The decidability of v-local catenativity and of other properties of DOL systems*

by

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Abstract. A number of problems concerning DOL systems is shown to be decidable. Among them (i) Given an alphabet Σ, a homomorphism h on Σ^* and x,y in Σ^+, does there exist $n \geq 0$ so that $h^n(x) = h^n(y)$? (ii) Given Σ,h,x,y as in (i), does there exist $r,s \geq 0$ so that $h^s(x) = h^r(y)$? (iii) Given a DOL system G and an integer vector v, is G v-locally catenative? (iv) Given a DOL system G and $d \geq 1$, does there exist an integer vector $v = (v_1, \ldots, v_t)$, $t \geq 1$, $1 \leq v_i \leq d$ for $1 \leq i \leq t$, such that G is v-locally catenative?

Introduction. L-systems were introduced by A.Lindenmayer as mathematical models of cellular development of organisms [1] and have been studied extensively since then. For biologists the most important are simple deterministic sequence generators, specifically DOL systems and locally catenative formulas, see [3]. It was shown in [3] that every locally catenative formula can be simulated by a propagating DOL system but not vice versa. The question whether we can decide for a given (propagating) DOL system whether it is locally catenative, i.e. whether it can be simulated by a locally catenative formula was left open and still remains an open problem.

Very recently [2], the following four simpler problems have been shown decidable.

(1) Given an ε-free homomorphism h on Σ^* and words x,y in Σ^+, it is decidable whether there exists $n \geq 0$ such that $h^n(x) = h^n(y)$.

(2) Given h,x,y as in (1), it is decidable whether there exist $r,s \geq 0$ so that $h^r(x) = h^s(y)$.

(3) Given a DOL system G and a vector v, it is decidable whether G is v-locally catenative.

(4) Given a DOL system G and $d \geq 1$, it is decidable whether G is locally catenative with depth d, i.e. whether there exists a vector v with depth at most d so that G is v-localy catenative.

We will use the crucial result from [2], namely that for alphabet Σ, $|\Sigma| = m$, ε-free homomorphism h and $x,y \in \Sigma^*$, there exists $n \geq 0$ so that $h^n(x) = h^n(y)$ iff $h^m(x) = h^m(y)$. We strengthen this result to arbitrary homomorphisms. Based on this result we give simple proofs of the four decidability results above without using the simplification of DOL systems introduced in [2]. The first two results will be extended to arbitrary homomorphisms. We also show three additional biologically motivated problems to be decidable, two of them suggested by A. Lindenmayer.

1. DOL systems with identical or matching homomorphisms

Definition. A DOL system G is a triple $G = (\Sigma, h, u)$ where Σ is a finite alphabet, h is a homomorphism on Σ^* and u is in Σ^+. The sequence generated by G, denoted E(G), is the sequence $u, h(u), h^2(u), \ldots$. $L(G) = \{h^n(u) : n \geq 0\}$ is the language generated by G. For elementary notions and notations of formal language theory see e.g. [4]. We use $|x|$ to denote the length of x if x is a string, and the cardinality of x if x is a set.

The following result restricted for nonerasing homomorphism has been shown in [2]. We will base our proof on this important result.

Lemma 1. Let h be any homomorphism on Σ^*, $x,y \in \Sigma^*$ and $|\Sigma| = m$. If there exists n so that $h^n(x) = h^n(y)$ then $h^m(x) = h^m(y)$.

Proof. Let $\Delta = \{a \in \Sigma : h^k(a) = \varepsilon$ for some $k\}$, let $|\Sigma-\Delta| = r$. If $r = 0$, i.e. $\Delta = \Sigma$, then $h^m(w) = \varepsilon$ for each $w \in \Delta^*$, hence $h^m(x) = h^m(y)$. If $r \geq 1$, we consider restrictions of h,x and y to $\Sigma-\Delta$. Formally, let g be the homomorphism defined by $g(a) = a$ for $a \in \Sigma-\Delta$, $g(a) = \varepsilon$ for $a \in \Delta$, and let $h'(a) = g(h(a))$ for each $a \in \Sigma-\Delta$, $x' = g(x)$ and $y' = g(y)$. If $h^n(x) = h^n(y)$, then also $h'^n(x') = h'^n(y')$ and therefore by [2,Theorem 2] $h'^r(x) = h'^r(y)$. Since each symbol in Δ is erased in at most m-r steps we have for each $k \geq 0$ $h^{m-r}(h'^k(x')) = h^{m-r+k}(x)$. Specifically, $h^m(x) = h^{m-r}(h'^r(x')) = h^{m-r}(h'^r(y')) = h^m(y)$. □

Corollary 1. Let $|\Sigma| = m$, h be a homomorphism on Σ^*, and $x,y \in \Sigma^*$. There exists $n \geq 0$ such that $h^n(x) = h^n(y)$ (and therefore $h^k(x) = h^k(y)$ for all $k \geq n$) iff $h^m(x) = h^m(y)$.

Note. The result for propagating homomorphism in [2] is shown for m-1 rather than m. This stronger version does not hold for the case when all symbols can be eventualy erased ($\Sigma = \Delta$) as shown by the following example. Let $\Sigma = \{a,b\}$, $h(a) = b$, $h(b) = \varepsilon$, $x = a$, $y = b$. We have $m-1 = 1$ and $h(x) = b$, $h(y) = \varepsilon$, but $h^k(x) = h^k(y)$ for all $k \geq 2$.

Theorem 1. Given a homomorphism h on Σ^* and two strings x,y is in Σ^+ it is decidable whether there exists $k \geq 0$ such that $h^k(x) = h^k(y)$.

Proof. By Corollary 1 such k exists iff $h^m(x) = h^m(y)$ where $m = |\Sigma|$.

□

<u>Theorem 2</u>. Given a homomorphism h on Σ^* and x,y is in Σ^+ it is decidable whether there exist $r,s\geq 0$ so that $h^r(x) = h^s(y)$.

<u>Proof</u>. Without loss of generality we may assume $r\geq s\geq m$ where $m = |\Sigma|$. By Corollary 1 $h^r(x) = h^s(y)$ iff $h^{r-s+m}(x) = h^m(y)$, i.e. iff $h^m(y) \in L(G)$, where G is the DOL system $G = (\Sigma,h,h^m(x))$. Clearly, decidable.

□

<u>Definition</u>. Two DOL systems $G_i = (\Sigma_i,h_i,x_i)$ are called ultimately equivalent if there exists $n\geq 0$ such that $h_1^k(x_1) = h_2^k(x_2)$ for all $k\geq n$.

The DOL equivalence problem has recently been shown decidable [5], however the ultimate equivalence problem for general DOL systems remains still open. We will show its decidability for a restricted case of matching homomorphisms.

<u>Definition</u>. Homomorphisms h_1 on Σ_1^* and h_2 on Σ_2^* are called <u>matching</u> if $h_1(a) = h_2(a)$ for each $a \in \Sigma_1 \cap \Sigma_2$.

<u>Theorem 3</u>. Let $G_i = (\Sigma_i,h_i,u_i)$ for i = 1,2 be two DOL systems with matching homomorphisms. It is decidable whether G_1 and G_2 are ultimately equivalent.

<u>Proof</u>. Let $\Delta = \Sigma_1 \cap \Sigma_2$. Clearly, if G_1 and G_2 are ultimately equivalent, then symbols in $\Sigma_i-\Delta$ must occur only in finitely many words of $L(G_i)$ for i = 1,2. Clearly, we can test whether this is the case and if so effectively find the smallest r such that $h_i^k(u_i) \in \Delta^*$ for all $k\geq r$, i = 1,2. Let $x_i = h_i^r(u_i)$ for i = 1,2 and $m = |\Delta|$. By Corollary 1, G_1 and G_2 are ultimately equivalent iff $h_1^{r+m}(u_1) = h_1^m(x_1) = h_2^m(x_2) = h_2^{r+m}(u_2)$ which is decidable.

□

Now, we give a solution to two biologically motivated problems suggested by A.Lindenmayer.

Theorem 4. Let h_1 on Σ_1^* and h_2 on Σ_2^* be two matching homomorphisms, and $x \in \Sigma_1^+$. We can effectively find the set $F = \{y \in \Sigma_2^+ :$ DOL systems (Σ_1,h_1,x) and (Σ_2,h_2,y) are ultimately equivalent$\}$. If h_1 and h_2 are ε-free then F is finite, otherwise regular.

Proof. Let $\Delta = \Sigma_1 \cap \Sigma_2$, and $\Pi = \{a \in \Sigma_2 : h_2^k(a) \in \Delta^*$ for all $k \geq s_a$, for some $s_a\}$. Clearly, Π can be effectively found as well as the smallest $s \geq 0$ such that $h_2^k(a) \in \Delta^*$ for each $a \in \Pi$ and all $k \geq s$. We can also find the smallest $t \geq 0$ so that $h_1^k(x) \in \Delta^*$ for all $k \geq t$. Let $r = \max(s,t)$. By definition of Π, we have $F \subseteq \Pi^*$, therefore by choise of r $h_2^r(F) \subseteq \Delta^*$. Also $h_1^r(x) \in \Delta^*$, therefore by Corollary 1 for $w \in h_2^r(F)$ there exists k so that $h_2^k(w) = h_1^{k+r}(x)$ iff $h_2^m(w) = h_1^{r+m}(x)$ where $m = |\Delta|$. Thus $h_2^r(F) = h_2^{-m}(h_1^{r+m}\{x\})$ and finally $F = h_2^{-(r+m)}(h_1^{r+m}(\{x\}))$. The assumption $F \in \Pi^*$ is satisfied since $h_1^{r+m}(x) \in \Delta^*$, $\Delta \subseteq \Pi$ because $h_2(\Delta^*) \subseteq \Delta^*$, and $h_2^{-1}(\Pi^*) \subseteq \Pi^*$ because of defition of Π . The last statement follows from the closure of regular sets under inverse homomorphisms, and finite sets under inverse ε-free homomorphisms.

□

The extension of DOL systems allowing finite number of starting strings is called FDOL systems. A FDOL system is a triple $G = (\Sigma,h,S)$ where Σ is an alphabet, h is a homomorphism on Σ^* and $S \subseteq \Sigma^+$, S finite. $L(G) = \{h^k(w) : w \in S$ and $k \geq 0\}$.

Theorem 5. Let $G_i = (\Sigma_i,h_i,S_i)$ for $i = 1,2$ be two FDOL systems with matching homomorphisms. Then it is decidable whether $L(G_1) \cap L(G_2) = \emptyset$.

Proof. Let $\Delta = \Sigma_1 \cap \Sigma_2$. Note that $h_i(a) \in \Delta^*$ for each $a \in \Delta$ and $i = 1,2$. Let $S_i' = \{w \in \Delta^* : w = h_i^r(v)$ for some $v \in S_i$, $r \geq 0$ such that $h_i^k \notin \Delta^*$ for all $0 \leq k < r\}$, S_i' is finite for $i = 1,2$. Clearly, $L(G_1) \cap L(G_2) \neq \phi$ iff there exist $x_1 \in S_1'$, $x_2 \in S_2'$ and $m,n \geq 0$ so that $h_1^m(x_1) = h_2^n(x_2)$. There are only finitely many $(|S_1'|\cdot|S_2'|)$ pairs x_1, x_2 to be checked and for each such pair the existence of m,n satisfying $h_1^m(x_1) = h_2^n(x_2)$ is decidable by Theorem 2.

□

2. Decidability of v-local catenativity for DOL systems

Definition. Let $v = (v_1,\ldots,v_t)$, where $t \geqq 2$ and $v_i \geqq 1$ for $1 \leqq i \leqq t$. Let $d = \max(v_1,\ldots,v_t)$. Let $G = (\Sigma,h,\sigma)$ be a DOL system, $E(G) = s_0,s_1,\ldots$. Let $c \geqq 0$. We say that G is v-locally catenative with cut c if $s_k =$ $= s_{k-v_1} s_{k-v_2} \cdots s_{k-v_t}$ for all $k \geqq d+c$.

We say that G is v-locally catenative if there exists c so that G is v-locally catenative with cut c. We say that G is locally catenative with depth d if there exists $v = (v_1,\ldots,v_t)$ with $\max(v_1,\ldots,v_t) = d$ so that G is v-locally catenative.

Theorem 6. It is recursively decidable whether a given DOL system G = $= (\Sigma,h,\sigma)$ is v-locally catenative for given $v = (v_1,\ldots,v_s)$.

Proof. Let $r = \max(v_1,\ldots,v_s)$. Let $\sigma_1 = h^r(\sigma)$ and $\sigma_2 = h^{r-v_1}(\sigma)h^{r-v_2}(\sigma)\ldots$ $\ldots h^{r-v_s}(\sigma)$. Let $s_k = h^k(\sigma)$ for $k \geqq 0$. Clearly, $s_k = h^{k-r}(\sigma_1)$ and $s_{k-v_1} s_{k-v_2} \cdots s_{k-v_s} = h^{k-r}(\sigma_2)$ for $k \geqq r$. Thus for $n \geqq r$ $s_k = s_{k-v_1} s_{k-v_2} \cdots$ $\cdots s_{k-v_s}$ for all $k \geqq n$ iff $h^n(\sigma_1) = h^n(\sigma_2)$. Whether such an n exists is decidable by Theorem 1, actually $h^n(\sigma_1) = h^n(\sigma_2)$ iff $h^m(\sigma_1) = h^m(\sigma_2)$ for $m = |\Sigma|$.

Theorem 7. Given a DOL system $G = (\Sigma, h, \sigma)$ and $d \geq 1$, it is decidable whether G is locally catenative with depth smaller than or equal to d.

Proof. In view of the proof of Theorem 6 we see that G is v-locally catenative iff it is v-locally catenative with cut $m = |\Sigma|$, i.e. iff $G' = (\Sigma, h, h^m(\sigma))$ is v-locally catenative with cut 0. Let $E(G') = s'_0, s'_1, \ldots$. Therefore, G' is locally catenative with depth $\leq d$ iff s'_d can be expressed as a concatenation of at least two strings from $S_{d-1} = \{s'_0, \ldots, s'_{d-1}\}$, possibly with repetitions, i.e. iff s'_d is in $S_{d-1}S^+_{d-1}$. Hence, we need to check the membership of s_d in a regular set which is decidable.

□

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