DECIDABILITY OF SEQUENCE EQUIVALENCE
FOR QUASI-SIMPLE DOL-SYSTEMS AND ALL
SYSTEMS OVER TWO LETTERS*

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THE DECIDABILITY OF SEQUENCE EQUIVALENCE FOR QUASI-SIMPLE DOL-SYSTEMS AND ALL SYSTEMS OVER TWO LETTERS*

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Abstract. The notion of a quasi-simple pair of DOL-systems is introduced. It is shown that for a quasi-simple pair of DOL-systems the sequence equivalence is decidable. As an application it is shown that the sequence equivalence problem is decidable for (all) DOL systems over two-letter alphabet.

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Introduction

This paper is a sequel of [1]. We assume that the reader is familiar with [1] and we will use its terminology and notation.

In the first section we will sharpen the linear algebraic part (first half of section 3) of [1]. The proofs are somewhat simplified, worked out in greater detail and the unnecessary assumptions are dropped. This will allow us to relax the sufficient conditions on a pair of equivalent DOL-systems to have bounded balance, namely, we will require that at least one member of the pair, rather than both, is so called primitive. Further, we note that the proof of Theorem 3.2 from [1] goes through unchanged using these weaker assumptions. Therefore, we can apply the Theorem 2.1 from [1] and we have an algorithm to decide whether two DOL-systems are sequence equivalent if at least one of them is primitive.

The case when neither system of a pair is primitive can in some instances be reduced to the case when one is. This is done by suitable composition of the homomorphisms. The pair of DOL-systems for which this is possible is called quasi-simple. Each pair of simple DOL systems from [1] is quasi-simple and so we have extended the family of pairs of DOL-systems for which we can decide sequence equivalence and therefore also language equivalence using the result from [2].

Finally, as an application of our main result we show that the sequence equivalence problem is decidable for (all) DOL-systems over two-letter alphabet. This is done by considering separately a few singular cases which are not quasi-simple.
1. Quasi-simple pairs of DOL-systems

Let $G = \langle \Sigma, h, \sigma \rangle$ be a DOL-system.

**Definition 1** We call $G$ irreducible if for each $a, b \in \Sigma$, $a \neq b$ there is

$1 \geq 1$ such that $h^i(a) = ubv$ for some $u, v \in \Sigma^*$. More precisely

denoting $m(a) = \bigcup_{i=0}^{\infty} \min(h^i(a))$ the condition for irreducibility is

$b \in m(a)$ for all $a, b \in \Sigma$, i.e., $m(a) = \Sigma$ for all $a \in \Sigma$.

**Definition 2** A real $n \times n$ matrix $M$ with non-negative elements $m_{ij} \geq 0$ is called

irreducible if there is no permutation of its rows and the same permutation

of its columns which puts $M$ into the form

$$M = \begin{pmatrix} N & O \\ P & Q \end{pmatrix},$$

where $N, Q$ are square matrices, and $O$ denotes a zero-matrix, possibly

rectangular. Equivalently, $M = (m_{ij})$ is irreducible if its indexes cannot be split

into two disjoint non-empty sets $\Sigma_1, \Sigma_2$ such that $a \in \Sigma_1$, $b \in \Sigma_2 \Rightarrow m_{ab} = 0$.

**Lemma 1** A DOL-system $G$ is irreducible iff its growth matrix is irreducible.

**Proof** Let $M = (m_{a,b})_{a,b \in \Sigma}$ be the growth matrix of $G$. Then $m_{ab} = 0$

iff $b \notin \min(h(a))$ by the definition of $M$. Let $\Sigma_1 \subset \Sigma$, then $m_{a,b} = 0$

for all $a \in \Sigma_1$, $b \notin \Sigma_1$ iff for all $a \in \Sigma_1 \min(h(a)) \subset \Sigma_1$, i.e. iff

$$h(\Sigma_1) \subset \Sigma_1.$$

Thus $M$ is irreducible if there is $\Sigma_1 \subset \Sigma$ such that (1), and such that

$\Sigma_1, \Sigma - \Sigma_1$ are both non-empty. Conversely if $a \in \Sigma_1$, $b \notin \Sigma_1$, then (1)

implies $b \notin m(a)$.

If a matrix $M$ is irreducible some of its powers might not be.
Definition 3 An $n \times n$ matrix $M$ is called primitive if all the matrices $M^i$, $i = 1, \ldots, n$ are irreducible.

If a matrix is primitive then all its powers are irreducible. Note that a matrix is primitive iff some of its powers are positive, i.e. using the terminology of [1] system $G$ is simple iff its growth matrix is primitive. Also note that in fact it is not necessary to examine all powers $M, M^2, \ldots, M^n$ but only $M^k$ for $k$ divides $n$.

Let $G_i = \langle \Sigma, h_i, \sigma \rangle$ be two DOL-systems. We define balance $\beta$ for $w \in \Sigma^*$ as in [1].:

\begin{align*}
(2) \quad \beta(w) = |h_1(w)| - |h_2(w)|.
\end{align*}

We will use the same symbol $\beta$ to denote the balance defined on vectors, if $[w]$ is the Parikh vector of $w$ we put $\beta([w]) = \beta(w)$. Let $M_1, M_2$ be the growth matrices of $G_1, G_2$. Obviously

\begin{align*}
(3) \quad \beta(x) = |xM_1 - xM_2| \quad \text{for a vector } x,
\end{align*}

where $|x| = \Sigma |x_i|$ (or simply $\Sigma x_i$ - as all $x_i \geq 0$).

This shows that $\beta(x)$ is well defined.

Lemma 2 Suppose that $G_1, G_2$ are two DOL-systems which are Parikh equivalent, and suppose that the growth matrix of $G_1$ is primitive. Then for every $\varepsilon > 0$ there is an $n_0 > 0$ such that for every $w \in \Sigma^*$

\begin{align*}
(4) \quad \beta(h_1^n(w)) \leq \varepsilon |h_1^n(w)| \quad \text{for all } n \geq n_0.
\end{align*}
Proof Translated into the language of linear algebra (4) becomes

\[ \beta(xM_1^n) \leq e|xM_1^n| \text{ for all vectors } x = (x_1, \ldots, x_n), \text{ such that } x_i \geq 0. \]

We will prove (5) for all non-negative vectors, not only those with \( x_i \) integers.

From the primitivity of \( M_1 \) it follows that \( M_1 \) has a positive characteristic value \( r \) such that \( r > \|r_1\| \) for all other characteristic values \( r_i \). Moreover, there is a single characteristic vector \( u \) belonging to \( r \), and \( u > 0 \) (i.e. \( u = (u_1, \ldots, u_n) \), \( u_i > 0 \) for all \( 1 \leq i \leq n \)). The same applies to the transposed matrix \( M_1^T \). Denote the single characteristic vector of \( M_1^T \) belonging to \( r \) by \( v \), thus \( v > 0 \).

Finally we will use \( (x,y) \) for the scalar product \( \sum x_iy_i \). As \( u,v > 0 \) \( (u,v) > 0 \).

Normalize them such that \( (u,v) = 1 \). If \( x \) is any vector and we write

\[ x = (x,v)u + w_0, \]

then \( (w_0,v) = 0 \); moreover, if \( x \geq 0 \) and \( x \neq 0 \) then \( (x,v) \neq 0 \). Multiplying (6) by \( M_1 \) we get \( xM_1 = (x,v)uM_1 + w_0M_1 = (x,v)r_1u + w_1 \), where \( w_1 = w_0M_1 \) and \( (w_1,v) = (w_0M_1,v) = (w_0,vM_1^T) = r(w_0,v) = 0 \). By induction

\[ xM_1^n = ar^n u + w_n, \text{ and } w_n \in W \]

where the space \( W = \{ w : (w,v) = 0 \} \) is invariant with respect to \( M_1 \), and \( a = (x,v) \neq 0 \).

Let us denote \( r_1 \) the modulus of largest characteristic number of \( M_1 \) on \( W \). Because \( u \notin W \) we have \( r > r_1 \). Let \( r_0 \) be such that \( r > r_0 > r_1 \), then
(8) \[ |w_n| = |w_0^{\infty}| < r_0^n |w_0| . \]

From this it follows that \( \frac{1}{a} \lim_{n \to \infty} \frac{xM_1^n}{r^n} = u \) for all \( x \neq 0, x \neq 0 \).

If we start in particular with \( s = [\emptyset] \), the Parikh vector of the initial string we see that

\[ u \in \{ s, sM_1, sM_2^2, \ldots \} = \{ s, sM_2, sM_2^2, \ldots \} \]

where the equality of sets follows from the assumption that \( G_1, G_2 \) are Parikh equivalent. But this gives \( uM_1 = uM_2 \), i.e. \( \beta(u) = 0 \).

Applying this to (7) and using (8) we get

\[ \beta(xM_1^n) = \beta(w_n) \leq \beta r_0^n |w_0| \]

where \( \beta = \max_1 \beta(\xi_i) \) for the unit vectors \( \xi_1, \xi_2, \ldots, \xi_n \). On the other hand,

\[ |xM_1^n| = |a r^n u + w_n| \geq a |r^n| |u| - r_0^n |w_0| \geq \frac{|(x,v)|}{2} r^n |u| \]

if \( n \) is large enough. Thus

\[ \frac{\beta(xM_1^n)}{|xM_1^n|} \leq \frac{r_0^n}{r^n} \max_1 \frac{|w_0|}{|(x,v)|} . \]

Here \( \frac{|w_0|}{|(x,v)|} \) is invariant with respect to the transformation \( x \to \alpha x \) and \( |(x,v)| \neq 0 \), so it is bounded by a constant. This proves (5).

**Lemma 3** Under the same assumptions as in Lemma 1, for every \( \varepsilon > 0 \) there is \( \lambda \) such that for every \( G_1 \)-prefix \( w \)

(9) \[ |w| > \lambda \Rightarrow \beta(w) \leq \varepsilon |w| . \]
Proof Using Lemma 2, given $\frac{\epsilon}{2}$ we find $n_0$. Let $w$ be any $G_1$-prefix, i.e. $h_1^n(\sigma) = wx$ for suitable $n,x$. If

\begin{equation}
 n \geq n_0
\end{equation}

then denote $u = h_1^{n-n_0}(\sigma)$, and let $u = u_1au_2$, $a \in \Sigma$ be such that $h_1^{n_0}(u_1)$ is a prefix of $w$ but $w$ is a proper prefix of $h_1^{n_0}(u_1a)$, i.e.

\[
w = h_1^{n_0}(u_1)x_1, h_1^{n_0}(u_1a) = wx_2, x_1, x_2 \in \Sigma^*.
\]

Now

\[
\beta(w) \leq \beta(h_1^{n_0}(u_1))\beta(x_1) \leq \frac{\epsilon}{2}|h_1^{n_0}(u_1)| + \beta(h_1^{n_0}(a)) \leq \frac{\epsilon}{2}|w| + \beta q_0
\]

where $q = \max \sum \Sigma m_{ij}$ is the length of the longest $h_1(\sigma), \sigma \in \Sigma$. To prove (9) it is sufficient to take $q \frac{\beta}{|w|} \leq \frac{\epsilon}{2}$, i.e. to take

\[\lambda > q_0 \max\left(\frac{2}{\epsilon} \beta, 1\right)\]

the second case in max-function guarantees (10).

**Lemma 4** Let again the assumptions be as in Lemma 1 then the balance is bounded for all $G_1$-prefixes.

**Proof** Theorem 3.2 of [1] proves this under the stronger assumption, namely that both $G_1, G_2$ are simple. However, only the following assumptions were actually used in the proof

a) $h_1^n(x)$ is exponentially growing for each $x \in \Sigma$

b) (9) is true for sufficiently long strings.

Both assumptions are true, so the proof goes through.
The following theorem has also been proved in [1].

**Theorem** If the pair of DOL-systems \( G_1, G_2 \) has bounded balance, then the semialgorithms for equivalence of \( G_1 \) and \( G_2 \) terminate.

From this it follows that the equivalence problem is decidable for such pairs \( G_1, G_2 \) in which at least one of the systems is primitive. This result can be somewhat strengthened as follows.

**Definition 4** Let \( G_1, G_2 \) be a pair of DOL-systems, \( G_i = \langle \Sigma, h_i, \sigma \rangle \). The pair is called *quasi-simple* if there exists

\[
(11) \quad h = h_{i_n}^{i_{n-1}} \ldots h_{i_2}^{i_1}, \text{ where } i_j \in \{1,2\} \text{ for } j = 1, \ldots, n
\]

such that \( G = \langle \Sigma, h, \sigma \rangle \) is simple.

**Theorem** The equivalence problem is decidable on the class of quasi-simple pairs of DOL-systems.

**Proof** We use the standard method to show that equivalence for the original pair can be established by establishing

(i) the equivalence for \( n \) pairs

\[
G_1^{(i)} = \langle \Sigma, h, \sigma^{(i)} \rangle, \quad G_2^{(i)} = \langle \Sigma, \tilde{h}, \sigma^{(i)} \rangle
\]

where \( \sigma^{(i)} = h_i^{i_1}(\sigma), i = 0,1,\ldots,n-1 \), and if \( h \) was given by (11) then

\[
\tilde{h} = h_{3-i_n}^{i_{n-1}} \ldots h_1;
\]

and \( j(i), \quad h_{i_j}^{j}(\sigma) = h_{i_j}^{j}(\sigma) \) for \( j = 1,2,\ldots,n-1 \).

Obviously, if \( G_1 \) is equivalent to \( G_2 \) then all \( G_1^{(i)} \) are equivalent to the corresponding \( G_2^{(i)} \). To show the converse assume that \( G_1 \) is not equivalent to \( G_2 \), so that there is \( j \) such that \( (h_{i_1}^{j}(\sigma) \neq h_{i_1}^{j}(\sigma)) \) and assume that our \( j \) is the smallest possible. In order to simplify the notation assume \( i_n = 1 \). Every \( \lambda \) can be written as
\[ \lambda = n \times \lambda + r \lambda \quad \text{with} \quad 0 \leq r < n \]

and we have \( h^\lambda_{\sigma}(r^\lambda_j) = h^\lambda_{\sigma}(r) \) for all \( \lambda \leq j \), similarly

\[ h^\lambda_{\sigma}(r^\lambda_j) = h^\lambda_{\sigma}(r^\lambda_j) \quad \text{for} \quad \lambda \leq j. \]

This shows that \( G_1^{(i)} \) is not equivalent to \( G_2^{(i)} \).

At the same time the systems \( G_1^{(i)} \) are irreducible so the problem for them is decidable.

Note that it is decidable whether a given pair of DOL-systems is quasi-simple. This is so because in the first place this depends only on the growths matrices of both systems and secondly not on actual elements of the matrices but only on whether the element is or is not zero.

Thus we may start with given growth matrices \( M_1, M_2 \). On each step take all the matrices from previous steps multiplied (on the right) by \( M_1 \) or \( M_2 \) considering two matrices different only if one has zero element where the second has a non-zero one (thus we consider in fact the matrices over a boolean semi-ring \( \{0,1\} \)). As there is only \( 2^{n^2} \) of such matrices we must either reach a matrix with all elements positive or reach a step when all matrices taken on this step were already considered. In the former case the pair is quasi-simple, in the latter it is not.

2. The case of two letter alphabet

**Theorem** Equivalence problem is decidable for DOL-systems with \( \#\Sigma = 2 \).

**Proof** It is sufficient to consider the case when neither of the two systems is primitive. For 2 symbols the system \( G \) is reducible if - after possible permutation - its growth matrix is of the form
\[ M_1 = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}. \]

we may assume \( c \neq 0 \), otherwise \( h(w) \) is a power of one letter only, and this case need not be considered (see [2]).

Let us consider separately the following three cases:

(i) \( a = 0 \). Let us write \( \Sigma = \{ \sigma_1, \sigma_2 \} \), \( h_1(\sigma_1) = \epsilon \), \( h_1(\sigma_2) = v \), and let \( (p, q) \) be the Parikh vector of the axiom \( w \). The sequence generated by \( G_1 \) is obviously \( L = (w, vq, vqc, vqc^2, \ldots) \).

Let \( G_2 = \langle \Sigma, g_2, w \rangle \) be a system generating the same sequence, i.e. in particular \( h_2(vq) = vqc \), but this implies \( h_2(v) = v^c \), and from this it follows that \( h_2(vqc^i) = vqc^{i+1} \). Thus to test whether \( G_1 \) eq \( G_2 \) it is enough to test whether \( h_1(w) = h_2(w) \) and \( h_1(v) = h_2(v) \).

(ii) \( b = 0 \). Let us write \( w = \sigma_1^{i_0} \sigma_2^{j_1} \sigma_1^{i_1} \cdots \sigma_2^{j_k} \sigma_1^{i_k} \), \( k \geq 0 \), where \( i_0, i_k \geq 0 \), \( i_s, j_s > 0 \), for the rest. In this case,

\[ L = (\ldots, \sigma_1^{i_0}, \sigma_2^{j_1}, \sigma_1^{i_1}, \cdots, \sigma_2^{j_k}, \sigma_1^{i_k}, \ldots | \ell = 0, 1, \ldots) \]

in particular, the number of "runs" of each symbol is constant, and both \( a \) and \( c \) are determined uniquely. Thus such a system is equivalent only with itself if both \( a, c > 1 \). The case \( a = 1 \) or \( c = 1 \) can give an additional equivalence of

\[ \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \text{with} \begin{pmatrix} 0 & 0 \\ 1 & b-1 \end{pmatrix} \]

which can easily be tested.
(iii) So we may now assume \( a \neq 0, b \neq 0, c \neq 0 \). There are two possibilities for \( G_2 \) which is also reducible.

a) \[
M_2 = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}.
\]
As before, we may assume that all elements \( d, e, f \) are \( \neq 0 \).

In this case \( M_1 M_2 \) is irreducible.

b) \[
M_2 = \begin{pmatrix} d & 0 \\ e & f \end{pmatrix}.
\]
From the Parikh equivalence we get \( f = c \).

We will show that

\[
(1) \quad a > 1 \Rightarrow a = d.
\]

From this, by symmetry we get \( d > 1 \Rightarrow a = d \), and as the cases \( a = 0 \) or \( d = 0 \) were already covered under (i), we may ignore them. On the other hand, \( a = d \) implies \( M_1 = M_2 \). Obviously, if the homomorphisms are same on one letter, they are same on the other one.

In order to prove (1) let us write \( \Sigma = \{ \sigma_1, \sigma_2 \} \),

\[
h_1(\sigma_1) = \sigma_1^a \\
h_2(\sigma_2) = \sigma_1^g \psi_1^h; \quad g, h \geq 0 \text{ and } \psi \text{ both starts and ends with } \sigma_2.
\]

[Note that we assume \( \sigma_2 \in h_2(\sigma_2) \) as \( c \neq 0 \).]

We will investigate substrings of \( w_k = h_1^k(w) \) consisting of \( \sigma_1 \)'s only - let us call such a substring *maximal* if it cannot be lengthened and still contains \( \sigma_1 \)'s only. In more details if \( w_k = xyz, y \in \{ \sigma_2 \}^+ \) and either

- \( x \) is empty - i.e. \( y \) is a prefix or \( x \) ends with \( \sigma_2 \) and at the same time
- \( z \) is empty - i.e. \( y \) is a suffix or \( z \) starts with \( \sigma_2 \).
Now if \( w_k \) contains a maximal string of the length \( \lambda \), then \( w_{k+1}, w_{k+2}, w_{k+3}, \ldots \) will contain a maximal string of the length \( a^\lambda + i, a^{2^\lambda + i}(a+1), a^{3^\lambda + i}(a^2+a+1), \ldots \)
where \( i = g \) if our string was a prefix, \( i = h \) for suffix, or \( i = g + h \) if the string was neither suffix nor prefix.

In order to see that such a sequence of string lengths determines a uniquely it is sufficient to consider the longest maximal string in \( w_k \), and take \( k \) large enough so that the length is larger than \( h_2(\sigma_2) \). This is possible if \( a > 1 \).

Finally, there is a case of \( M \) irreducible, but \( M_1^2 \) reducible, in this case \( M_1 = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \) and this case can be reduced to the case (ii) by considering \( M_1^2 \) instead of \( M_1 \). See [2] for more details.
References

