WEIGHTED GROWTH FUNCTIONS OF DOL-SYSTEMS & GROWTH FUNCTIONS OF PARALLEL GRAPH REWRITING SYSTEMS

by

Karel Culik II

Research Report CS-74-24

Department of Computer Science

University of Waterloo
Waterloo, Ontario, Canada

December, 1974

* This work has been supported by the National Research Council of Canada, Grant No. A7403.

ABSTRACT

The weighted growth functions of DOL systems are introduced and studied with the main goal to characterise growth functions of deterministic propagating graph OL-systems. For graph systems either the number of nodes, or number of edges, or their sum can be considered as criterion of growth. It is shown that in the first and the last case we get the same growth functions as for string systems. In the case that the number of edges only is considered, the growth functions are exactly the same as the weighted growth functions of the corresponding string systems.

1. Introduction

In [1] mathematical models for development of simple filamentous (one-dimensional) multi-cellular organisms were introduced. These models have been intensively investigated in recent years, in particular, the growth functions of the DOL-systems were studied in [2,3].

Recently, a general model for the development of multi-dimensional organisms was given in [4], where organisms are represented as graphs and parallel rewriting on graphs is studied. The main goal of this paper is to study the growth properties of the deterministic propagating graph rewriting systems from [4] (DPGOL-systems). In the case of a string rewriting system we are observing the lengths of strings in the sequence generated by the system. In the case of graph-rewriting we can consider either the number of nodes, or the number of edges or the sum of both of them, as a criterion of growth. For biological considerations the number of nodes is of essential interest, since nodes represent cells in these models. As for edges, the average number of edges per node would also be interesting.

We will show that if we consider the number of nodes only, or the total number of both nodes and edges, then we get exactly the same growth functions as for the corresponding string systems. However, if we consider the number of edges only, then we can get more complicated type of growth. To characterise these new growth functions we will introduce "weighted growth" of string systems. Intuitively, the weighted growth means that we consider cells of various types to be of various "sizes" including possibly "zero size". Several results concerning various types of weighted growth are shown in section 3, all of them only for integer nonnegative weights. We will see that it makes an important difference whether or not the weight zero is allowed. The weighted growth equivalence is shown to be decidable for DOL-systems. We also note that instead of weighted growth of DOL-sequences we can equivalently consider non-weighted growth of homomorphic images of DOL-sequences

Finally, we will show that the edge-growth functions of deterministic propagating graph OL-systems are exactly the same as the weighted growth functions of DOL-systems or DPOL-systems.

2. Preliminaries

Given an alphabet Σ, Σ^* is the set of all strings over Σ, $\Sigma^+ = \Sigma^* - \{\varepsilon\}$, ε being the empty string. The length of w in Σ is denoted by $|w|$, in particular $|\varepsilon| = 0$.

A DOL-system G is a triple (Σ, P, σ) where:

Σ is a finite set, the alphabet of G.

P is a finite subset of $\Sigma \times \Sigma^*$, such that for each a in Σ there is exactly one w so that $(a,w) \in P$. Elements of P are called productions and are written in the form $a \mapsto w$.

σ in Σ^+ is the axiom.

The set of productions specifies the homomorphisms h_G on Σ^* defined by $h_G(a) = w$ if $a \mapsto w$ is in P.

The sequence generated by G is denoted s(G) and defined to be the sequence of strings $x_0, x_1, x_2, \ldots$ where $x_0 = \sigma$ and $x_{k+1} = h_G(x_k)$ for $k \geq 0$.

If $P \subseteq \Sigma \times \Sigma^+$, then G is called a propagating DOL-system (PDOL-system).

The family of all sequences generated by DOL-systems (PDOL-systems) is denoted by DOL (PDOL).

In [4] a model of the development of multidimensional organisms is introduced. Organisms are represented by directed labeled graphs, so from mathematical point of view parallel rewriting systems on graphs are considered.

Very roughly speaking, a deterministic propagating graph OL-system (DPGOL-system) G generates a sequence of graphs $s(G) = X_0, X_1, X_2, \ldots$ in the following way. The graph X_0 is the axiom of G. For each $k \geq 0$, X_{k+1} is obtained from X_k by first replacing every node of X_k (uniquely) by a graph, according to "productions" of G; and then connecting the new graphs by additional edges into one graph X_{k+1}, again uniquely, according to "connection rules". It is not surprising that graph generating systems are much more complicated than corresponding string systems and we do not have the space here to give even an informal but reasonably rigorous definition of DPGOL-systems. We have to refer the reader to [4] for definitions and notation concerning this subject. We only need to add the following.

The sequence (of abstract e-graphs) generated by a DPGOL-system $G = (\Sigma,\Delta,P,C,S)$ is denoted by s(G) and defined to be the sequence $A_0,A_1,A_2,\ldots$ where $A_0 = S$ and $A_k \underset{G}{\Rightarrow} A_{k+1}$ for all $k \geq 0$. The family of all sequences generated by DPGOL-systems is denoted by DPGOL.

3. Weighted growth functions of DOL-systems

Let Σ be a finite alphabet and let s be an infinite sequence of strings over Σ, i.e. $s = x_0, x_1, \ldots$ where $x_i \in \Sigma^*$ for $i \geq 0$. Let N be the set of natural numbers, $N = \{0,1,\ldots\}$. The function $f_s : N \mapsto N$, defined by $f_s(i) = |x_i|$ for $i \geq 0$, is called the growth function of s.

A function $\rho : \Sigma \mapsto N$ is called a weighting function on Σ. For w in Σ^* and $a \in \Sigma$ let $n_a(w)$ be the number of occurrencies of a in w.

The ρ-weighted growth function of s, denoted by f_s^ρ, is defined by $f_s^\rho(i) = \sum_{a \in \Sigma} n_a(x_i)\rho(a)$ for all i in N.

For a family of sequences F, $p \in N$, and $q \in N \cup \{\infty\}$ let

$$F(F) = \{f_s : s \in F\}$$

$$W_p^q(F) = \{f_s^\rho : s \text{ over } \Sigma \text{ is in } F \text{ and } \rho : \Sigma \mapsto \{k \in N : p \leq k \leq q\}\}.$$

Clearly, $W_1^1(F) = F(F)$.

Lemma 1 $W_0^\infty(\text{DOL}) = W_0^1(\text{PDOL})$.

Proof. Inclusion $W_0^1(\text{PDOL}) \subseteq W_0^\infty(\text{DOL})$ is trivial.

If f is in $W_0^\infty(\text{DOL})$, then there exists a DOL-system G, $G = (\Sigma, P, \sigma)$, and a function $\rho : \Sigma \mapsto N$ so that $f = f_{s(G)}^\rho$. Let c,d be two new symbols not in Σ. Let homomorphism h on Σ is defined by $h(a) = ad^{\pi(a)}$ where $\pi(a) = 0$ if $\rho(a) \leq 1$ and $\pi(a) = \rho(a)-1$ otherwise.

We construct PDOL-system $G' = (\Sigma', P', \sigma')$ where

i) $\Sigma' = \Sigma \cup \{c, d\}$,

ii) $\sigma' = h(\sigma)$,

iii) $P' = \{a \mapsto h(w) : a \mapsto w \in P,\ w \neq \varepsilon\} \cup$

$\cup \{a \mapsto c : a \mapsto \varepsilon \in P\} \cup \{c \mapsto c;\ d \mapsto c\}$.

Let $\mu:\Sigma' \mapsto \{0,1\}$ be defined by $\mu(a) = 1$ for a in Σ and $\rho(a) \geq 1$, $\mu(a) = 0$ for a in Σ and $\rho(a) = 0$, $\mu(c) = 0$ and $\mu(d) = 1$. Clearly, $f^{\mu}_{s(G')} = f^{\rho}_{s(G)}$. Thus, we have the second inclusion $W^{\infty}_{0}(DOL) \subseteq W^{1}_{0}(PDOL)$. □

<u>Theorem 1</u> $W^{1}_{0}(PDOL) = W^{\infty}_{0}(PDOL) = W^{1}_{0}(DOL) = W^{\infty}_{0}(DOL)$.

<u>Proof</u> The following inclusions are obvious:

$$W^{1}_{0}(PDOL) \subseteq W^{\infty}_{0}(PDOL) \subseteq W^{\infty}_{0}(DOL)$$

$$W^{1}_{0}(PDOL) \subseteq W^{1}_{0}(DOL) \subseteq W^{\infty}_{0}(DOL).$$

So the equalities follow immediately by Lemma 1. □

<u>Theorem 2</u> $W^{\infty}_{1}(DOL) = F(DOL)$.

<u>Proof</u> Let f be in $W^{\infty}_{1}(DOL)$. By definition there is DOL-system $G = (\Sigma,P,\sigma)$ and weighting function $\rho:\Sigma \mapsto N-\{0\}$ so that $f = f^{\rho}_{s(G)}$. Let $\bar{\Sigma} = \{\bar{a}:a \in \Sigma\}$, and let $h:\Sigma^{*} \mapsto (\Sigma \cup \bar{\Sigma})^{*}$ be the homomorphism defined by $h(a) = a\bar{a}^{\rho(a)-1}$ for each a in Σ.

We construct DOL-system $G' = (\Sigma \cup \bar{\Sigma},P',h(\sigma))$ where $P = \{a \mapsto h(w):a \mapsto w \in P\} \cup \{\bar{a} \mapsto \varepsilon:a \in \Sigma\}$. Clearly, $s(G') = h(s(G))$. Thus $f^{\rho}_{s(G)} = f_{s(G')}$ and $W^{\infty}_{1}(DOL) \subseteq F(DOL)$. Since the reverse inclusion is trivial the proof is completed. □

<u>Theorem 3</u> $F(PDOL) \subsetneq W^{\infty}_{1}(PDOL) \subsetneq W^{\infty}_{1}(DOL) = F(DOL) \subsetneq W^{\infty}_{0}(DOL)$.

<u>Proof</u> The equation $W^{\infty}_{1}(DOL) = F(DOL)$ is proved in Theorem 2, all the inclusions are obvious. It remains to show that they are proper.

There are only monotonic functions in F(PDOL) but clearly not in $W^{\infty}_{1}(PDOL)$ so the first inclusion is proper.

Let function $f:N \mapsto \{0,1\}$ be defined by $f(0) = 1$, $f(i) = 0$ for $i \geq 1$. Clearly, f is in $W_1^\infty(DOL)-W_1^\infty(PDOL)$.

Finally, let $G = (\{a,b\},\{a \mapsto bb,\ b \mapsto aa\},a)$ and let function $\rho:\{a,b\} \mapsto \{0,1\}$ be defined by $\rho(a) = 1$, $\rho(b) = 0$. Let $s = s(G)$. Clearly, $f_s^\rho(2i) = 2^i$ and $f_s^\rho(2i+1) = 2^{i+1}$ for $i \in N$. Since G is a DOL system f_s^ρ is in $W_0^\infty(DOL)$ but from the properties of growth functions of DOL systems [2,3] it follows easily that f_s^ρ is not in $F(DOL)$. Thus the last inclusion has been shown to be proper. $\square$

It is easy to see that weighted growth functions of family of sequences F is the same as non-weighted growth functions of homomorphic images of sequences in F. We will formulate and prove this observation formally.

Let $s = x_0,x_1,\ldots$ be a sequence over Σ and $h:\Sigma^* \mapsto \Delta^*$ be a homomorphism. We write $h(s) = h(x_0),h(x_1),\ldots$. Let F be a family of sequences of strings. We write

$$HF = \{h(s):s \in F \text{ and } h \text{ is a homomorphism}\}$$

$$\bar{H}F = \{h(s):s \in F \text{ and } h \text{ is an } \varepsilon\text{-free homomorphism}\}.$$

Theorem 4 For every family F of strings

$$W_0^\infty(F) = F(HF) \text{ and } W_1^\infty(F) = F(\bar{H}F).$$

Proof Let $f \in W_0^\infty(F)$. By definition there are s over Σ in F and weighting function $\rho:\Sigma \mapsto N$ so that $f = f_s^\rho$. Let h be the homomorphism on Σ^* defined $h(a) = a^{\rho(a)}$ for each a in Σ $(a^0 = \varepsilon)$. Clearly, $f_{h(s)} = f_s^\rho$ and thus $W_0^\infty(F) \subseteq F(HF)$.

Let $f \in F(HF)$. By definition there are sequence s over Σ and homomorphisms h on Σ such that $f = f_{h(s)}$. Let $\rho:\Sigma \to N$ be defined by $\rho(a) = |h(a)|$. Clearly, $f_s^{\rho} = f_{h(s)}$ and thus $F(HF) \subseteq W_0^{\infty}(F)$.

In the above constructions, h is ε-free if $\rho(a) \neq 0$ for all a in Σ in the first part and vice versa in the second part. So also the second equation holds.

<u>Corollary 1</u> For every family of strings F

$$W_0^{\infty}(HF) = W_0^{\infty}(F) = F(HF) \text{ and } W_1^{\infty}(HF) = W_1^{\infty}(F) = F(\bar{H}F).$$

<u>Proof</u> Clearly, $HHF = HF$ and therefore also $F(HHF) = F(HF)$. So by Theorem 4 $W_0^{\infty}(HF) = F(HHF) = F(HF) = W_0^{\infty}(F)$. Similarly for the second equations.

In [2] and [3] it was shown that the growth functions of DOL-systems can be expressed in a special form. This characterisation was then used to show the decidability of the growth equivalence problem for DOL-systems. We will generalise these results to the case of weighted growth functions.

For a DOL-system $G = (\Sigma,P,\sigma)$ with $\Sigma = \{a_1,\ldots,a_k\}$ and for a weighting function $\rho:\Sigma \mapsto N$ we define the following matrices. The <u>initial vector</u>, π, is the k-dimensional row vector $(\pi_1,\ldots,\pi_k)$ where $\pi_i = n_{a_i}(\sigma)$. The <u>growth matrix</u>, A, is the k-dimensional square matrix whose (i,j)-th entry $a_{i,j} = n_{a_j}(w)$ for $a_i \mapsto w$ in P. The weighting vector, θ, is the k-dimensional column vector $(\theta_1,\ldots,\theta_k)$ where $\theta_i = \rho(a_i)$ for $i = 1,\ldots,k$. The following theorem is a direct consequence of the definitions.

Theorem 5 For any DOL-system G and any weighting function ρ the ρ-weighted growth function of s(G) can be expressed in the form $f^{\rho}_{s(G)} = \pi A^n \theta$ where A^0 is the identity matrix I. □

The generating function of a function f^{ρ}_s is defined to be the formal sum $F^{\rho}_s(x) = \sum_{n=0}^{\infty} f^{\rho}_s(n)x^n$. Obviously, $F^{\rho}_{s_1} = F^{\rho}_{s_2}$ iff $f^{\rho}_{s_1} = f^{\rho}_{s_2}$.

Lemma 2 For any DOL-system G and any weighting function ρ, $F^{\rho}_{s(G)} = \pi(I-Ax)^{-1}\theta$, where π and θ are defined above and I is the identity matrix.

Proof Clearly, the matrix I-Ax is nonsingular. By Theorem 5 we have $F^{\rho}_s(x) = \sum_{n=0}^{\infty} \pi A^n \theta x^n$. We complete the proof by using the matrix equation $(I-Ax)^{-1} = \sum_{n=0}^{\infty} A^n x^n$.

Now, we are prepared to show that ρ-weighted growth equivalence is decidable for DOL-systems.

Theorem 6 Given two DOL-systems G_1, G_2 with alphabet Σ and weighting function $\rho:\Sigma \mapsto N$, it is decidable whether $f^{\rho}_{s(G_1)} = f^{\rho}_{s(G_2)}$.

Proof The generating functions of $f^{\rho}_{s(G_1)}$ and $f^{\rho}_{s(G_2)}$ are of the form $p(x)/q(x)$ where p and q are polynomials with integer coefficients.

Clearly, the generating function $F^{\rho}_{s(G_i)} = p_i(x)/q_i(x)$ where p_i and q_i are polynomials with integer coefficients for i = 1,2. So $f^{\rho}_{s(G_1)} = f^{\rho}_{s(G_2)}$ iff $F^{\rho}_{s(G_1)} = F^{\rho}_{s(G_2)}$ iff $p_1(x)q_2(x) = p_2(x)q_1(x)$. We can, of course, decide whether two polynomials are identical.

4. The growth functions of DPGOL-systems

Let s be an infinite sequence of abstract e-graphs, $s = x_0, x_1, \ldots$, $x_i \in [\Sigma,\Delta]_*$. We will have three different growth functions associated with the sequence s depending on whether we count nodes, edges or both of x_i for each $i \geq 0$.

The node growth function of s, denoted v_s, is defined by $v_s(i)$ being the cardinality of $V_\Sigma^{x_i}$ (set of all non-environmental nodes of x_i) for each $i \geq 0$. The edge growth function, denoted by e_s, is defined by $e_s(i)$ being the cardinality of $E_\Delta^{x_i}$ (set of all edges of x_i) for each $i \geq 0$. Finally, the total growth function is denoted t_s where $t_s(i) = v_s(i)+e_s(i)$ for each $i \geq 0$.

Now, we will consider families of graph growth functions corresponding to families of graphs. Given a family of abstract e-graphs F we write

$$V(F) = \{v_s : s \in F\},$$
$$E(F) = \{e_s : s \in F\},$$
$$T(F) = \{t_s : s \in F\}.$$

Theorem 7 V(DPGOL) = F(PDOL).

Proof Let $G = (\Sigma,\Delta,P,C,S)$ be a DPGOL system. We choose any fixed ordering of the symbols of Σ. Let μ be the mapping from $[\Sigma,\Delta]_*$ to Σ^* which maps each abstract e-graph A over Σ,Δ to the alphabetically ordered string of all the occurrences of symbols from Σ as node labels in A. We construct the PDOL-system $G' = (\Sigma,P',\mu(\sigma))$ where $P' = \{a \mapsto \mu(A) : a \mapsto A \in P\}$.

Clearly, $f_{s(G')} = v_{s(G)}$. Thus we have V(DPGOL) $\subseteq$ F(PDOL) and the proof is completed since the reverse inclusion is obvious. $\square$

Theorem 8 T(PDGOL) = F(PDOL).

Proof Let $G = (\Sigma,\Delta,P,C,S)$ be a DPGOL system. Let $\Omega = \Sigma \times \Delta \times \Sigma$ and let there be some fixed ordering of Ω. Let μ_1 be the mapping from $[\Sigma,\Delta]_+$ to Ω^* which maps an abstract e-graph A over Σ,Δ to the alphabetically ordered string w in Ω^* such that for every edge of A labeled h with source and target nodes labeled a,b, respectively, there is one occurrence of (a,h,b) in w. Let μ_2 be the mapping from $[\Sigma,\Delta]_+$ to $\Sigma^+\Omega^*$ defined by $\mu_2(A) = \mu(A)\mu_1(A)$ for each A in $[\Sigma,\Delta]_+$ where μ is as defined in the proof of Theorem 7.

We construct PDOL system $G' = (\Sigma \cup \Omega, P', \sigma')$ where $\sigma' = \mu_2(\sigma)$ and $P' = \{a \mapsto \mu_2(A) : a \mapsto A \in P\} \cup \{(a,h,b) \mapsto w : (a,h,b) \in \Omega, a \xrightarrow{h} b \underset{G}{\Rightarrow} B, \mu_1(B) = w\}$.

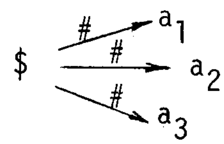
Clearly, $f_{s(G')} = t_{s(G)}$. Thus T(PDGOL) $\subseteq$ F(PDOL). The reverse inclusion is easy to show, we can simulate the growth function of a PDOL system with the node-growth function of the PDGOL-system which generates a sequence of graphs without any edges.

Theorem 9 E(DPGOL) = W_0^1(PDOL).

Proof Let $G = (\Sigma,\Delta,P,C,S)$ be a DPGOL-system. Let G' be the corresponding DPOL system from the proof of Theorem 8 and let ρ be the weighting function defined by $\rho(a) = 0$ for all x in Σ and $\rho(x) = 1$ for all x in Ω. Clearly, $f^{\rho}_{s(G')} = e_{s(G)}$. Thus E(DPGOL) $\subseteq W_0^1$(PDOL).

Let $H = (\Sigma,P,\sigma)$ be a DPOL system and $\rho : \Sigma \mapsto \{0,1\}$ be a weighting function. First, we define two auxiliary mappings ν,η from Σ^+ to $[\Sigma \cup \{\$\},\{\#\}]_+$, where \$, # are not in Σ. The mapping ν maps a string

$a_1 \ldots a_n$ in Σ^+ to the abstract e-graph with n isolated nodes labeled $a_1, \ldots, a_n$. The mapping η maps a string $a_1 \ldots a_n$ in Σ^+ to the abstract e-graph



Now, we construct DPGOL system $G' = (\Sigma \cup \{\$\}, \{\#\}, P', C, S)$ where $S = \eta(\sigma)$, $P' = \{a \mapsto \nu(w) : a \mapsto w \in P\} \cup \{\$ \mapsto \$\}$ and $C = \{\# \mapsto S_s \xrightarrow{\#} a_t : \rho(a) = 1\} \cup$ $\cup \{\# \mapsto \lambda : \rho(a) = 0\}$, λ being the empty stencil. Clearly, $e_{s(G')} = f^{\rho}_{s(G)}$. Thus $W^1_0(\mathrm{PDOL}) \subseteq E(\mathrm{PDGOL})$. □

By Theorem 1 family $W^1_0(\mathrm{PDOL})$ is equal to a number of other growth function families. So we have for example the following corollary of Theorem 7.

<u>Corollary 2</u> $E(\mathrm{PDGOL}) = W^{\infty}_0(\mathrm{DOL})$. □

To summarize our results about growth functions of DPGOL-systems, we have shown that if we count only nodes or both nodes and edges we are getting the same type of growth as for string DPOL-systems (without weighting symbols). On the other hand, if we count only edges we get the same type of growth as if we allow weight zero for symbols in string DPOL-systems.

ACKNOWLEDGEMENTS

The author is grateful to A. Lindenmayer for the discussions of the presented material.

References

[1] A. Lindenmayer, Mathematical models for cellular interactions in development, Parts I and II, J. of Theoret. Biology, v.18, 280-315, 1968.

[2] A.L. Szilard, Growth functions of Lindenmayer systems, Techn. rep. No.4, 1971, Dep. of Comp. Science, University of Western Ontario, London, Canada, 1971.

[3] A. Par and A. Salomaa, Integral sequential word functions and growth equivalence of Lindenmayer systems, Information and Control, v.23, 313-343, 1973.

[4] K. Culik II and A. Lindenmayer, Parallel rewriting on graphs and multidimensional development, submitted to Acta Informatica, also Res. report CS-74-22, Dep. of Comp. Science, University of Waterloo, Waterloo, Ontario, Canada, 1974.