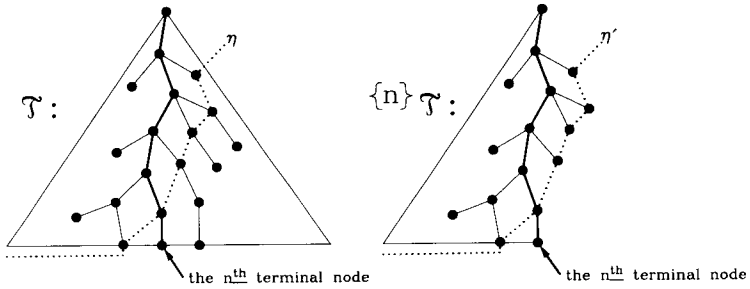


Fig. 9, illustrating theorem 2.14. The restriction η' of η to $^{\{n\}}\mathcal{T}$ contains an internal node of $\mathcal{T}$. Consequently $\eta' = \eta$ and η' is a LCCS of $^{\{n\}}\mathcal{T}$.

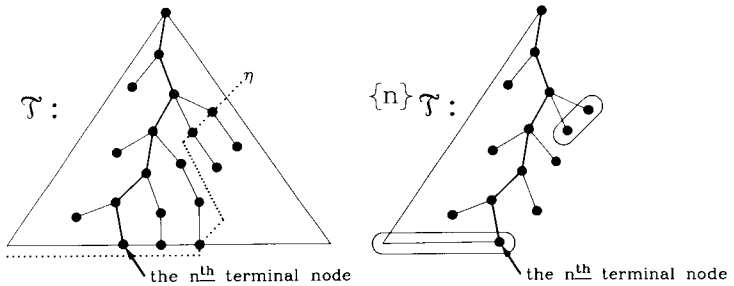


Fig. 10. In this case the restriction of η to ${}^{\{n\}}\mathcal{T}$ does not contain an internal node of $\mathcal{T}$. Hence the restriction of η to ${}^{\{n\}}\mathcal{T}$ (encircled above right) need not be a LCCS of ${}^{\{n\}}\mathcal{T}$.

the leftmost internal node of η then its parent z_g belongs by definition to ${}^{\{n\}}\mathcal{T}$. If the leftmost internal node of η is instead one of $z_{g+1}, \cdots, z_r$ then since z_g is left of that node z_g again must belong to ${}^{\{n\}}\mathcal{T}$. In either case θ is a LCCS of $\mathcal{T}$ at level ℓ whose restriction to ${}^{\{n\}}\mathcal{T}$ contains the internal node z_g. It follows from the induction hypothesis that θ is a LCCS of ${}^{\{n\}}\mathcal{T}$ at level ℓ. By definition, then, η is a LCCS of ${}^{\{n\}}\mathcal{T}$ having level $\ell+1$. In particular the restriction η' of η to ${}^{\{n\}}\mathcal{T}$ is in fact η itself. Hence $\eta' = \eta$ and the theorem is established. ■

If the restriction of η to ${}^{\{n\}}\mathcal{T}$ does <u>not</u> contain an internal node then it need not be a LCCS of ${}^{\{n\}}\mathcal{T}$. Such a situation is depicted in figure 10.

Theorem 2.15. [11] Let $\mathcal{T}$ be a grammatical tree over some cfg G, let n be a positive integer, and let $s = |\mathit{fr}(\mathcal{T})|$. Let $\eta = (x_1 \cdots x_k)$ be a RCCS in ${}^{[n]}\mathcal{T}$ and let $y_r, \cdots, y_s$ be all the leaves of $\mathcal{T}$ which are right of x_k; accordingly we assume $x_k \llcorner y_r \llcorner \cdots \llcorner y_s$. Then the sequence

$$\theta = (x_1 \cdots x_k \; y_r \cdots y_s)$$

is a RCCS of $\mathcal{T}$.

Theorem 2.16. Let $\mathcal{T}$ be a grammatical tree and n a positive integer. If η is a LCCS of ${}^{\{n\}}\mathcal{T}$ then η is a LCCS of $\mathcal{T}$ as well.

Proof: The proof is by induction on the level ℓ of η.

Basis ($\ell = 0$): It must be the case that η is the root node, which is a LCCS of $\mathcal{T}$ by definition.

Induction Step: Assume that the theorem holds for all LCCS's of level l or less. Let θ be a LCCS of ${}^{\{n\}}\mathcal{T}$ at level $l+1$ and let η be the LCCS of ${}^{\{n\}}\mathcal{T}$ at level l from which it is formed. By the induction hypothesis η is a LCCS of $\mathcal{T}$. By definition, then, θ is a LCCS of $\mathcal{T}$.

We will need the following special case of theorem 2.16.

Theorem 2.17. Let $G = (N,\Sigma,P,S)$ be a cfg and $\mathcal{T}$ a derivation tree over G. Let n be a positive integer. Then *leaves*$({}^{\{n\}}\mathcal{T})$ is a LCCS of $\mathcal{T}$.

Proof: According to fact 2.6 *leaves*$({}^{\{n\}}\mathcal{T})$ is a LCCS of ${}^{\{n\}}\mathcal{T}$. It then follows from theorem 2.16 that *leaves*$({}^{\{n\}}\mathcal{T})$ is a LCCS of $\mathcal{T}$ as well. ■

Finally, we will need to define what it means for trees, or parts of trees, to be equal.

Definition 2.18. Two trees $\mathcal{T}$ and $\mathcal{T}'$ are said to be *structurally isomorphic*, written $\mathcal{T} \approx \mathcal{T}'$, iff there exists a bijection $\mathcal{T} \to \mathcal{T}' : x \to x'$ between the nodes of $\mathcal{T}$ and $\mathcal{T}'$ such that

- $x \sqsubset y$ iff $x' \sqsubset y'$
- $x \sqcap y$ iff $x' \sqcap y'$

(Note that we use the symbols $\sqsubset$ and $\sqcap$ to represent the descendancy and left–right relations in both trees.) Intuitively, $\mathcal{T}$ and $\mathcal{T}'$ are identical except for labeling. If the structural isomorphism preserves labeling ($\lambda(x) = \lambda(x')$) then we say that the trees are *isomorphic* and write $\mathcal{T} = \mathcal{T}'$.

3. A Left Part Theorem

Our goal is to establish iteration theorems for the LL(k) languages. Our first such theorem will be founded on an argument about derivation trees, and in particular on a characterization of derivation trees over LL(k) grammars, which is our immediate goal. Our starting point is the following result, which is analogous to Geller's Extended LR(k) Theorem [9].

Theorem 3.1. (The Extended LL(k) Theorem). Let $G = (N,\Sigma,P,S)$ be an LL(k) grammar. For any $A \in N$; $w, x, y \in \Sigma^*$; and $\gamma \in V^*$, if

(1) $S \Rightarrow_L^{\pi} wA\gamma \Rightarrow_L^{*} wx$

(2) $S \Rightarrow_L^{*} wy$

(3) $x/k = y/k$

then

(4) $S \Rightarrow_L^{\pi} wA\gamma \Rightarrow_L^{*} wy$

Proof: Assume for the sake of contradiction that (1), (2) and (3) hold, but not (4). Since the leftmost derivations of wx and wy have the initial left sentential form S in common, and (4) does not hold, derivations (1) and (2) diverge before reaching $wA\gamma$. Let $uB\delta$ be the last left sentential form they have in common (where $u \in \Sigma^*$, $B \in N$, and $\delta \in V^*$). Then for some $\sigma \in P^*$ and $v \in \Sigma^*$ such that $w = uv$ we have

$S \Rightarrow_L^{\sigma} uB\delta \Rightarrow_L u\beta_1\delta \Rightarrow_L^{*} uvA\gamma \Rightarrow_L^{*} uvx = wx$

$S \Rightarrow_L^{\sigma} uB\delta \Rightarrow_L u\beta_2\delta \Rightarrow_L^{*} uvy = wy$

for distinct rules $B \rightarrow \beta_1$ and $B \rightarrow \beta_2$ of G. Since $x/k = y/k$, we must have $(vx)/k = (vy)/k$. It follows that $\beta_1 = \beta_2$ since G is LL(k), contradicting the assumption that $uB\delta$ is the last common sentential form, so that (4) must hold. ■

This theorem describes a property of derivation trees as well as of derivations. Let wx and wy be strings in the language generated by an LL(k) grammar G. Then the portions of the derivation trees $\mathcal{T}^{wx}$ and $\mathcal{T}^{wy}$ for wx and wy which have been filled in at the time the last symbol of w is exposed in leftmost derivations of wx and wy will be the same. Our left part theorem is a somewhat stronger formalization of this intuition. It is convenient to begin with the following preliminary result.

Lemma 3.2. Let $G = (N,\Sigma,P,S)$ be a reduced LL(k) grammar and let $\mathcal{T}$ and $\mathcal{T}'$ be two grammatical trees over G such that $rtl(\mathcal{T}) = rtl(\mathcal{T}') = B$, where B is a variable, terminal or Λ. Let n be a non-negative integer. If for some variable A and terminal strings u, v and v′ such that $A \Rightarrow^{*} uBv$ and $A \Rightarrow^{*} uBv'$ we have $[\ell r(\mathcal{T})v]/(n+k) = [\ell r(\mathcal{T}')v']/(n+k)$ then ${}^{\{n+1\}}\mathcal{T} = {}^{\{n+1\}}\mathcal{T}'$.

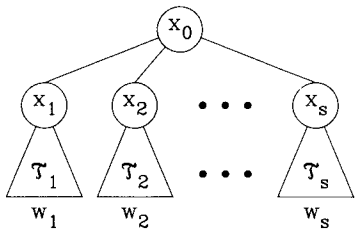
Proof: The proof proceeds by means of an induction on the height h of the higher of the two trees $\mathcal{T}$ and $\mathcal{T}'$. Let $rtn(\mathcal{T}) = x_0$ and $rtn(\mathcal{T}') = x_0'$.

Basis ($h = 0$): Both $\mathcal{T}$ and $\mathcal{T}'$ consist of a single node. Suppose that $\lambda(x_0) = \lambda(x_0')$. Trivially we have $\mathcal{T} = \mathcal{T}'$, whence ${}^{\{n+1\}}\mathcal{T} = {}^{\{n+1\}}\mathcal{T}'$.

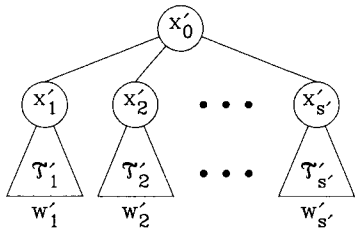
Induction Step: Assume that the lemma is true for trees of height $\leq h$, and call this assumption hypothesis H. We shall extend H to trees of height $\leq (h+1)$.

Without loss of generality assume that $\mathcal{T}$ has height $h+1$. Then x_0 is an internal node of $\mathcal{T}$ so that $B \in N$. Since $\lambda(x_0) = \lambda(x_0')$ and $\mathit{fr}(\mathcal{T}') \in \Sigma^*$ ($\mathcal{T}'$ is a grammatical tree) x_0' must be an internal node of $\mathcal{T}'$.

Let $\mathcal{T}$ be the tree



and let $\mathcal{T}'$ be the tree



Our hypothesis is that

$A \Rightarrow^* uBv$

$A \Rightarrow^* uBv'$

$\lambda(x_0) = \lambda(x_0') = B$

$[\mathit{fr}(\mathcal{T})v]/(n+k) = [\mathit{fr}(\mathcal{T}')v']/(n+k)$

for some variable A and some u, v, $v' \in \Sigma^*$.

Claim A. The elementary subtrees rooted in x_0 and x_0' are isomorphic. That is,

- $s = s'$
- $\lambda(x_i) = \lambda(x_i'), \qquad 1 \leq i \leq s$

Proof of Claim A: By definition $(x_1, \cdots, x_s)$ is a CS of $\mathcal{T}$ and $(x_1', \cdots, x_{s'}')$ is a CS of $\mathcal{T}'$. Hence by fact 2.7

$\lambda(x_0) = B \Rightarrow \lambda(x_1 \cdots x_s) \Rightarrow^* w_1 \cdots w_s$

$\lambda(x_0') = B \Rightarrow \lambda(x_1' \cdots x_{s'}') \Rightarrow^* w_1' \cdots w_{s'}'$

Since G_A is LL(k) (theorem 1.8) and

$(w_1 \cdots w_s v)/(n+k) = (w_1' \cdots w_{s'}' v')/(n+k)$

it follows from theorem 1.6 that

$$\lambda(x_1 \cdots x_s) = \lambda(x'_1 \cdots x'_s)$$

and the claim is established. □

Claim B. If for some $\ell \leq s$ we have

(a) $\mathcal{T}_i = \mathcal{T}'_i, \qquad 1 \leq i < \ell$

(b) $|w_1 \cdots w_{\ell-1}| = |w'_1 \cdots w'_{\ell-1}| = m \leq n$

then for $n' = n-m$ we have ${}^{\{n'+1\}}\mathcal{T}_\ell = {}^{\{n'+1\}}\mathcal{T}'_\ell$.

Proof of Claim B: Observe that $\mathcal{T}_\ell$ and $\mathcal{T}'_\ell$ have height $\leq h$. If we can satisfy the conditions of hypothesis H then we will immediately obtain the desired result. If $\ell = s = 1$ and $\lambda(x_1) = \Lambda$ then the claim follows trivially. We may therefore assume that x_ℓ is not a Λ-node. From Claim A we know that $\lambda(x_\ell) = \lambda(x'_\ell)$. Let $C = \lambda(x_\ell)$. Since x_ℓ is not a Λ-node we have $C \in V$.

By assumption there exist derivations

$$A \Rightarrow^* uBv$$

$$A \Rightarrow^* uBv'$$

Since $\mathcal{T}$ and $\mathcal{T}'$ are grammatical trees there exist derivations

$$B \Rightarrow^* w_1 \cdots w_{\ell-1} C w_{\ell+1} \cdots w_s$$

$$B \Rightarrow^* w'_1 \cdots w'_{\ell-1} C w'_{\ell+1} \cdots w'_s$$

(facts 2.5 and 2.7) so that

$$A \Rightarrow^* uw_1 \cdots w_{\ell-1} C w_{\ell+1} \cdots w_s v$$

$$A \Rightarrow^* uw'_1 \cdots w'_{\ell-1} C w'_{\ell+1} \cdots w'_s v' \qquad (3)$$

Since $w_i = w'_i$, $1 \leq i < \ell$, we may write

$$z = w_1 \cdots w_{\ell-1} = w'_1 \cdots w'_{\ell-1}$$

$$A \Rightarrow^* uzCw_{\ell+1} \cdots w_s v \qquad (2)$$

$$A \Rightarrow^* uzCw'_{\ell+1} \cdots w'_s v' \qquad (3)$$

It follows from (b) that $n' = n-m$ is a non-negative integer. Since

$$(w_1 \cdots w_s v)/(n+k) = (w'_1 \cdots w'_s v')/(n+k)$$

and $w_i = w'_i$, $1 \leq i < \ell$, we must have

$$(w_\ell \cdots w_s v)/(n'+k) = (w'_\ell \cdots w'_s v')/(n'+k)$$

or

$$[\mathcal{L}_r(\mathcal{T}_\ell) w_{\ell+1} \cdots w_s v]/(n'+k) = [\mathcal{L}_r(\mathcal{T}'_\ell) w'_{\ell+1} \cdots w'_s v']/(n'+k) \qquad (4)$$

In view of (2), (3), (4), and the fact that $\mathcal{T}_\ell$ and $\mathcal{T}'_\ell$ have height at most h we may invoke H to conclude that ${}^{\{n'+1\}}\mathcal{T}_\ell = {}^{\{n'+1\}}\mathcal{T}'_\ell$, as desired. □

Claim C. If for some $\ell \leq s$ no tree among $\mathcal{T}_1, \cdots, \mathcal{T}_\ell$ contains the $(n+1)^{\underline{st}}$ terminal node of $\mathcal{T}$ and no tree among $\mathcal{T}'_1, \cdots, \mathcal{T}'_\ell$ contains the $(n+1)^{\underline{st}}$ terminal node of $\mathcal{T}'$ then $\mathcal{T}_j = \mathcal{T}'_j$, for each j in the range $1 \leq j \leq \ell$.

Proof of Claim C: The argument is an induction on j.

Basis ($j = 0$): Vacuous.

Induction Step: ($j \geq 1$) Assume that the claim is true for indices $1, \cdots, (j-1)$. Then condition (a) of Claim B is satisfied for $\ell = j$. Since neither $\mathcal{T}_j$ nor $\mathcal{T}'_j$ contain the $(n+1)^{\underline{st}}$ terminal node of $\mathcal{T}$ and $\mathcal{T}'$, respectively, we have

$$|w_1 \cdots w_{j-1}| = |w'_1 \cdots w'_{j-1}| = m \leq n - |w_j|$$

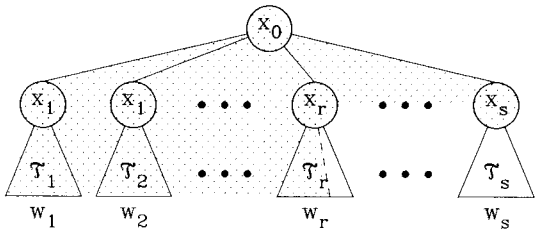
and, for $n' = n-m$,

$$n' \geq |w_j| \tag{5}$$

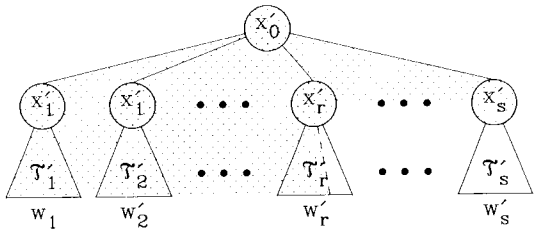
$$n' \geq |w'_j| \tag{6}$$

so that condition (b) of Claim B is satisfied and we may conclude that ${}^{\{n'+1\}}\mathcal{T}_j = {}^{\{n'+1\}}\mathcal{T}'_j$. In fact from (5) and (6) it follows that ${}^{\{n'+1\}}\mathcal{T}_j = \mathcal{T}_j$ and that ${}^{\{n'+1\}}\mathcal{T}'_j = \mathcal{T}'_j$, whence $\mathcal{T}_j = \mathcal{T}'_j$. □

Now let r be the least index such that at least one of $\mathcal{T}_r$ and $\mathcal{T}'_r$ contains the $(n+1)^{\underline{st}}$ terminal node of $\mathcal{T}$ and $\mathcal{T}'$, or (s+1) if no such index exists. It follows from Claims B and C that there are isomorphisms $\mathbf{f}_i$ establishing $\mathcal{T}_i = \mathcal{T}'_i$, $1 \leq i < r$, and an isomorphism $\mathbf{f}_r$ establishing ${}^{\{n'+1\}}\mathcal{T}_r = {}^{\{n'+1\}}\mathcal{T}'_r$, where $m = |w_1 \cdots w_{r-1}| = |w'_1 \cdots w'_{r-1}|$ and $n' = n-m$. Now ${}^{\{n+1\}}\mathcal{T}$ is the shaded portion of



and ${}^{\{n+1\}}\mathcal{T}'$ is the shaded portion of



If we define the mapping **f** by

- $\mathbf{f}(x_0) = x'_0$
- $\mathbf{f}(x_i) = x'_i,\ 1 \leq i \leq s$
- $\mathbf{f}(p) = \mathbf{f}_i(p),\ 1 \leq i < r$, if p is a node of τ_i
- $\mathbf{f}(p) = \mathbf{f}_r(p)$ if p is a node of ${}^{\{n'+1\}}\tau_r$ and $r \leq s$

then it follows easily from Claim A and the above argument that **f** is a label-preserving structural isomorphism between ${}^{\{n+1\}}\tau$ and ${}^{\{n+1\}}\tau'$, so that ${}^{\{n+1\}}\tau = {}^{\{n+1\}}\tau'$ and the proof is complete. ■

Lemma 3.2 is actually the forward direction of the Left Part Theorem, which we ar now prepared to prove.

Theorem 3.3. (The LL(k) Left Part Theorem) A reduced cfg G is LL(k) iff the following condition holds for all $n \geq 0$: if τ and τ' are grammatical trees over G such that

(1) $rtl(\tau) = rtl(\tau')$

(2) $\ell r(\tau)/(n+k) = \ell r(\tau')/(n+k)$

then ${}^{\{n+1\}}\tau = {}^{\{n+1\}}\tau'$.

Proof →: Lemma 3.2 suffices to establish the forward direction. Suppose that G = (N,Σ,P,S) is a reduced LL(k) grammar and that τ and τ' are any two grammatical trees over G such that

(1) $rtl(\tau) = rtl(\tau')$

(2) $\ell r(\tau)/(n+k) = \ell r(\tau')/(n+k)$

Let $A = rtl(\tau) = rtl(\tau') = B$ and $u = v = v' = \Lambda$. For the derivations $A \Rightarrow^* uBv$ and $A \Rightarrow^* uBv'$ we use the trivial derivation $A \Rightarrow^* A$. Since $v = v' = \Lambda$,

$$[\ell r(\tau)v]/(n+k) = [\ell r(\tau')v']/(n+k)$$

follows immediately from (2). We have now satisfied the hypothesis of lemma 3.2, and may therefore conclude that ${}^{\{n+1\}}\tau = {}^{\{n+1\}}\tau'$, as desired.

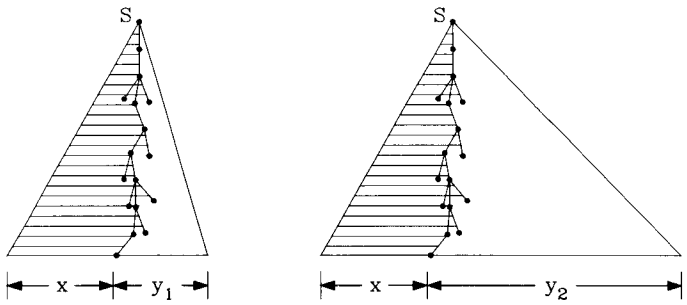


Fig. 11, illustrating the Left Part Theorem for LL languages. The left $\{|x|+1\}$-parts of derivation trees for xy_1 and xy_2 are shown shaded. These left parts are the portions of the respective trees which have been filled in at the time all of $x(y_1/1)$ and $x(y_2/1)$ have been exposed. If the grammar is LL(k) and $y_1/k = y_2/k$ then these left parts are necessarily identical.

Proof ⇐: Let $G = (N,\Sigma,P,S)$ be a reduced cfg with the property that if $\mathcal{T}$ and $\mathcal{T}'$ are any two grammatical trees over G such that

(1) $rtl(\mathcal{T}) = rtl(\mathcal{T}')$

(2) $fr(\mathcal{T})/(n+k) = fr(\mathcal{T}')/(n+k)$

then ${}^{\{n+1\}}\mathcal{T} = {}^{\{n+1\}}\mathcal{T}'$. We intend to show that G must necessarily be an LL(k) grammar. For suppose that G is not LL(k). In view of theorem 1.7 there must exist a pair of derivations

$$S \overset{\ell}{\Rightarrow}_L uA\beta \Rightarrow_L u\alpha\beta \overset{*}{\Rightarrow}_L uv$$

$$S \overset{\ell}{\Rightarrow}_L uA\beta \Rightarrow_L u\alpha'\beta \overset{*}{\Rightarrow}_L uv'$$

such that $v/k = v'/k$ and $\alpha \neq \alpha'$. Let $\mathcal{T}$ and $\mathcal{T}'$ be derivation trees over G for uv and uv', respectively, and let $n = |u|$ so that $(uv)/(n+k) = (uv')/(n+k)$. Since $rtl(\mathcal{T}) = S = rtl(\mathcal{T}')$, $fr(\mathcal{T}) = uv$, and $fr(\mathcal{T}') = uv'$ there exists by assumption an isomorphism $\mathbf{f}$ establishing ${}^{\{n+1\}}\mathcal{T} = {}^{\{n+1\}}\mathcal{T}'$. Let

$$\eta = (z_1 \cdots z_g \cdots z_r)$$

$$\eta' = (z'_1 \cdots z'_{g'} \cdots z'_{r'})$$

be the unique LCCS's at level ℓ in $\mathcal{T}$ and $\mathcal{T}'$ (fact 2.3) having the label $uA\beta$, in which z_g and $z'_{g'}$ are the leftmost internal nodes (so that they are labeled with A). Since $n = |u|$ and $u = \lambda(z_1, \cdots, z_{g-1})$ the $(n+1)^{\underline{st}}$ terminal node of $\mathcal{T}$ is either one of the nodes $z_{g+1}, \cdots, z_r$ or is

descended from one of the nodes $z_g, \cdots, z_r$. Similarly the $(n+1)^{st}$ terminal node of $\mathfrak{T}'$ is either one of the nodes $z'_{g'+1}, \cdots, z'_{r'}$ or is descended from one of the nodes $z'_{g'}, \cdots, z'_{r'}$. Accordingly the restrictions of η and η' to ${}^{\{n+1\}}\mathfrak{T}$ and ${}^{\{n+1\}}\mathfrak{T}'$ each contain an internal node – z_g and $z'_{g'}$, respectively. According to theorem 2.14 it follows that η and η' are LCCS's of ${}^{\{n+1\}}\mathfrak{T}$ and ${}^{\{n+1\}}\mathfrak{T}'$ at level ℓ. Since $\mathbf{f}$ preserves labeling it must be the case that $\mathbf{f}(\eta)$ is a LCCS of ${}^{\{n+1\}}\mathfrak{T}'$ at level ℓ having label $uA\beta$. But η' is also a LCCS of ${}^{\{n+1\}}\mathfrak{T}'$ having level ℓ. Since there can be at most one such LCCS (fact 2.3) we must have $\mathbf{f}(\eta) = \eta'$. It follows that $g = g'$, $\mathbf{f}(z_g) = z'_{g'}$, and that the elementary subtrees rooted in z_g and $z'_{g'}$ are isomorphic. That is to say, if $x_1, \cdots, x_s$ are the children of z_g and $x'_1, \cdots, x'_{s'}$ are the children of $z'_{g'}$ then $s = s'$ and

$$\lambda(\, x_1 \cdots x_s \,) \;=\; \lambda(\, x'_1 \cdots x'_{s'} \,)$$

But

$$\lambda(\, x_1 \cdots x_s \,) \;=\; \alpha$$

$$\lambda(\, x'_1 \cdots x'_{s'} \,) \;=\; \alpha'$$

so that $\alpha = \alpha'$, which we assumed was not the case. Consequently G must be LL(k). ■

4. Iteration Theorems

Armed with the Left Part Theorem our intent is to establish some pumping properties of the LL(k) languages. Roughly speaking, we will develop the argument used in establishing Ogden's lemma to obtain the usual decomposition of the derivation tree for a string w belonging to $\mathcal{L}$(G) in which we have distinguished a sufficient number of positions. This induces the usual factorization of w as $w_1w_2w_3w_4w_5$. By looking at derivation trees for w and for any other string w_1w_2u in $\mathcal{L}$(G) such that $(w_3w_4w_5)/k = u/k$, and applying the Left Part Theorem appropriately, we will obtain our first iteration theorem. We will need the following definitions.

Definition 4.1. Let $w \in \Sigma^*$. If $w_1w_2w_3w_4w_5 = w$ then the sequence (w_1,w_2,w_3,w_4,w_5) is said to be a *factorization* of w.

Definition 4.2. Let $w \in \Sigma^*$. Suppose that $w = a_1a_2 \cdots a_n$, where each $a_i \in \Sigma$. Any index i, $1 \leq i \leq |w|$, is called a *position* in w. For example, the

symbol occupying position 3 of the string aacbda is c. Next let $\mathcal{K}$ be any set of positions in a terminal string w. Any factorization $\varphi = (w_1,w_2,w_3,w_4,w_5)$ of w induces a natural "partition" $\mathcal{K}/\varphi$ of $\mathcal{K}$ into:

$$\mathcal{K}/\varphi = \{ \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5 \}$$

where

$$\mathcal{K}_i = \{ k \in \mathcal{K} \text{ such that the } k^{th} \text{ symbol is part of } w_i \}$$
$$= \{ k \in \mathcal{K} \mid |w_1 \cdots w_{i-1}| < k \leq |w_1 \cdots w_i| \}$$

Thus $\mathcal{K}_i$ selects out of $\mathcal{K}$ those positions which appear in w_i. We call the elements of $\mathcal{K}$ *distinguished positions* (or *dp's*). The following notation will also be convenient.

Definition 4.3. Let $u_i \in \Sigma^*$, $1 \leq i \leq r$, for some alphabet Σ. Then

$$\prod_{i=1}^{r}(u_i) = u_1u_2\cdots u_{r-1}u_r$$

We are now ready to proceed.

Theorem 4.4. (The 1st LL Iteration Theorem) Let L be an LL(k) language. There exists an integer p such that given a string w in L and p or more distinguished positions $\mathcal{K}$ in w we may write

$$\varphi = (w_1,w_2,w_3,w_4,w_5)$$
$$\mathcal{K}/\varphi = \{ \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5 \}$$

where

(1) $w_2 \neq \Lambda$

(2) a: Either w_1, w_2 & w_3 each contain dp's ($\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3 \neq \emptyset$),
or w_3, w_4 & w_5 each contain dp's ($\mathcal{K}_3$, $\mathcal{K}_4$, $\mathcal{K}_5 \neq \emptyset$),

b: and $w_2w_3w_4$ contains at most p dp's ($|\mathcal{K}_2 \cup \mathcal{K}_3 \cup \mathcal{K}_4| \leq p$).

(3) a: Let $n = |w_1w_2|$ and suppose that w' is any string in L such that $w'/(n+k) = w/(n+k)$. Then there is a factorization (w_1,w_2,w_3',w_4',w_5') of w' such that

(i) $w_1w_2^rw_3\prod_{i=1}^{r}(u_i)w_5$

(ii) $w_1w_2^rw_3'\prod_{i=1}^{r}(u_i)w_5$

(iii) $w_1w_2^rw_3\prod_{i=1}^{r}(u_i)w_5'$

(iv) $w_1w_2^rw_3'\prod_{i=1}^{r}(u_i)w_5'$

are in L for all $n \geq 0$ and for all strings $\prod_{i=1}^{r}(u_i)$ in which $u_i = w_4$ or $u_i = w_4'$, $1 \leq i \leq r$.

b: Furthermore, if $\prod_{i=1}^{r}(\bar{u}_i)$ is a catenation of words $\bar{u}_i \in \{w_4, w_4'\}$ such that

$$\prod_{i=1}^{r}(u_i) = \prod_{i=1}^{r}(\bar{u}_i)$$

then $u_i = \bar{u}_i$, $1 \leq i \leq r$.

Proof: Let G = (N,Σ,P,S) be an arbitrary reduced LL(k) grammar generating L. The methods used by Ogden [19] (or see Harrison and Havel [11]) suffice to establish the existence of an integer p such that for any string w in L in which p or more positions $\mathcal{K}$ are distinguished there is a factorization $\varphi = (w_1,w_2,w_3,w_4,w_5)$ such that (2) holds and for some variable A ∈ N

$$S \Rightarrow^{*} w_1Aw_5 \Rightarrow^{+} w_1w_2Aw_4w_5 \Rightarrow^{+} w_1w_2^rAw_4^rw_5 \Rightarrow^{+} w_1w_2^rw_3w_4^rw_5$$

for all non-negative integers n. Since no LL(k) grammar is left recursive (1) holds. To complete our proof we must show that φ satisfies (3) as well.

Let $n = |w_1w_2|$ and consider any string w′ in L such that $w'/(n+k) = w/(n+k)$. Let $\mathcal{T}$ and $\mathcal{T}'$ be the derivation trees for w and w′, respectively. Since $w/(n+k) = w'/(n+k)$ we may invoke the Left Part Theorem to obtain ${}^{\{n+1\}}\mathcal{T} = {}^{\{n+1\}}\mathcal{T}'$. (Refer to figure 12.)

Consider $\mathcal{T}$. Let x_j and x_k be the internal nodes of $\mathcal{T}$ corresponding to the A's in w_1Aw_5 and $w_1w_2Aw_4w_5$. We know that $w_3 \neq \Lambda$ since $\mathcal{K}_3 \neq 0$. Therefore the subtree rooted in x_k has a terminal node among its leaves. The leftmost such terminal node π is labeled with $(w_3w_4w_5)/1$ and is contained in ${}^{\{n+1\}}\mathcal{T}$; it is, in fact, the $(n+1)^{\underline{st}}$ terminal node of $\mathcal{T}$. Since the nodes x_j and x_k defined above lie on the root-leaf path to π they also belong to ${}^{\{n+1\}}\mathcal{T}$. (They appear in figure 12a labeled by A). Let f be the isomorphism of the Left Part Theorem. It follows that

$$A = \lambda(x_j) = \lambda(f(x_j))$$
$$A = \lambda(x_k) = \lambda(f(x_k))$$

Let η and θ now be the unique LCCS's of $\mathcal{T}$ in which the leftmost internal nodes are x_j and x_k, respectively (fact 2.5). We may write

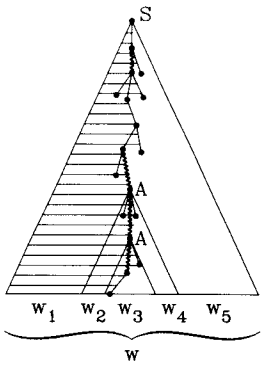


Fig. 12a: $\mathfrak{T}$.

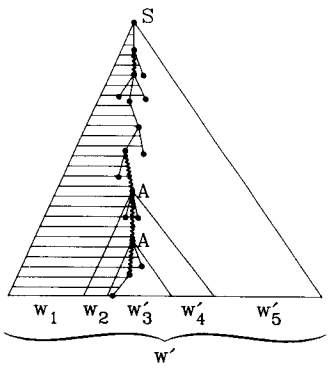


Fig. 12b: $\mathfrak{T}'$.

Fig. 12. Derivation trees for w and w′, in which the left $\{|w_1w_2|+1\}$–parts are shaded. As a result of the fact that G is LL(k) and $(w_1w_2w_3w_4w_5)/(|w_1w_2|+k) = (w_1w_2w'_3w'_4w'_5)/(|w_1w_2|+k)$ the left $\{|w_1w_2|+1\}$–parts are isomorphic. In particular, the two nodes labeled A in ${}^{\{n+1\}}\mathfrak{T}$ must appear in the same position in ${}^{\{n+1\}}\mathfrak{T}'$.

$$\eta = (a_1 \cdots a_{\underline{a}} \; x_j \; z_1 \cdots z_{\underline{z}}) \tag{7}$$

$$\theta = (a_1 \cdots a_{\underline{a}} \; b_1 \cdots b_{\underline{b}} \; x_k \; y_1 \cdots y_{\underline{y}} \; z_1 \cdots z_{\underline{z}}) \tag{8}$$

Since x_j and x_k are both internal nodes of $\mathfrak{T}$ which belong to ${}^{\{n+1\}}\mathfrak{T}$, η and θ are also LCCS′s of ${}^{\{n+1\}}\mathfrak{T}$ (theorem 2.14). Since ${}^{\{n+1\}}\mathfrak{T} = {}^{\{n+1\}}\mathfrak{T}'$, $\mathbf{f}(\eta)$ and $\mathbf{f}(\theta)$ are LCCS′s of ${}^{\{n+1\}}\mathfrak{T}'$, and hence of $\mathfrak{T}'$ (theorem 2.16). Again because ${}^{\{n+1\}}\mathfrak{T} = {}^{\{n+1\}}\mathfrak{T}'$ we may conclude that $\lambda(\eta) = \lambda(\mathbf{f}(\eta))$ and $\lambda(\theta) = \lambda(\mathbf{f}(\theta))$. In particular,

$$w_1 = \lambda(a_1 \cdots a_{\underline{a}}) = \lambda(\mathbf{f}(a_1 \cdots a_{\underline{a}}))$$

$$w_2 = \lambda(b_1 \cdots b_{\underline{b}}) = \lambda(\mathbf{f}(b_1 \cdots b_{\underline{b}}))$$

and for some $\alpha, \beta \in V^*$

$$\alpha = \lambda(y_1 \cdots y_{\underline{y}}) = \lambda(\mathbf{f}(y_1 \cdots y_{\underline{y}}))$$

$$\beta = \lambda(z_1 \cdots z_{\underline{z}}) = \lambda(\mathbf{f}(z_1 \cdots z_{\underline{z}}))$$

Now by invoking theorem 2.8 we obtain from $\mathfrak{T}$ the derivations

$$S \Rightarrow_L^* \lambda(a_1 \cdots a_{\underline{a}})\lambda(x_j)\lambda(z_1 \cdots z_{\underline{z}}) = w_1 A \beta \tag{9}$$

$$A = \lambda(x_j) \Rightarrow_L^* \lambda(b_1 \cdots b_{\underline{b}})\lambda(x_k)\lambda(y_1 \cdots y_{\underline{y}}) = w_2 A \alpha \tag{10}$$

$$A = \lambda(x_k) \Rightarrow_L^* w_3 \tag{11}$$

$$\alpha \Rightarrow_L^* w_4 \tag{12}$$

$$\beta \Rightarrow_L^* w_5 \tag{13}$$

and from $\mathfrak{T}'$ the derivations

$$S \Rightarrow_L^* \lambda(\mathbf{f}(a_1 \cdots a_{\underline{a}}))\lambda(\mathbf{f}(x_j))\lambda(\mathbf{f}(z_1 \cdots z_{\underline{z}})) = w_1 A \beta \quad (14)$$

$$A = \lambda(\mathbf{f}(x_j)) \Rightarrow_L^* \lambda(\mathbf{f}(b_1 \cdots b_{\underline{b}}))\lambda(\mathbf{f}(x_k))\lambda(\mathbf{f}(y_1 \cdots y_{\underline{y}})) = w_2 A \alpha \quad (15)$$

$$A = \lambda(\mathbf{f}(x_k)) \Rightarrow_L^* w_3' \quad (16)$$

$$\alpha \Rightarrow_L^* w_4' \quad (17)$$

$$\beta \Rightarrow_L^* w_5' \quad (18)$$

for some terminal strings w_3', w_4' and w_5' such that $w_3'w_4'w_5' = u$. By suitably combining these derivations we can obtain any of the strings specified in (3a). For example, to obtain strings of the form

(i) $w_1 w_2^r w_3 \prod_{i=1}^{r} (u_i) w_5$

begin with (9), followed by r applications of (10), followed by (11), followed by a suitable mixture of (12) & (17), and finish with (13). (Season to taste.)

Next we establish (3b). If $w_4 = w_4'$ then (3b) follows trivially. Therefore assume that $w_4 \neq w_4'$, so that (12) and (17) are distinct leftmost derivations. For the sake of simplicity we restrict our attention now to strings of type (i). Let R be the set

$$\{(9)\}\ \{(10)\}^r\ \{(11)\}\ \{(12)+(17)\}^r\ \{(13)\}$$

Notice that a string in R uniquely specifies the leftmost derivation of a type (i) word in L. In particular, let $\mathfrak{p}_i$, $1 \leq i \leq r$, be defined by

$\mathfrak{p}_i = (12)$ if $u_i = w_4$

$\mathfrak{p}_i = (17)$ if $u_i = w_4'$

Then given a string of type (i), which determines the sequence $\mathfrak{p}_i$,

$$\{(9)\}\ \{(10)\}^r\ \{(11)\}\ \prod_{i=1}^{r} \{\mathfrak{p}_i\}\ \{(13)\}$$

is a leftmost derivation of the word. If there exist <u>two</u> catenations

$$\prod_{i=1}^{r} (u_i) \quad \text{and} \quad \prod_{i=1}^{r} (\bar{u}_i)$$

and corresponding sequences $\mathfrak{p}_i$ and $\bar{\mathfrak{p}}_i$ such that

$$\prod_{i=1}^{r} (u_i) = \prod_{i=1}^{r} (\bar{u}_i)$$

and for which $u_i \neq \bar{u}_i$, for some i in the range $1 \leq i \leq r$, so that $\mathfrak{p}_i \neq \bar{\mathfrak{p}}_i$, then there are two distinct strings in R, representing two distinct leftmost derivations of the same string in L. But then G is an

ambiguous grammar, which cannot be the case since G is LL(k). Hence (3b) follows for a string of type (i).

We can extend (3b) to strings of type (ii), (iii) and (iv) by analogous arguments – the details are omitted. ■

Before proceding with a formal development of a second pumping lemma for the LL(k) languages, we sketch the intuition underlying our argument. (Refer to figure 13.) Suppose that uv and uvy, $|v| = k$, are strings in some language L generated by a Λ–free LL(k) grammar G. Leftmost derivations of uv and uvy must proceed identically at least until all of u has been exposed; that is the meaning of the Extended LL(k) Theorem. After exposing the rightmost terminal of u in a leftmost derivation of either uv or uvy there can be no more than k variables remaining in the left sentential form since G is Λ–free and $|v| = k$. Judicious use of this fact, together with the Left Part Theorem and the argument of the 1st Iteration Theorem, is sufficient for our purposes.

We will need the following result, which is due to Rosenkrantz and Stearns.

Theorem 4.5. Given an LL(k) grammar $G = (N,\Sigma,P,S)$ we can construct an LL(k+1) grammar $G' = (N',\Sigma,P',S')$ such that $\mathcal{L}(G') = \mathcal{L}(G)$ and G′ is Λ–free unless $\Lambda \in \mathcal{L}(G)$, in which case G′ contains the single Λ–rule $S' \to \Lambda$ and S′ does not appear in the right hand side of any rule in P′.

Proof. Using the arguments found in Rosenkrantz and Stearns [21], pages 236–241 (or see Aho and Ullman [2], pages 674–681), we may obtain a Λ–free grammar $G'' = (N'',\Sigma,P'',S'')$ generating $\mathcal{L}(G) - \{\Lambda\}$. If $\Lambda \notin \mathcal{L}(G)$ then set $G' = G''$.

Suppose, however, that L contains Λ. Then we form a new grammar G′ whose start symbol is S′ and whose rules are the rules of G″ together with $S' \to S'' \mid \Lambda$, where S′ is a new variable not in V″. It is trivial to prove that G′ is also LL(k+1) and generates exactly $\mathcal{L}(G)$. ■

Theorem 4.6. (The 2nd LL Iteration Theorem) Let L be an LL(k–1) language, $k \geq 1$. There exists an integer p such that for any two distinct strings x and xy in L, if $|x| \geq k$ and p or more positions in y are distinguished, then there is a factorization $\varphi = (w_1,w_2,w_3,w_4,w_5)$ of xy such that (1) – (3) of the 1st LL Iteration Theorem hold and $|w_1| \geq |x| - k$.

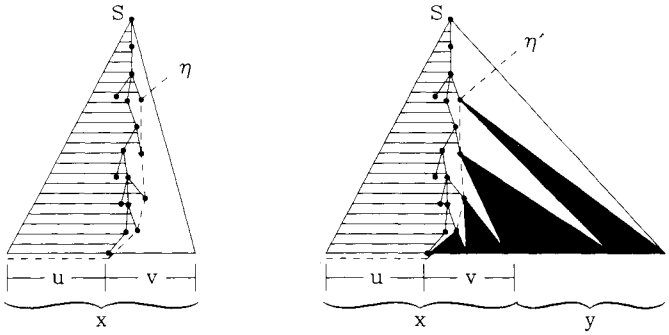


Fig. 13a. $\mathfrak{T}$. Fig. 13b. $\mathfrak{T}'$.

Fig. 13. The solidly shaded areas indicate the leaves descended from a particular internal node of $\mathfrak{T}'$ which is a leaf of the left $\{|u|+1\}$–part of $\mathfrak{T}'$. The dashed lines mark the frontier of the left $\{|u|+1\}$–parts for each tree. This is the left sentential form obtained at the time u(v/1) is exposed.

Proof. In view of theorem 4.5 we may assume that L is generated by some LL(k) grammar $G = (N,\Sigma,P,S)$ which is Λ–free, except possibly for an $S \to \Lambda$ rule, in which case S does not appear in any right part.

For any variable A let $G_A = (N,\Sigma,P,A)$ be the cfg obtained from G by changing the start symbol to A, let p_A be the constant obtained from the $1^{\underline{st}}$ Iteration Theorem for the language $\mathscr{L}(G_A)$ (which is also LL(k) – see theorem 1.8), and let

$$p' = \max\{\, p_A \mid A \in N \,\}$$
$$p = kp' + 1$$

Suppose that x and xy are strings belonging to L, where $|x| \geq k$ and p or more positions are distinguished in y. Let us write x as uv, where $|u| = n$ and $|v| = k$, and let $\mathfrak{T}$ and $\mathfrak{T}'$ be derivation trees for uv and uvy. (See figure 13.) Let $\eta = \mathit{leaves}(\{n+1\}\mathfrak{T})$ and $\eta' = \mathit{leaves}(\{n+1\}\mathfrak{T}')$.

Since $x/(n+k) = (xy)/(n+k) = x$, it follows from the Left Part Theorem that $\{n+1\}\mathfrak{T} = \{n+1\}\mathfrak{T}'$, whence η and η' are isomorphic and $\lambda(\eta) = \lambda(\eta')$. It follows from theorem 2.17 that η and η' are LCCS's of $\mathfrak{T}$ and $\mathfrak{T}'$, respectively. Consequently we may write

$$S \Rightarrow_L^* u\gamma = \lambda(\eta) \Rightarrow_L^* uv$$
$$S \Rightarrow_L^* u\gamma = \lambda(\eta') \Rightarrow_L^* uvy$$

for some γ in V^* (fact 2.8). Since $|v| = k \geq 1$ these derivations involve no Λ-rules. It follows that $|\gamma| \leq k$ since $|v| = k$ and $\gamma \Rightarrow_L^* v$.

Now write γ as $X_1X_2 \cdots X_s$ $(s \leq k)$. Let $(z_1,z_2,\cdots,z_s)$ be the factorization of vy such that $X_i \Rightarrow_L^* z_i$, $1 \leq i \leq s$. Suppose that there are p′ or fewer dp′s in each z_i. Then there are at most $sp' \leq kp' < p$ dp′s in vy, which is not the case. Hence some particular z_i contains more than $p' \geq p_{X_i}$ dp′s. Now the string z_i belongs to the language $\mathcal{L}(G_{X_i})$, which (as we noted above) is an LL language. Also, we have distinguished p_{X_i} or more positions in this string. It follows from the 1st Iteration Theorem that there is a factorization $(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5)$ of z_i such that (1) – (2) of theorem 4.4 hold with respect to $\mathcal{L}(G_{X_i})$ and for some variable B we have the derivation

$$X_i \Rightarrow^* \sigma_1 B \sigma_5 \Rightarrow^+ \sigma_1\sigma_2 B \sigma_4\sigma_5 \Rightarrow^+ \sigma_1\sigma_2^r B \sigma_4^r\sigma_5 \Rightarrow^+ \sigma_1\sigma_2^r\sigma_3\sigma_4^r\sigma_5$$

in G_i. From this it follows that the factorization

$$(uz_1 \cdots z_{i-1}\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5 z_{i+1} \cdots z_s) = (w_1,w_2,w_3,w_4,w_5)$$

satisfies (1) – (2) with respect to L. Since u is necessarily a prefix of w_1 it is clear that $|w_1| \geq |x| - k$. If we let

$$n = |uz_1 \cdots z_{i-1}\sigma_1\sigma_2|$$

and consider any other string w′ in L such that $w'/(n+k) = w/(n+k)$, the argument used to deduce (3) in theorem 4.4 may be used to deduce property (3) here, and the proof is complete. ■

5. Applications

We begin by showing that every LL(k) grammar is LR(k). This is not a new result; Brosgol [8] obtained a rigorous proof *via* LR(k) grammar theory by embedding Λ-rules in the grammar. It is more often argued intuitively from a consideration of LL(k) and LR(k) derivation trees that this result is obvious (see Aho and Ullman [2], for example). Using the LL(k) Left Part Theorem we can now make the tree argument rigorous.

Theorem 5.1. Every LL(k) grammar is LR(k), $k \geq 0$.

Proof: Let G be an arbitrary LL(k) grammar. If $k = 0$ then $\mathcal{L}(G)$ is $\emptyset$ or a singleton set, both of which are trivially LR(0). We therefore assume in the remainder of this proof that $k \geq 1$. Also, $S \Rightarrow_R^+ S$ is impossible since G is unambiguous. Hence if G is not LR(k) then for some w, w′, x $\in \Sigma^*$; α, α', β, β' $\in V^*$; A, A′ $\in$ N, there exist derivations

$$S \Rightarrow_R^* \alpha Aw \Rightarrow_R \alpha\beta w$$
$$S \Rightarrow_R^* \alpha' A' x \Rightarrow_R \alpha'\beta' x = \alpha\beta w'$$

such that $w/k = w'/k$ and $(A \to \beta, |\alpha\beta|) \neq (A' \to \beta', |\alpha'\beta'|)$. Without loss of generality we may assume that G is reduced. Let $z \in \mathcal{L}(\alpha\beta)$, let $\mathcal{T}$ be the derivation tree for zw, let $\mathcal{T}'$ be the derivation tree for zw', and let $n = |z|$. Since G is LL(k) and $(zw)/(n+k) = (zw')/(n+k)$, we may apply the Left Part Theorem to obtain ${}^{\{n+1\}}\mathcal{T} = {}^{\{n+1\}}\mathcal{T}'$. Let $\mathbf{f}$ be the mapping which effects the isomorphism. Let $\eta = (u_1, \cdots, u_s)$ be the unique RCCS of $\mathcal{T}$ having the label αAw (theorems 1.2 and 2.9). Let u_i be the node of η labeled by the A explicitly shown in αAw, and let

$$\theta = (u_1 \cdots u_{i-1} \; v_1 \cdots v_h \; u_{i+1} \cdots u_s)$$

be the RCCS formed from η by expanding u_i, so that $\lambda(v_1 \cdots v_h) = \beta$ and $\lambda(\theta) = \alpha\beta w$. (Refer to figure 14a.) Let $a = w/1$ $(a \in \Sigma_\Lambda)$. Since $w/k = w'/k$ and $k \geq 1$, we also have $a = w'/1$. Consider ${}^{[n+1]}\mathcal{T}$: $\ell_r({}^{[n+1]}\mathcal{T}) = za$. Let $\chi = (u_1, \cdots, u_r)$ be the restriction of η to ${}^{[n+1]}\mathcal{T}$ and recall that $\lambda(\eta) = \alpha Aw$. If $a \in \Sigma$ then $i < r$ and u_i belongs to ${}^{[n+1]}\mathcal{T}$. If $a = \Lambda$ (because $w = \Lambda$) then ${}^{[n+1]}\mathcal{T} = \mathcal{T}$, so that $r = s$, $\chi = \eta$, and $\lambda(\chi) = \alpha Aa = \alpha Aw = \alpha A$. In either case $\lambda(\chi) = \alpha Aa$ $(i \leq r \leq s)$, so that u_i appears in χ. Next let

$$\psi = (u_1 \cdots u_{i-1} \; v_1 \cdots v_h \; u_{i+1} \cdots u_r)$$

be the restriction of θ to ${}^{[n+1]}\mathcal{T}$, so that $\lambda(\psi) = \alpha\beta a$. χ and ψ are RCCS's of ${}^{[n+1]}\mathcal{T}$ (theorem 2.13), ψ being obtained in one step from χ by rewriting u_i. Since ${}^{\{n+1\}}\mathcal{T} = {}^{\{n+1\}}\mathcal{T}'$ under $\mathbf{f}$ we must also have ${}^{[n+1]}\mathcal{T} = {}^{[n+1]}\mathcal{T}'$ under $\mathbf{f}$. If we let $\chi' = \mathbf{f}(\chi)$ and $\psi' = \mathbf{f}(\psi)$ then

$$\lambda(\chi) = \lambda(\chi') = \alpha Aa$$
$$\lambda(\psi) = \lambda(\psi') = \alpha\beta a$$

and in view of the isomorphism χ' and ψ' must be RCCS's of ${}^{[n+1]}\mathcal{T}'$, ψ' being obtained in one step from χ' by rewriting $\mathbf{f}(u_i)$. Now extend χ' to form a RCCS ξ in $\mathcal{T}'$ by appending to χ' (in left-to-right order) all of the leaves of $\mathcal{T}'$ which are right of $\mathbf{f}(u_r)$ (theorem 2.15), so that $\lambda(\xi) = \alpha Aw'$. Similarly extend ψ' to obtain a RCCS ζ in $\mathcal{T}'$ such that $\lambda(\zeta) = \alpha\beta w'$. Since there are no internal nodes to the right of u_i in η, there can be no internal nodes to the right of u_i in χ, and no internal nodes to the right of $\mathbf{f}(u_i)$ in χ'. Since ξ is obtained from χ' by appending leaves, $\mathbf{f}(u_i)$ is also the rightmost internal node of ξ. Hence ζ is a RCCS of $\mathcal{T}'$ which can be obtained from the RCCS ξ of $\mathcal{T}'$ in one

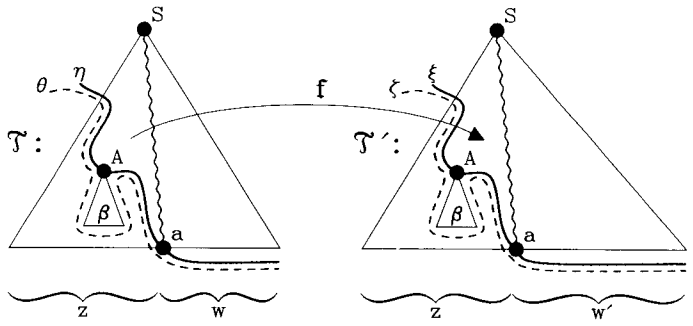


Figure 14a, illustrating the proof of theorem 5.1. In $\mathcal{T}$ we show η and θ, the unique RCCS's of $\mathcal{T}$ labeled αAw and $\alpha\beta$w′. In $\mathcal{T}'$ we show RCCS's ξ and ζ, the extensions of χ' and ψ' (see figure 14b below) to $\mathcal{T}'$ from $^{[n+1]}\mathcal{T}'$. The isomorphism $\mathbf{f}$ maps $^{[n+1]}\mathcal{T}$ onto $^{[n+1]}\mathcal{T}'$.

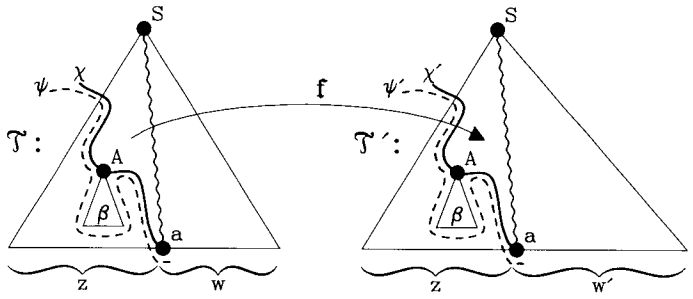


Figure 14b, illustrating the proof of theorem 5.1. In $\mathcal{T}$ we show the restrictions χ and ψ of η and θ to $^{[n+1]}\mathcal{T}$. In $\mathcal{T}'$ we show the isomorphic images χ' and ψ' of χ and ψ under $\mathbf{f}$. Since $^{[n+1]}\mathcal{T} = {}^{[n+1]}\mathcal{T}'$ we have $\lambda(\chi) = \lambda(\chi') = \alpha$Aa and $\lambda(\psi) = \lambda(\psi') = \alpha\beta$a.

step by rewriting $\mathbf{f}(u_i)$. We must have

$$r\mathcal{U}(\mathcal{T}) \Rightarrow_R^* \lambda(\xi) \Rightarrow_R \lambda(\zeta)$$

(fact 2.8). That is,

$$S \Rightarrow_R^* \alpha Aw' \Rightarrow_R \alpha\beta w'$$

Since we also know that

$$S \Rightarrow_R^* \alpha'A'x' \Rightarrow_R \alpha'\beta'x' = \alpha\beta w'$$

and that G is unambiguous (theorem 1.2) it must be the case that $\alpha = \alpha'$, $\beta = \beta'$, and A = A′ so that $(A \rightarrow \beta, |\alpha\beta|) = (A' \rightarrow \beta', |\alpha'\beta'|)$ which is a contradiction. Hence G is, in fact, an LR(k) grammar. ■

We next consider a number of results which follow easily from our iteration theorems. Theorems 5.2, 5.3, 5.4, 5.5 and 5.6 each illustrate a different way in which possessing the LL(k) property restricts the form of strings in a language; each of the proofs illustrates a different way in which the iteration theorems may be used. We consider only languages which are LR(k) since every LL(k) language is LR(k); if a language is not even LR(k) then other tools already exist for demonstrating this which incidently demonstrate that the language also fails to be LL.

Theorem 5.2. The LR language $L_1 = \{ a^n b^n, a^n c^n \mid n \geq 1 \}$ is not LL.

Proof: (Figure 15.) Assume that L_1 is LL(k) and let p be the constant obtained for L_1 from the 1st Iteration Theorem. Consider the string $w = a^p a^k b^{p+k}$ in which the first block of p a's are distinguished. From theorem 4.4 we obtain the usual factorization $\varphi = (w_1,w_2,w_3,w_4,w_5)$ of w. Since φ satisfies theorem 4.4 we must have $w_2 \in a^+$ and $w_4 \in b^+$, and w_3 must begin with at least k a's since w_4 cannot contain any distinguished positions. Now consider $a^{p+k}c^{p+k}$, which we can write as w_1w_2u for some $u \in a^k a^* c^+$. Note that $u/k = (w_3w_4w_5)/k = a^k$. It follows that for some w_3', w_4' and w_5' we have $u = w_3'w_4'w_5'$ and $w_1w_2^2w_3'w_4'w_4w_5' \in L_1$. But $w_4 \in b^+$ and $w_3'w_4'w_5' \in a^+c^+$, and there are no strings containing both b's and c's in L_1. ■

Theorem 5.3. The LR language $L_2 = \{ a^n 0 b^n, a^n 1 b^{2n} \mid n \geq 1 \}$ is not LL.

Proof: (Figure 16.) Assume that L_2 is LL(k) and let p be the constant obtained for L_2 from the 1st Iteration Theorem. Consider the string $w = a^p a^k 1 b^{2(p+k)}$ in which the first block of p a's are distinguished. From theorem 4.4 we obtain a factorization $\varphi = (w_1,w_2,w_3,w_4,w_5)$ of w. Since φ satisfies theorem 4.4 we must have $w_2 \in a^+$ and $w_4 \in b^+$, $2 \leq 2|w_2| = |w_4| < 2p$, and w_3 must begin with at least k a's. Now consider $a^{p+k}0b^{p+k}$, which may be written as w_1w_2u for some $u \in a^k a^* 0 b^*$. Note that $u/k = (w_3w_4w_5)/k$. It follows from theorem 4.4 that for some w_3', w_4' and w_5' we have $u = w_3'w_4'w_5'$, $|w_2| = |w_4'|$ and $w_1w_2^2w_3'w_4'w_4w_5' \in L_2$. Let $\#_a$ and $\#_b$ be the number of a's and b's in this string. Then $p+k+|w_2| = \#_a < p+k+2|w_2| = \#_b < 2(p+k+|w_2|) = 2\#_a$, so that this string contains an illegal number of b's and cannot belong to L_2. ■

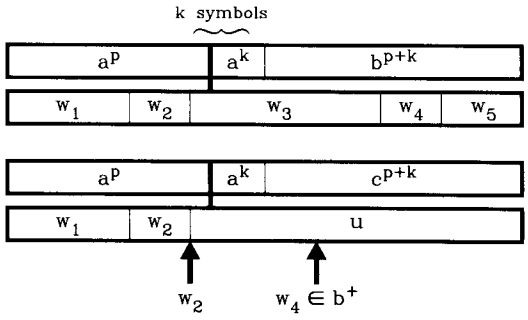


Fig.15. An application of Theorem 4.4 to the language $a^nb^n + a^nc^n$.

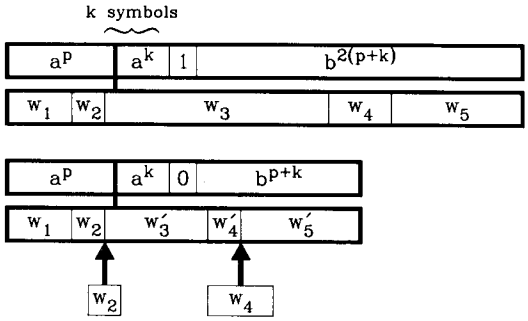


Fig. 16. An application of Theorem 4.4 to the language $a^n0b^n + a^n1b^{2n}$.

Theorem 5.4. The LR language $L_3 = \{ a^nda^ne, a^nfa^ng \mid n \geq 1 \}$ is not LL.

Proof: (Figure 17.) Assume that L_3 is LL(k) and let p be the constant obtained for L_3 from the 1st Iteration Theorem. Consider the string $w = a^pa^kda^{p+k}e$ in which the first block of p a's are distinguished. From theorem 4.4 we obtain a factorization $\varphi = (w_1,w_2,w_3,w_4,w_5)$ of w such that $w_1w_2^nw_3w_4^nw_5$ is in L_3 for every $n \geq 0$. In view of this we must have $w_2 \in a^+$, $w_4 \in a^+$ and $w_3 \in a^*da^*$. As usual we also have $(w_3w_4w_5)/k = a^k$. Now consider $a^{p+k}fa^{p+k}g$, which we may write as w_1w_2u for some u. It is necessarily the case that $u/k = (w_3w_4w_5)/k$. It follows from theorem 4.4 that for some w_3', w_4' and w_5' we have $u = w_3'w_4'w_5'$, $w_3' \in a^*fa^*$, w_5' ends in g and $w_1w_2^nw_3w_4^nw_5'$ is in L_3 for every $n \geq 0$. But these strings have the form a^+da^+g, and therefore cannot belong to L_3. ■

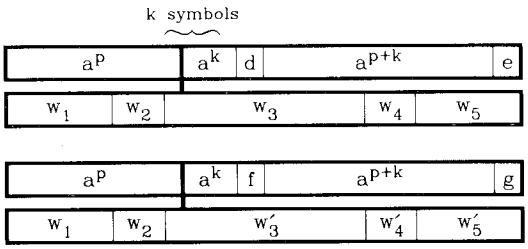


Fig. 17. An application of Theorem 4.4 to the language $a^n d a^n e + a^n f a^n g$. If this language is LL then it must contain the strings $w_1 w_2^n w_3 w_4^n w_5' \in a^+ d a^+ g$, which it does not.

Theorem 5.5. The LR language $L_4 = \{ a^m b^{m+n} \mid m \geq 1,\ 0 \leq n \leq m \}$ is not LL.

Proof: Assume that L_4 is LL(k) and let p be the constant obtained for L_4 from the 1st Iteration Theorem. Without loss of generality assume that $p \geq k$. Consider the string $a^p b^p$ in which the a's are distinguished. From theorem 4.4 we obtain a factorization $\varphi = (w_1, w_2, w_3, w_4, w_5)$ of $a^p b^p$ such that $w_1 w_2^n w_3 w_4^n w_5$ is in L_4 for every $n \geq 0$. It follows that w_2 and w_4 cannot both consist entirely of a's, for then we could obtain strings having more a's than b's. Also, w_2 and w_4 cannot consist entirely of b's for then we could obtain strings with too many b's. Clearly neither w_2 nor w_4 can contain both a's and b's. Hence w_2 must consist entirely of a's and w_4 entirely of b's. Furthermore, $|w_2| \leq |w_4|$, for otherwise we could again obtain strings with more a's than b's for a suitably large value of n. In particular, $w_1 w_3 w_5$ is in L_4. Let $i = |w_2|$; we know that $i \geq 1$. If w_4 contains more than i b's then $w_1 w_3 w_5$ will contain more a's than b's, which is not allowed. Therefore $|w_2| = |w_4|$; we have $w_2 = a^i$ and $w_4 = b^i$.

Now consider the string $a^p b^{2p}$. Since $w_1 w_2 \in a^+$ and $p \geq k$ it must be the case that $a^p b^p / (|w_1 w_2| + k) = a^p b^{2p} / (|w_1 w_2| + k)$. Hence there is a factorization $(w_1, w_2, w_3', w_4', w_5')$ of $a^p b^{2p}$ such that $w_1 w_2^n w_3' w_4'^n w_5'$ is in L_4 for every $n \geq 0$, so that $w_4 \in b^+$. In particular $w_1 w_3' w_5'$ belongs to L_4. Let $\#_a$ be the number of a's in $w_1 w_3' w_5'$. Define $\#_b$ similarly, and let $j = |w_4'|$. Since we must have $\#_b \leq 2\#_a$ we must have $(2p - j) \leq 2(p - i)$. It follows that $j \geq 2i > i$. Hence $w_4 \neq w_4'$. But $w_4 w_4' = w_4' w_4 = b^{i+j}$, which is a violation of condition (3b) of theorem 4.4. Hence L_4 cannot be LL. ∎

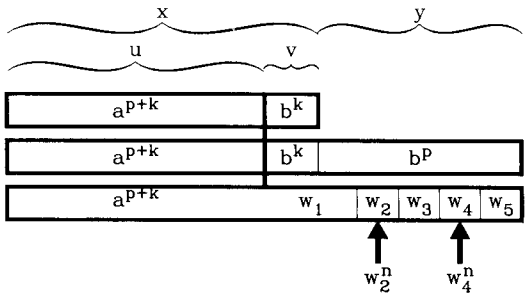


Fig. 18. An application of Theorem 4.6 to the language $a^m b^n$, $m \geq n \geq 0$. Because $a^{p+k}b^{p+k}$ is sufficiently longer than $a^{p+k}b^k$ a pumping must occur among the b's.

Theorem 5.6. The LR language $L_5 = \{ a^m b^n \mid m \geq n \geq 0 \}$ is not LL.

Proof: (Figure 18.) Suppose that L_5 is LL(k−1) for some k and let p be the constant obtained by applying the 2nd Iteration Theorem to L_5. Consider the two strings $a^{p+k}b^k$ and $a^{p+k}b^{p+k}$, and distinguish the final p b's in the latter string. According to the 2nd Iteration Theorem $a^{p+k}b^{p+k}$ has a factorization $(a^{p+k}w_1,w_2,w_3,w_4,w_5)$ such that

- $w_2 \neq \Lambda$
- $a^{p+k}w_1w_2^nw_3w_4^nw_5 \in L_4$ for every $n \geq 0$

From this we can deduce that $w_2w_4 \in b^+$ so that for a sufficiently large value of n we can obtain a string with more b's than a's – a string which cannot belong to L_5. ■

Note that it is possible to prove theorem 5.6 using the 1st Iteration Theorem and the technique applied in theorem 5.5.

Using L_5 we easily obtain the following result.

Theorem 5.7. The LL languages are not closed under right quotient with a regular set.

Proof: It is easy to see that the language a^nb^n is an LL language, and b^* is obviously a regular set. However

$$a^nb^n / b^* = \{ a^mb^n \mid m \geq n \geq 0 \}$$

is not an LL language, as we have just seen. ■

The 2nd Iteration Theorem is by its very nature not applicable to LL languages which are prefix-free. Thus theorem 4.6 could not be used to

prove any of theorems 5.2, 5.3 and 5.4. It is not known, however, whether there are languages which satisfy the 1st Iteration Theorem but which the 2nd Iteration Theorem can show are not LL, nor is it known whether one can always establish that a language fails to be LL *via* theorem 4.4 when that is the case.

L_1 and L_5 are from Rosenkrantz and Stearns [21]. L_2 is taken from van Leeuwen [14]. L_3 is taken from Bordier and Saya [7]. L_5 abstracts the fatal difficulty, insofar as LL(k) grammars are concerned, with the infamous dangling-ELSE introduced by the original Algol report [16] (and eliminated in the revised report [17]). Constructs such as

IF <bexp> THEN IF <bexp> THEN <stmt> ELSE <stmt>

in which the ELSE-clause might plausibly belong to either IF-THEN are allowed in PL/I [20] and Pascal [12]. The ambiguity is customarily resolved by associating an ELSE with the last previous unmatched THEN. It is claimed without proof by Aho, Johnson and Ullman [1] that such constructs are not LL; applying the argument of theorem 5.6 allows us to establish this rigorously. A direct proof such as ours is necessary since the family of LL languages is not closed under homomorphisms or gsm mappings [21].

Theorem 5.8. The dangling IF-THEN-ELSE construct does not appear in any LL language.

Since this construct is, however, easily handled by a recursive descent compiler operating without backup, it follows that the LL(k) languages are only a subset of the family of languages which can be compiled by this technique, and are therefore not a perfect model of this family.

Conclusions

Theorems 4.4 and 4.6 provide a powerful and reasonably general technique for establishing that languages are not LL(k) when that is the case. Previous results of this kind ([7], [14] and [21]) have generally been based on more complicated and less satisfying *ad hoc* arguments.

We leave open the question of whether satisfying the conditions of theorem 4.4 is sufficient to ensure that a language is LL(k), although we do not believe that to be the case. The task of characterizing a family

of languages by means of an iteration theorem appears, in general, to be a difficult one. Although a number of iteration theorems have been established for several language classes, in only one case is the result known to be sufficient as well as necessary.[22]

Finally, our arguments illustrate the advantages to be obtained from the careful analysis of derivation trees, various properties of which we have presented.

Acknowledgements

A stronger version of theorem 4.4 is presented here than was reported in [4], and the author is indebted to Bill Ogden, who also suggested the proof of theorem 5.5, for the improvement. Theorem 4.6 was inspired by an observation of Jan van Leeuwen's [14]. The suggestions and observations of Kellogg Booth, Kimberly King and especially Professor Michael Harrison are also keenly appreciated.

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