POLYHEDRAL NEOPOLARITIES

by

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ABSTRACT

Let $(S, \underset{\sim}{+})$ be a finite commutative semigroup, H a subset of S, and b an element of S. We define the semigroup program (H, b, c) as minimize ct over $(t_h \in \mathbb{N}: t_h \geq 0, h \in H)$ satisfying $\underset{h \in H}{\underset{\sim}{\Sigma}} t_h \cdot h = b$ (where $\underset{\sim}{\Sigma}$ is the iteration of $\underset{\sim}{+}$ and $t_h \cdot h = \underset{i=1}{\overset{t_h}{\underset{\sim}{\Sigma}}} h$); (H, b, c) is a master semigroup program when H is the whole S. These definitions include Gomory's groups and master group programs. They also include the integer covering programs: minimize ct over integer solutions $t \geq 0$ of $At \geq b$ where A is a matrix and b is a vector both with non-negative integer entries, the columns A_j of A are all different and less than or equal to b; the semigroup here is $(\{s: 0 \leq s \leq b, s \text{ integer}\}, \underset{\sim}{+})$ with $s \underset{\sim}{+} r = \min(s + r, b)$ (the minimum is taken component by component).

We call $E(H, b)$ the convex hull of the solutions t of $\underset{h \in H}{\underset{\sim}{\Sigma}} t_h \cdot h = b$. $E(H, b)$ is a polyhedron, that is the set of solutions x of some finite systems $(a^i x \geq \alpha_i: i \in I)$. Knowing any one of these systems, the program (H, b, c) would become an ordinary linear program over $E(H, b)$.

When P is a polyhedron we define its β-polar polyhedron P^β to be the set $\{y: xy \geq 1$ for all $x \in P\}$. We call P β-closed when $P = P^{\beta\beta}$. We characterize the β-closed polyhedra and show the relation between minimal systems of linear inequalities defining a β-closed polyhedron P and extreme points and extreme rays of P^β. These results extend the results of Fulkerson on blocking

polyhedra.

We characterize those semigroup programs with $P(H, b)$ β-closed, in this case we use the results of polarity to obtain minimal systems for $P(H, b)$, thereby extending Gomory's theory characterizing $P(H, b)$ for group programs.

For master semigroup programs, systems of inequalities are also provided by the extreme points of more highly structured programs.

$\bar{s}$ is a b-complementor of s when $s \underset{\sim}{+} \bar{s} = b$ and $s \underset{\sim}{+} r = h \underset{\sim}{+} \bar{s} = b$ implies $r \underset{\sim}{+} h = b$, $(S, \underset{\sim}{+})$ is b-complementary when every element in S has a b-complementor. For a master b-complementary semigroup program (S, b, c) we obtain the following result: $(x \geq 0;\ v^i x \geq 1: i \in I)$ is a minimal system for $E(S, b)$ when $v^i: i \in I$ are the extreme points of $\{\pi \geq 0: \pi_b = 1;\ \pi_s + \pi_r \geq \pi_{s \underset{\sim}{+} r},\ s, r \in S;\ \pi_s + \pi_{\bar{s}} = 1,\ \bar{s}$ a b-complementor of $s\}$.

We give some algorithms to solve semigroup programs based on similar ones for group programs.

We obtain analogous results for integer packing programs maximize ct over non-negative integer solutions t of $At \leq b$, where A, b are defined as in covering programs. Let $S = \{a: 0 \leq a \leq b,\ a \text{ integer}\}$, a master packing program have all elements of S as columns of A, for this program: $(x \geq 0;\ v^i x \leq 1: i \in I)$ is a minimal system for the convex hull of integer solutions of $(t \geq 0;\ At \leq b)$ when $v^i: i \in I$ are the extreme points of $\{\pi \geq 0: \pi_b = 1;\ \pi_s + \pi_r \leq \pi_{s+r}$ for all $s, r \in S,\ s + r \leq b;\ \pi_s + \pi_{b-s} = 1$ for all $s \in S\}$.

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Chapter 1

INTRODUCTION

1.1. Summary of Results*

This thesis is concerned mainly with semigroup programs. Our interest in semigroup programs derives from their relation to integer programming as we will show in this introduction.

Let $(S, \underset{\sim}{+})$ be a finite commutative semigroup and let H be a subset of S and b an element of S. For any vector $c \in \mathbb{R}_+^H$ ($\mathbb{R}_+^H$ is the set of non-negative real vectors $(x_h \geq 0 \colon h \in H)$), we define the semigroup program (H, b, c) over $(S, \underset{\sim}{+})$ as

(1.1.1) minimize ct, over

(1.1.2) non-negative integer vectors $t = (t_h \colon h \in H)$,

(1.1.3) satisfying $\underset{h \in H}{\underset{\sim}{\Sigma}} t_h \cdot h = b$,

where $\underset{\sim}{\Sigma}$ is the iteration of $\underset{\sim}{+}$ and $k \cdot s$ is defined for any non-negative integer k and any s in S as $\underset{\sim}{\overset{k}{\underset{i=1}{\Sigma}}} s$.

A semigroup program (H, b, c) over $(S, \underset{\sim}{+})$ is called a master semigroup program when H is the whole S .

Gomory introduced (commutative) group programs in [G3], [G4] and [G5]. A group program (H, b, c) over $(S, \underset{\sim}{+})$ is a semigroup program with $(S, \underset{\sim}{+})$ being a finite commutative group.

The results in this thesis about semigroup programs

* The notation used in this section is given in section 1.3.

are not new for group programs, they appear in Gomory [G5] and Gomory and Johnson [G6]. But the development here is different and we believe our presentation clarifies the concepts involved.

Furthermore semigroup programs also include (integer) covering programs. A covering program (A, b, c) is defined for any vector c in $\mathbb{R}^J$ as:

(1.1.4) minimize ct, over

(1.1.5) non-negative integer vectors $t = (t_j: j \in J)$,

(1.1.6) satisfying $At \geq b$.

Where A is the matrix $(a_j^i: i \in I, j \in J)$, b is the vector $(b_i: i \in I)$ and both A and b have non-negative integer entries.

The results obtained for covering programs using the results obtained for semigroup programs are new. Covering programs and group programs motivated our interest in semigroup programs in general.

Group programs were introduced because of a relationship to integer programming and with the hope that they will be easier to solve than the integer programs (this hasn't been proved) and because they will give insight on the structure of the integer programs. These same statements can be claimed about semigroup programs and their relation to covering programs.

We will outline here the relationship between group programs and integer programming. The relation with

Smith Normal Forms can be found in Hu [H1] or Garfinkel and Nemhauser [G2].

Let A be the matrix $(a^i \in \mathbb{N}^J : i \in I)$ and b be the vector $(b_i : i \in I)$, we assume rank of A equals $|I|$.

(1.1.7) An <u>integer program</u> is the program given by

(1.1.8) maximize $\sum_{j \in J} c_j x_j$

over vectors $(x_j \in \mathbb{R} : j \in J)$ satisfying

(1.1.9) $Ax = b$,

(1.1.10) $x \geq 0$,

(1.1.11) x_j integer for all $j \in J$.

(1.1.12) Let x^0 be an optimal basic solution of the linear program (1.1.7) to (1.1.10) with (1.1.11) omitted. Let K be a subset of J satisfying $|K| = I$, rank of A_K equals rank of A and $K \supseteq \{j \in J : x_j^0 \neq 0\}$. Let $L = J - K$. We can now formulate the integer program (1.1.7) as

(1.1.13) maximize $c_K A_K^{-1} b - (c_K A_K^{-1} A_L - c_L) x_L$

over solutions of

$$x_K = A_K^{-1}(b - A_L x_L),$$

$$x_K,\ x_L \geq 0, \quad x_j \text{ integer for all } j \in J.$$

(1.1.14) Let $d_L = c_K A_K^{-1} A_L - c_L$, $H_L = A_K^{-1} A_L$ and $\bar{b} = A_K^{-1} b$, d_L is non-negative because by choice of K

we have $x_K^0 = A_K^{-1}b$ and $x_L^0 = 0$. Consider the relaxation of (1.1.13) in which the inequalities $x_K \geq 0$ are omitted, this relaxation is equivalent to

(1.1.15) minimize $d_L x_L$ over solutions x_K, x_L satisfying

(1.1.16) $x_K = \bar{b} - H_L x_L$, $x_L \geq 0$, integer, x_K integer.

(1.1.17) Any solution x of (1.1.9) to (1.1.11) satisfies (1.1.16), moreover $cx = c_K\bar{b} - d_L x_L$. If x' is an optimal solution of (1.1.15), (1.1.16) and $x'_K \geq 0$ then x' is a solution of (1.1.9) to (1.1.11) and $cx \leq cx'$. That is when x' is an optimal solution of (1.1.15) and (1.1.16) and $x'_K \geq 0$, it is also an optimal solution of the integer program (1.1.7).

The interest in group programs is because system (1.1.15), (1.1.16) is equivalent to a group program.

(1.1.18) We have $x_K = \bar{b} - H_L x_L$ is integer if and only if $H_L x_L = \bar{b}$ (modulo 1).

We denote by H_f: $f \in L$ the columns of H_L.

(1.1.19) Let $H = \{h \in [0, 1]^I : h = H_f \text{ (modulo 1) for some } f \in L\}$.

We define $r \underset{\sim}{+} s \equiv r + s$ (modulo 1) for any $r, s \in \mathbb{R}^I$.

Let $S = \{s \in [0, 1]^I : s = \sum_{h \in H} x_h h \text{ (modulo 1)}, x_h \in \mathbb{N}_+\}$.

Then $(S, \underset{\sim}{+})$ is a finite group generated by H. (For a proof see [G2], Chapter 7).

(1.1.20) Let $b' \in [0, 1]$ satisfy $b' = \bar{b}$ (modulo 1) and $e_h = \min_{f \in H}\{d_f : h = H_f \text{ (mod 1)}\}$. A consequence of our discussion is that to solve (1.1.15), (1.1.16) is equivalent to solve the group program (H, b', e) over $(S, \underset{\sim}{+})$ and (by (1.1.17)) may solve the integer program (1.1.7).

We will describe now the relationship of covering programs with semigroup programs.

Given a covering program (A, b, c) we will make the non-restrictive assumptions.

(1.1.21) $b > 0$, since $b_i = 0$ implies $a^i t \geq b_i$ for all $t \in \mathbb{N}_+^J$.

(1.1.22) $a^i \neq 0$: $i \in I$, since $a^i = 0$ implies no t in $\mathbb{N}_+^J$ satisfies (1.1.6) when $b_i > 0$. Notice that with this assumption the set of solutions to (1.1.5) and (1.1.6) is always non-empty.

(1.1.23) We assume that any column A_j of A satisfies $A_j \leq b$. Because if $A_j \nleq b$ then, for any $t \in \mathbb{N}_+^J$, we have $At \geq b$ if and only if $A_{J-\{j\}} t_{J-\{j\}} + A'_j t_j \geq b$, where A'_j is the vector $(\min\{a_j^i, b_i\}: i \in I)$.

(1.1.24) We assume no two columns of A are equal. Because otherwise, keeping only one with minimum cost, the optimal solutions of the new program are optimal solutions of the old one.

(1.1.25) Let <u>A(b)</u> denote the set $\{a \in \mathbb{N}_+^I: a \leq b\}$. For any $a^1, a^2 \in A(b)$ let $a^1 \underset{\sim}{+} a^2$ denote the vector $(\min(a_i^1 + a_i^2, b_i): i \in I)$ in $A(b)$.

$(A(b), \underset{\sim}{+})$ is a semigroup called a covering semigroup.

Now taking H to be the set of columns of A we have that the covering program (A, b, c) is equivalent to the semigroup program (H, b, c) over $(A(b), \underset{\sim}{+})$ because $(t_j: j \in J)$ satisfies (1.1.5) and (1.1.6) if and only if $(t_{A_j}: t_{A_j} = t_j, j \in J)$ satisfies (1.1.2) and (1.1.3).

For any semigroup program (H, b, c) we denote by T(H, b) the set of solutions of (1.1.2) and (1.1.3). And we denote by E(H, b) the convex hull of T(H, b). Assuming that E(H, b) is the set of solutions x of some finite system $(a^i x \geq \alpha_i: i \in I)$, knowing any one of these systems, the program (H, b, c) would become an ordinary linear program over E(H, b).

In Chapter 2 we give the foundations of polyhedral theory used in later chapters.

A polyhedron P is the set of solutions of a finite system of linear inequalities, we call this a defining system of inequalities for P. When $P = \{x \in \mathbb{R}^J: Ax \geq b\}$ we denote by RAY(P) the set $\{x \in \mathbb{R}^J: Ax \geq 0\}$.

Given a defining system of inequalities for P we call the system irredundant when deleting any inequality in the system the set of solutions is bigger than P.

In section 2.5 we give a summary of the elemental properties about finite commutative semigroups that we need.

In Chapter 3 we discuss three polarity types. For any set P in $\mathbb{R}^J$ the γ-polar P^γ is the set $\{y: xy \geq 0$ for all $x \in P\}$, the α-polar P^α is the set

$\{y: xy \le 1$ for all $x \in P\}$ and the β-polar P^β is the set $\{y: xy \ge 1$ for all $x \in P\}$. The results for the first two are not new and can be found in several texts (Stoer and Witzgall [S3], Rockafellar [R1]), but the results in β-polarity are new and extend results of Fulkerson [F1] in blocking polyhedra.

P^β is a polyhedron when P is a polyhedron.

We call P β-closed when $P = P^{\beta\beta}$. We obtain the following characterization of β-closed polyhedra.

Theorem (3.3.11). Let P be a non-empty polyhedron strictly contained in $\mathbb{R}^J$. Then the following statements are equivalent:

(i) P is β-closed.

(ii) P is contained in RAY(P) and $0 \notin P$.

(iii) P is the set of solutions x of a finite system of the form

$$\begin{cases} a^m x \ge 1 \ , \ m \in M \ , \ M \text{ is not empty,} \\ r^n x \ge 0 \ , \ n \in N \ . \end{cases}$$

We also show the relation between system of inequalities for a β-closed polyhedron P and generating sets for P^β. Specially when P is full dimension we obtain:

Theorem (3.3.24). Let P be a full dimension β-closed polyhedron strictly contained in $\mathbb{R}^J$. Then the system

$$\begin{cases} a^m x \ge 1 \ , \ m \in M, \\ r^n x \ge 0 \ , \ n \in N; \end{cases}$$

is an irredundant defining system for P if and only if $\{a^m: m \in M\}$ is the set of extreme points of P^β and $\{r^n: n \in N\}$ is the set of extreme rays of P^β satisfying $r^n \neq \lambda a^m$ for any $\lambda \geq 0$ and all $m \in M$, (that is the ray generated by r^n contains no extreme point).

Let A be the matrix $(a^i \in \mathbb{R}^J: i \in I)$ and let b be the vector $(b_i \in \mathbb{R}: i \in I)$. Let P be the polyhedron $\{x \in \mathbb{R}^J: Ax \geq b\}$. For any $I' \subseteq I$, the set $\{x \in P: a^i x = b_i$ for all $i \in I'\}$ is a face of P. A facet of P is a face of P with dimension one less than dimension of P. When P is full dimension and $Ax \geq b$ is an irredundant system for P, the facets of P are the sets $\{x \in P: a^i x = b_i\}$ for any $i \in I$ (for a proof of this see Pulleyblank [P1]). Hence to characterize irredundant defining systems for P is equivalent to characterize the facets of P when P is full dimension. However, we believe that the importance of the facets is that they provide irredundant systems. Thus we will refer to irredundant systems rather than to facets. In Chapter 4 section 4.1 we define K-unbounded sets for any nonempty set $K \subseteq J$ by:

T in $\mathbb{N}_+^J$ is K-unbounded when it satisfies

(i) $0 \notin T$ and $T \neq \emptyset$,

(ii) if $t \in T$ then $t_j = 0: j \in J - K$,

(iii) for all $t \in T$ and for all $j \in K$ there is $k \geq 1$ such that $t + k\delta^j \in T$.

Where $\mathbb{N}_+^J$ is the set of non-negative integer vectors in $\mathbb{R}^J$ and δ^j are the vector with all components zero but the jth which is 1.

K-unbounded sets play a role in semigroup programs because the sets of solutions of group and covering programs belong to this class.

We obtain the following results:

Theorem (4.1.8). For any J-unbounded set T in $\mathbb{R}^J$, the convex hull P of T is a full-dimension β-closed polyhedron with the following irredundant defining system:

$$\begin{cases} vx \geq 1, & \text{for all extreme points } v \text{ of } P^\beta, \\ x_j \geq 0, & \text{for all } j \in J \text{ such that } \min_{t \in T}\{t_j\} = 0. \end{cases}$$

Furthermore the set P^β is the set of solutions π to the system

$$\begin{cases} t\pi \geq 1, & \text{for any minimal element } t \text{ of } T, \\ \pi_j \geq 0, & \text{for all } j \in J. \end{cases}$$

Theorem (4.1.12). Let T be a J-unbounded set in $\mathbb{R}^J$ and K a non-empty subset of J. Let $X^K = \{x \in \mathbb{R}^J : x_j = 0 \text{ for all } j \in J - K\}$ and T^K denotes $T \cap X^K$. The following relations hold between the convex hull P of T and the convex hull P^K of T^K.

(i) $P^K = P \cap X^K$.

(ii) The set V^K of extreme points of P^K equals the set $V \cap X^K$ where V is the set of extreme points of P.

Theorem (4.1.8) extends to J-unbounded sets theorem (7) to (9) of Gomory [G5]. Theorem (4.1.12) extends theorems (12) and (13) of Gomory [G5] about the relation between a group program and its master group program when we regard T as an integer master program for T^K.

These extensions will carry over to semigroup programs (H, b, c) over $(S, \underset{\sim}{+})$ whose $T(H, b)$ are H-unbounded and in this case theorem (4.1.12) applied to any semigroup program (H', b, c) over $(S, \underset{\sim}{+})$ when $H' \subseteq H$ since we can consider $T(H', b)$ as $T(H, b) \cap X^{H'}$.

In section 4.2 we introduce unbounded semigroup programs. Given a semigroup program (H, b, c), we call it unbounded (or H-unbounded) when $T(H, b)$ is H-unbounded.

Let $(S, \underset{\sim}{+})$ be a finite commutative semigroup. For any $s, b \in S$ we define $b \sim s$ to be the set $\{x \in S: s \underset{\sim}{+} x = b\}$.

The following conditions are imposed in a semigroup program (H, b, c) over $(S, \underset{\sim}{+})$ to avoid trivial cases. $(S, \underset{\sim}{+})$ is generated by H, b is not the identity of the semigroup and $b \sim h$ is non-empty for all $h \in H$. With these assumptions we obtain:

<u>Theorem (4.2.16)</u>. A semigroup program (H, b, c) over $(S, \underset{\sim}{+})$ is unbounded if and only if there is $k > 1$ such that $b = k \cdot b$.

We consider a polyhedral neopolarity between two families of polyhedra to be a relation which assigns to each polyhedron P in the first family, one polyhedron Q in the second family together with a rule to obtain a defining system for P knowing the extreme rays and extreme points of Q. The better known examples of polyhedral neopolarities are provided by the polyhedral polarities and this justifies the name.

In Chapter 5 we introduce several polyhedral neopolarities for unbounded master semigroup programs.

We can apply the results of section 4.1 to unbounded semigroup programs. But also we can use the semigroup structure to obtain more information about the extreme points of $E^{\beta}(H, b)$. These points together with inequalities $x_h \geq 0$ provide an irredundant system for $E(H, b)$.

Let $(S, \underset{\sim}{+})$ be a finite commutative semigroup and let b be an element in S. For all $\bar{s}, s \in S$ we call $\bar{s}$ a b-complementor of s when $s \underset{\sim}{+} \bar{s} = b$ and for all $r, h \in S$ we have $s \underset{\sim}{+} r = h \underset{\sim}{+} \bar{s} = b$ implies $r \underset{\sim}{+} h = b$. The semigroup is called b-complementary when every element has a b-complementor and uniquely b-complementary when every element has one and only one b-complementor.

It is easy to verify that for any element b in a group, the group is uniquely b-complementary because the unique b-complementor of any element s is $b - s$.

It is also easy to verify that any covering semigroup $(A(b), \underset{\sim}{+})$ is uniquely b-complementary because for any $s \in A(b)$ the unique b-complementor of s is $b - s$.

We obtain:

<u>Lemma (5.2.23)</u>. A semigroup program (H, b, c) over $(S, \underset{\sim}{+})$ is H-unbounded when $(S, \underset{\sim}{+})$ is b-complementary.

Let (H, b, c) be an H-unbounded semigroup program over $(S, \underset{\sim}{+})$. We show that the extreme points of $E^{\beta}(H, b)$ are minimal points of $E^{\beta}(H, b)$ and that any minimal point π of $E^{\beta}(H, b)$ satisfies:

Subadditivity: if $h, r, h \underset{\sim}{+} r \in \check{H}$ then $\pi_h + \pi_r \geq \pi_{h \underset{\sim}{+} r}$.

Monotony: if $b \sim h \subseteq b \sim r$ and $h, r \in H$ then $\pi_h \leq \pi_r$.

Complementarity: if $h, \bar{h} \in H$ and $\bar{h}$ is a b-complementor of h then $\pi_h + \pi_{\bar{h}} = 1$.

We also obtain $\pi_b = 1$ when $b \in H$.

For groups the monotony condition is trivial because when S is a group we have $b \sim s = \{b - s\}$ for all $s \in S$. Therefore $b \sim h \subseteq b \sim r$ is equivalent to $h = r$.

For covering programs the monotony condition is relevant because when $h, r \in A(b)$ and $h \leq r$ we have $b \sim h \subseteq b \sim r$, since for any $s \in b \sim h$ we have $b \leq h + s \leq r + s$, hence $s \in b \sim r$.

In master semigroup programs subadditivity has strong implications.

<u>Theorem (5.2.11)</u>. Let (S, b, c) be an S-unbounded master semigroup program. If $\pi \in \mathbb{R}^S_+$ satisfies $\pi_b = 1$ and $\pi_h + \pi_r \geq \pi_{h \underset{\sim}{+} r}$ for all $h, r \in S$ then we have $\pi x \geq 1$ for all $x \in E(S, b)$, i.e. $\pi \in E^{\beta}(S, b)$.

This result for group programs is in Gomory and Johnson [G6] (theorem I.5).

As a consequence of theorem (5.2.11) we obtain several characterizations of defining systems for $E(S, b)$ using the extreme points of highly structured programs. In particular we obtain:

Theorem (5.2.25). Let (S, b, c) be a master semigroup program and let $(S, \underset{\sim}{+})$ be b-complementary. Then

$$\begin{cases} v^i x \geq 1 & , i \in I , \\ x_s \geq 0 & , s \in S , \end{cases}$$

is an irredundant defining system for $E(S, b)$ when $v^i : i \in I$ are the extreme points of

$$\begin{cases} \pi_b = 1, \\ \pi_s + \pi_h \geq \pi_{s \underset{\sim}{+} h} \quad \text{for all} \quad s, h \in S, \\ \pi_s + \pi_{\bar{s}} = 1 \quad \text{for all} \quad \bar{s} \quad \text{a b-complementor of } s, \\ \pi_s \geq 0 \quad \text{for all} \quad s \in S. \end{cases}$$

And the following conjecture:

Conjecture (5.2.35). Theorem (5.2.25) is still valid with the weaker assumption (S, b, c) is S-unbounded instead of $(S, \underset{\sim}{+})$ being b-complementary.

Theorem (5.2.25) for group programs is in Gomory [G5] (theorem 18).

In Chapter 6 we describe a family of uniquely b-complementary semigroups larger than groups and covering semigroup and give some examples to show the difficulty of characterizing uniquely b-complementary semigroups as a sum of a given family of semigroups.

In Chapter 7 we introduce (integer) packing programs and obtain results similar to those obtained for semigroup programs.

A packing program (A, b, c) is defined as

(1.1.26) maximize ct, over

(1.1.27) non-negative integer vectors $t = (t_j : j \in J)$

(1.1.28) satisfying $At \le b$.

Where A and b are defined as in covering programs. Let A(b) be the set $\{a \in \mathbb{N}_+^I : a \le b\}$. A master packing program is a packing program (A, b, c) where the set of columns of A is the set A(b).

Using α-polarity we obtain:

<u>Theorem (7.2.14)</u>. Let (A, b, c) be a master packing program. Then

$$\begin{cases} v^m x \le 1 , & m \in M , \\ x_j \ge 0 , & j \in J . \end{cases}$$

is an irredundant defining system for the convex hull of solutions of (1.1.27) and (1.1.28) when $v^m : m \in M$ are the extreme points of

$$\begin{cases} \pi_j = 1 \text{ when } A_j = b, \\ \pi_j + \pi_k \le \pi_e \text{ when } j,k,e \in J \text{ and } A_j + A_k = A_e, \\ \pi_j + \pi_k = 1 \text{ when } j,\ k \in J \text{ and } A_j + A_k = b, \\ \pi_j \ge 0 \text{ for all } j \in J. \end{cases}$$

Finally, in Chapter 8, we discuss some algorithms to solve semigroup programs. These algorithms are based on similar ones for group programs.

1.2. Remarks

We don't think that this thesis exhausts the topic of semigroup programs, on the contrary we believe that this work opens a field where several questions appear, some in relation with Integer Programming and the possible use of the theory exposed here in a better understanding of this field or to obtain better algorithms to solve integer programs. Others are questions opened by the semigroup programs themselves. Examples of this latter type of question are,

Given a semigroup $(S, \underset{\sim}{+})$ and a loop element b in S, is there an extension of S to a b-complementary semigroup?

or

What is a characterization of reduced b-complementary semigroups as sum of a small family of b-complementary semigroups? In Chapter 6 we show the difficulty of such characterization.

The extension of packing programs to packing semigroup programs is under way and uses not only a semigroup structure but also a partial order over the elements of the semigroup, we are now trying to put as few conditions as possible in this partial order. We hope the results will be available very soon.

The definition of K-unbounded sets is still too restrictive, from the results in polarity of polyhedra we really are interested in sets of integer vectors which convex hull is β-closed and these sets are far more than the K-unbounded. Therefore a more general definition is desirable.

Respect to algorithms, those known for group programs (and therefore their extension to semigroup programs) are theoretically inefficient, although they may be useful empirically. Thus research for more efficient algorithms for semigroup programs will be useful.

Respect to empirical results most of the work remains to be done, no experimental data has been used and no implementation in Euristical programs have been intended, surely a work for covering programs as the one reported in Garry et al [G7] for group programs is necessary. From the theoretical point of view, it is worth noting that the importance of the characterization of systems of inequalities (redundant or not) for the convex hull of the solutions of an integer program seems to reside mainly in the possibility of utilization of the strong L.P. Duality Theorem and the Complementary Slackness Theorem as in Edmonds' paper [E1]. However it is noticeable that most algorithms for group programs only use subadditivity constraints but no complementarity and algorithms for covering programs can also use monotony, more efforts are due.

Knapsack programs, that is packing or covering programs with one equation have special theoretical interest and special structure also. Knapsack covering programs correspond to semigroup programs over subsemigroups of cyclic semigroups and they are always b-complementary (see section 6.2). Special algorithms for cyclic semigroup programs gain relevance and their study is worthwhile.

In conclusion, we believe that the theory exposed in this thesis is interesting by itself and by the implications it could have in the important and elusive area of Integer Programming.

We want to point out that in Gomory's paper [G5] (as well as in other papers motivated by it) there are a lot of other results about group programs, some about the relation of group programs with integer programs, others about group programs themselves. We have checked several of the latter type and found that they have a semigroup counterpart. We have not included them here because we consider they are not so important or we don't need them in the line followed in this thesis. However, we want to clarify the relation that other results may have. From the results in this thesis we can conclude:

(1.2.1) It seems that to solve group programs is as difficult as to solve covering programs, maybe more, since we have some structure in the characterization of facets of master covering polyhedra which is not present for master group polyhedra. Hence the study of differences and similarities between group programs, semigroup programs and covering programs may give insight to obtain efficient algorithms for one of these classes.

(1.2.2) Some properties of semigroup programs and their polyhedra that have little theoretical interest may be very useful in the implementation of heuristic algorithms. From this point of view, it is important to keep in mind that most properties of group programs translate to semigroups.

We close with a quotation from Gomory ([G5], p. 452) (here P is the convex hull of the solutions of an integer program).

"Since any algorithm for the integer programming problem, whether related to linear programming, branch and bound, exhaustive search, or whatever, must end up finding a vertex of P, information on P seems relevant to any approach to the integer programming problem. Yet information about P is very difficult to obtain."

1.3. General Notation

We use the symbol "$\equiv$" to indicate a definition and reserve the symbol "=" for denote the equality of two objects.

If X and Y are sets we denote the union and intersection of X and Y by $X \cup Y$ and $X \cap Y$ respectively. We let $X - Y$ denote the set theoretic difference, that is

$$X - Y \equiv \{x \in X: x \notin Y\} .$$

We denote the empty set by $\emptyset$.

We let $|X|$ denote the cardinality of X.

We use $X \subseteq Y$ to denote "X is a subset of Y" and we use $X \subset Y$ to denote "X is a proper subset of Y" (thus $X \neq Y$).

We denote the cartesian product of two sets X and Y by $X \times Y$. Thus

$$X \times Y \equiv \{(x, y): x \in X, y \in Y\}.$$

When J is a finite set, we denote the cartesian product of $|J|$ copies of a set X by X^J. Thus

$$X^J \equiv \{(x_j: j \in J): x_j \in X \text{ for all } j \in J\}.$$

We let $\mathbb{R}$ denote the set of real numbers.

Chapter 2

FOUNDATIONS

2.1. Linear Algebra

Let J be a finite set. We let $\mathbb{R}^J \equiv \{(x_j: j \in J): x_j \in \mathbb{R}$ for all $j \in J\}$. We let 0 denote the vector which is zero in every component, δ^j the vector which is zero in every component but j which is equal to 1, and $\bar{1}$ the vector whose components are all 1.

(2.1.1) A set $X \subseteq \mathbb{R}^J$ is said to be linearly independent if whenever $\sum_{x \in X} \alpha_x x = 0$ for some $(\alpha_x \in \mathbb{R}: x \in X)$ we have $\alpha_x = 0$ for all $x \in X$. Otherwise X is linearly dependent.

(2.1.2) Let $X \subseteq \mathbb{R}^J$. A basis of X is a maximal linearly independent subset of X. The following result is well known.

(2.1.3) Theorem. (Birkhoff and MacLean [B1], Ch. 7, §4). All bases of $X \subseteq \mathbb{R}^J$ have the same cardinality called the rank of X, and the rank of X is no greater than $|J|$. □

(2.1.4) If $x, y \in \mathbb{R}^J$ we let xy denote $\sum_{j \in J} x_j y_j$.

(2.1.5) The null space of $X \subseteq \mathbb{R}^J$, is defined to be $\{y \in \mathbb{R}^J: yx = 0$ for all $x \in X\}$. The following is a basic result.

(2.1.6) Theorem. (Birkhoff and McLean [B1], Ch VIII, Theorem 11). For any $X \subseteq R^J$, the rank of X plus the rank of the null space of X equals $|J|$. □

(2.1.7) If $x, y \in \mathbb{R}^J$, we say $x \leq y$ if $x_j \leq y_j$ for all $j \in J$. We say $x < y$ if $x_j < y_j$ for all $j \in J$.

(2.1.8) We denote by $\mathbb{R}^J_+$ the set $\{x \in \mathbb{R}^J: 0 \leq x\}$.

(2.1.9) Let I, J be finite sets. If $A \in \mathbb{R}^{I \times J}$ is the matrix $(a^i \in \mathbb{R}^J: i \in I)$ then for any $S \subseteq J$ we let A_S denote $((a^i_j: j \in S): i \in I)$. If S is a single element v we abbreviate $A_{\{v\}}$ by A_v. $a^i: i \in I$ are the rows of A and $A_j: j \in J$ are the columns of A. If $x \in \mathbb{R}^J$ we define the product Ax to be the vector $y = (y_i: i \in I) \in \mathbb{R}^I$ where $y_i = a^i x$ for all $i \in I$.

(2.1.10) <u>Theorem</u>. (Birkhoff and MacLean [B1], p. 221, Corollary 3). The rank of the set $\{a^i: i \in J\}$ equals the rank of $\{A_j: j \in J\}$ and is called the rank of A. □

(2.1.11) The vectors $x^k \in \mathbb{R}^J: k \in K$ are <u>affinely independent</u> when for any $h \in K$, the vectors $x^k - x^h$: $k \in K - \{h\}$ are linearly independent.

(2.1.12) <u>Proposition</u>. Let $X \subseteq \mathbb{R}^J$ and 0 belongs to X. Then the maximum number p of affinely independent vectors in X equals the maximum number q of linearly independent vectors in X plus 1.

Proof: Let $A = \{a^i : i \in I\}$ be a set of q linearly independent vectors in X. Clearly $0 \notin A$ and $A \cup \{0\}$ is affinely independent.

Let $y^0, y^1, \ldots, y^q$ be affinely independent points of X, where $p = q + 1$ is the maximum number of affinely independent points in X. Then $y^1 - y^0, \ldots, y^q - y^0$ are linearly independent. If $y^0 = 0$ then $y^1, \ldots, y^q$ are linearly independent. Otherwise

$$(2.1.13) \qquad 0 - y^0 = \sum_{e=1}^{q} \lambda_e (y^e - y^0)$$

because $0 \in X$ and choice of $y^0, y^1, \ldots, y^q$. Moreover some λ_e are non-zero (being $y^0 \neq 0$), say $\lambda_q \neq 0$. We claim that $y^0, y^1, \ldots, y^{q-1}$ are linearly independent. Let

$$\sum_{e=1}^{q-1} \mu_e y^e = 0 . \text{ We have}$$

$$\sum_{e=0}^{q-1} \mu_e y^e = \sum_{e=1}^{q-1} \mu_e (y^e - y^0) + \left(\sum_{e=0}^{q-1} \mu_e\right) y^0 \text{ , let } \alpha = \sum_{e=0}^{q-1} \mu_e \text{ ,}$$

using (2.1.13).

$$\sum_{e=0}^{q-1} \mu_e y^e = \sum_{e=1}^{q-1} \mu_e (y^e - y^0) - \alpha \sum_{e=1}^{q} \lambda_e (y^e - y^0) =$$

$$\sum_{e=1}^{q-1} (\mu_e - \alpha\lambda_e)(y^e - y^0) - \alpha\lambda_q (y^q - y^0) = 0. \text{ Hence}$$

$(\mu_e - \alpha\lambda_e) = 0$ for $1 \leq e \leq q - 1$ and $\alpha\lambda_q = 0$ (because $y^1 - y^0, \ldots, y^q - y^0$ are linearly independent. Since $\lambda_q \neq 0$ and $\alpha\lambda_q = 0$ we have $\alpha = 0$, therefore $\mu_e = 0$ for $1 \leq e \leq q - 1$, and $\mu_0 = \alpha - \sum_{e=1}^{q-1} \mu_e = 0$. We have proved that $\sum_{e=0}^{q-1} \mu_e y^e = 0$ implies $\mu_e = 0$ for $0 \leq e \leq q - 1$, thus $y^0, \ldots, y^{q-1}$ are linearly independent. □

2.2. Systems of Inequalities

Let I and J be a finite set. We associate with any matrix $A = (a^i \in \mathbb{R}^J : i \in I)$ and any vector b in $\mathbb{R}^I$ the (finite) system of (linear) inequalities:

(2.2.1) $Ax \geq b$, i.e. $a^i x \geq b_i$ for all $i \in I$,

and we denote by *$\langle A, b\rangle$* the system (2.2.1). The inequality $a^i x \geq b_i$ will be denoted by *$\langle a^i, b_i\rangle$*.

(2.2.2) The *rank of the system* $\langle A, b\rangle$ is the rank of A by definition.

(2.2.3) A *polyhedron* is defined as the set of solutions of a finite system of linear inequalities. Given any system $\langle A, b\rangle$, the *solution set* of $\langle A, b\rangle$, written *$P(A, b)$*, is the set $\{x \in \mathbb{R}^J : Ax \geq b\}$. Thus $P(A, b)$ is a polyhedron. We call $\langle A, b\rangle$ a *defining system* for a polyhedron P when $\langle A, b\rangle$ is a finite system of linear inequalities and P equals $P(A, b)$.

Clearly any finite system of linear inequalities and linear equations can be represented in the form (2.2.1) since $ax \leq b$ is equivalent to $(-a)x \geq (-b)$, and $ax = b$ is equivalent to $ax \geq b$ and $ax \leq b$.

(2.2.4) Let $x \in P(A, b)$ and $I^x = \{i \in I : a^i x = b_i\}$. We call the *equality system* of x , written $EQ(x)$, the system $\langle (a^i \in \mathbb{R}^J : i \in I^x), (b_i : i \in I^x)\rangle$. The set I^x is called the *equality index* of x.

(2.2.5) Let $x \in P(A, b)$. x is called a <u>basic solution</u> if the rank of the system $EQ(x)$ is $|J|$.

Let $c \in \mathbb{R}^J$. Consider the (primal) <u>linear program</u>:

$$\text{(2.2.6)} \qquad \text{minimize } cx \quad \text{for } x \in \mathbb{R}^J \text{ satisfying}$$

(2.2.7) $a^i x \geq b_i$ for all $i \in I$, where I is finite.

The <u>dual linear program</u> is

$$\text{(2.2.8)} \qquad \text{maximize } \sum_{i \in I} b_i y_i \quad \text{for } y = (y_i \in \mathbb{R}: i \in I) \text{ satisfying}$$

(2.2.9) $y_i \geq 0$ for all $i \in I$

(2.2.10) $\sum_{i \in I} y_i a^i = c.$

A vector $x \in \mathbb{R}^J$ satisfying (2.2.7) is called a <u>feasible solution</u> to the primal program. A vector $y \in \mathbb{R}^I_+$ which satisfies (2.2.10) is called a <u>feasible dual solution</u>.

A feasible primal solution x^0 which minimizes $c \cdot x$ for all feasible primal solutions is called an <u>optimal primal solution</u>; an <u>optimal dual solution</u> is defined analogously .

The following is a fundamental theorem of linear programming (see Dantzig [D.1], p. 120, Theorem 1).

(2.2.11) <u>Theorem</u>. For any linear program exactly one of the following situations occurs.

i) There exists no feasible solution.

ii) For any $\gamma \in \mathbb{R}$ there is a feasible solution x such that $c \cdot x < \gamma$.

iii) There is an optimal feasible solution. □

The following theorems give the relationship between the values of cx and by for primal and dual feasible solutions.

(2.2.12) Weak L.P. Duality Theorem. (Dantzig [D1], p. 130). If x is a feasible primal solution and y is a feasible dual solution then $cx \geq by$.□

(2.2.13) Corollary. If for any $\beta \in R$ there is a feasible dual solution y such that $by \geq \beta$ then there is no feasible primal solution. □

(2.2.14) Strong L.P. Duality Theorem. (Dantzig [D1], p. 129, Theorem 1, p. 134, Theorems 2, 3). If there is a feasible primal solution and a lower bound of cx over all feasible primal solutions then there is an optimal primal solution x^0 and an optimal dual solution y^0 and $cx^0 = by^0$.□

(2.2.15) We say that $\langle a, b \rangle$ is a valid inequality for $P(A, b)$ whenever $ax \geq b$ for all $x \in P(A, b)$.

(2.2.16) Let $A = (a^i \in \mathbb{R}^J : i \in I)$, $b = (b_i : i \in I)$ and $e \in I$. We call $\langle a^e, b_e \rangle$ redundant if $P(A, b)$ is equal to $P(\bar{A}, \bar{b})$ where $\langle \bar{A}, \bar{b} \rangle = \langle (a^i \in \mathbb{R}^J : i \in I-\{e\}), (b_i : i \in I-\{e\}) \rangle$, otherwise we say that $\langle a^e, b_e \rangle$ is irredundant. The system $\langle A, b \rangle$ is irredundant if each of its inequalities are irredundant.

(2.2.17) Clearly $\langle a^e, b_e \rangle$ is redundant if and only if $\langle a^e, b_e \rangle$ is a valid inequality for $P(\bar{A}, \bar{b})$.

We will now characterize all valid inequalities for $P(A, b)$, this characterization is an immediate consequence of the Strong L.P. Duality Theorem.

(2.2.18) Lemma. Let $A = (a^i \in \mathbb{R}^J : i \in I)$, $b = (b_i : i \in I)$ and $P(A, b) \neq \emptyset$. Then $\langle a, \beta \rangle$ is a valid inequality for $P(A, b)$ if and only if there exists

(2.2.19) $\lambda \in \mathbb{R}_+^I$ such that $a = \sum_{i \in I} \lambda_i a^i$ and $\beta \leq \sum_{i \in I} \lambda_i b_i$.

Proof: Let $\langle a, \beta \rangle$ be a valid inequality for $P(A, b)$. Consider the primal program minimize ax over $x \in P(A, b)$, since $P(A, b) \neq \emptyset$ there is a feasible primal solution. Moreover $ax \geq \beta$ for all $x \in P(A, b)$ since $\langle a, \beta \rangle$ is a valid inequality for $P(A, b)$ hence β is a lower bound of ax over all primal solution x .

By the Strong L.P. Duality Theorem there exists an optimal dual solution $\lambda \in \mathbb{R}_+^I$ such that

$\sum_{i \in I} \lambda_i b_i = \min_{x \in P(A,b)} ax \geq \beta$, moreover $a = \sum_{i \in I} \lambda_i a^i$ because λ is dual feasible.

In the other way, let λ satisfy (2.2.19). For all $x \in P(A,b)$ we have

$ax = \sum_{i \in I} \lambda_i a^i x \geq \sum \lambda_i b_i \geq \beta$. Hence $\langle a, \beta \rangle$ is a valid inequality for $P(A, b)$. □

(2.2.20) Complementary Slackness Theorem. (Dantzig, [D1], p. 135, 136). A feasible solution x^0 to (2.2.6)-(2.2.7) and a feasible solution y^0 to (2.2.8)-(2.2.10) are optimal if and only if $y_i > 0$ implies $a^i x = b_i$ for all $i \in I$. □

2.3. Convex Sets

(2.3.1) A convex set C is a set of points such that

$$\left.\begin{array}{l} x^1 \in C,\ x^2 \in C \\ 0 \le \lambda \le 1 \end{array}\right\} \text{iff } \lambda x^1 + (1 - \lambda) x^2 \in C \ .$$

(2.3.2) A point x is called convex combination of $\{x^k : k \in K\}$, where K is finite, if $x = \sum_{k \in K} \lambda_k x^k$ for some $(\lambda_k \ge 0 : k \in K)$ such that $\sum_{k \in K} \lambda_k = 1$.

It is easy to justify this name by showing that the set of linear combinations of $\{x^k : k \in K\}$ is convex.

(2.3.3) The convex hull of a set $X \subseteq \mathbb{R}^J$, denoted by CONV(X), is the set of all convex combinations of any finite subset of X. Again it is easy to show that CONV(X) is convex.

(2.3.4) C is called a polytope if $C = \text{CONV}(X)$ where X is finite, the set X is called a generating set for C, furthermore if X is a minimal generating set for C, we say that X is a basis of C.

(2.3.5) Let C be a convex set and let $x \in C$,we call x an <u>extreme point</u> of C if there exists no two different points $x^1, x^2 \in C$ satisfying

$$x = \lambda x^1 + (1 - \lambda)x^2 , \quad 0 < \lambda < 1.$$

(2.3.6) We call C <u>pointed</u> if it has an extreme point.

It is easy to verify that the following sets are convex.

(2.3.7) The solution set of any inequality $ax \geq b$.

(2.3.8) The intersection of two convex sets.

(2.3.9) The solution set of any finite system $P(A,b)$.

(2.3.10) The <u>sum</u> of two convex sets C^1 and C^2 , where the sum $C^1 + C^2$ is defined as the set

$$\{c^1 + c^2 : c^1 \in C^1, c^2 \in C^2\}.$$

(2.3.11) A subset C of $\mathbb{R}^J$ is called a <u>convex cone</u> it it is closed under the operations of addition and multiplication by nonnegative scalars, that is,

(2.3.12) if $x, y \in C$ then $x + y \in C$,

(2.3.13) if $x \in C$ and $\lambda \in \mathbb{R}_+$ then $\lambda x \in C$.

From this definition any convex cone is convex.

<u>Examples</u>

(2.3.14) The set of all nonnegative vectors $\mathbb{R}^J_+$ is a convex cone called the <u>nonnegative orthant</u> of $\mathbb{R}^J$.

(2.3.15) For any vector $r \neq 0$ in $\mathbb{R}^J$ the set of vectors of the form λr , $\lambda \in \mathbb{R}_+$ is a convex cone, called a <u>ray</u> and denoted by $[r]$. Thus

$$[r] = \{x: x = \lambda r , \lambda \in \mathbb{R}_+\}.$$

(2.3.16) The set of all solutions x of the inequality $ax \geq 0$ is a convex cone.

(2.3.17) If C^1 and C^2 are convex cones, their sum $C^1 + C^2$ (as defined in (2.3.10)) is also a convex cone.

(2.3.18) If C^1 and C^2 are convex cones, their <u>intersection</u> $C^1 \cap C^2$ is again a convex cone.

It is easy to verify that the sets in (2.3.14) to (2.3.18) satisfy the conditions (2.3.12) and (2.3.13).

(2.3.19) A convex cone C is called a <u>finite cone</u> if it is the sum of a finite number of rays, that is,

$$C = \text{CONE}(\{r^k: 1 \leq k \leq n\}) \equiv [r^1]+\ldots+[r^n] = \{x: x = \sum_{k=1}^{n} \lambda_k r^k, \lambda_k \geq 0 \text{ for } 1 \leq k \leq n\}.$$

(2.3.20) The vectors r^1 to r^n are called <u>generators</u> of C. By definition $\text{CONE}(\emptyset) = \{0\}$.

(2.3.21) A minimal set of generators is called a (cone) <u>basis</u> of C. Any sum of a finite number of rays is a finite cone, by (2.3.15) and (2.3.17).

(2.3.22) A <u>polyhedral cone</u> C is the set of solutions of a finite system of linear homogeneous inequalities. That is, $C = \{x \in \mathbb{R}^J : a^i x \geq 0,\ i \in I\}$ where $a^i \in \mathbb{R}^J$ for all $i \in I$ and I is finite.

By (2.3.16) and (2.3.18) the polyhedral cones are convex cones. From the definition of polyhedra (see 2.2.3) we see that polyhedral cones are a special case of polyhedra.

(2.3.23) Theorem. (Gale [G1], pp. 56-58, Theorems 2.12, 2.13). C is a finite cone if and only if C is a polyhedral cone. □

Let C be a convex cone. We call $[r] \subseteq C$ an extreme ray of C if there exists no two different rays $[r^1], [r^2] \subseteq C$ such that $[r] = [r^1 + r^2]$.

(2.3.24) Theorem. (Gale [G1], p. 65, Theorem 2.16). Let $C = P(A, 0)$ be a polyhedral cone and let r be non-zero. $[r] \subseteq C$ is an extreme ray of C if and only if the set of rows a^i of A such that $a^i r = 0$ has rank $|J| - 1$. □

(2.3.25) Theorem. (Gale [G1], p. 63, Theorem 2.15). If the matrix A has rank $|J|$ then the cone $P(A, 0)$ is the sum of its extreme rays. □

(2.3.26) Corollary. With the conditions of the theorem above, the set of extreme rays of $P(A, 0)$ is the unique basis of $P(A, 0)$. Because from the definition of extreme ray they have to be in any basis and (2.3.25) says that there are enough. □

(2.3.27) P is a <u>finite convex set</u> if P is the sum of a polytope and a finite cone, i.e. for K and L finite,

$$P = \mathrm{CONV}(\{x^k : k \in K\}) + \mathrm{CONE}(\{r^e : e \in L\}).$$

The pair $(\{x^k : k \in K\}, \{r^e : e \in L\})$ is called a <u>generating set</u> for P. Let (V, R) be a generating set for P. We call (V, R) a <u>basis</u> for P if for any $V^1 \subseteq V$, $R^1 \subseteq R$ such that $V^1 \neq V$ or $R^1 \neq R$ then (V^1, R^1) is not a generating set for P.

(2.3.28) An <u>extreme ray</u> of P is an extreme ray of $\mathrm{CONE}(\{r^e : e \in L\})$.

2.4. <u>Polyhedra</u>.

Let I and J be finite sets, let $A = (a^i \in \mathbb{R}^J : i \in I)$ and let $b = (b_i : i \in I)$.

(2.4.1) A <u>polyhedron</u> P is defined as the solution set of any finite system of linear inequalities. Hence

$P = P(A, b) \equiv \{x \in \mathbb{R}^J : Ax \geq b\}$ is a polyhedron.

We take A, b, I and J to be defined as above throughout the rest of this chapter.

If there is $i \in I$ such that $a^i = 0$, then either $b_i > 0$ in which case $P(A, b) = \emptyset$ or else $b_i \leq 0$ and $a^i x \geq b_i$ for all $x \in \mathbb{R}^J$, hence $a^i x \geq b_i$ can be deleted from the system. Therefore we will henceforth assume that $a^i \neq 0$ for all $i \in I$ (that is, the matrix A has no zero rows).

(2.4.2) With any polyhedron $P(A, b)$, we associate a polyhedral cone RAY(P(A, b)) given by

$$\text{RAY}(P(A, b)) = P(A, 0).$$

Clearly if $x \in P(A, b)$ and $r \in P(A, 0)$ then $x + r \in P(A, b)$. Hence $\{x\} + \text{RAY}(P) \subseteq P$ for any x in the polyhedron P.

(2.4.3) Let P in $\mathbb{R}^J$ be a polyhedron. We define the dimension of P to be k if the largest affinely independent subset of P has cardinality $k + 1$. In view of (2.1.3) and definition (2.1.11) the dimension of $P(A, b)$ is not greater than $|J|$. We say that a polyhedron has full dimension when its dimension is $|J|$.

(2.4.4) When $0 \in P$ we have dimension of P equals rank of P, (by (2.1.3) and (2.1.12)).

We denote by $\langle x, t\rangle$ where $x \in \mathbb{R}^J$ and $t \in \mathbb{R}$, the vectors of $\mathbb{R}^J \times \mathbb{R}$.

For the rest of this chapter let $V = \{v^m \in \mathbb{R}^J : m \in M\}$ and $R = \{r^n \in \mathbb{R}^J : n \in N\}$ with M and N finite sets.

(2.4.5) For any finite convex set $Q = \text{CONV}(V) + \text{CONE}(R)$ in R^J we denote by $\bar{Q}$ the finite cone $\text{CONE}((V \times \{1\}) \cup (R \times \{0\}))$ in $\mathbb{R}^J \times \mathbb{R}$.

(2.4.6) With the association of (2.4.5), $x \in Q$ if and only if $\langle x, 1\rangle \in \bar{Q}$. Because

$$x \in Q \ \underline{\text{iff}}\ x = \sum_{m\in M} \lambda_m v^m + \sum_{n\in N} \mu_n r^n \ \text{ for some } \sum_{m\in M} \lambda_m = 1$$

$$\text{and } \lambda_m, \mu_n \geq 0 \ \underline{\text{iff}}\ \langle x, 1\rangle = \sum_{m\in M} \lambda_m \langle v^m, 1\rangle + \sum_{n\in N} \mu_n \langle r^n, 0\rangle$$

$$\text{for some } \lambda_m, \mu_n \geq 0 \ \underline{\text{iff}}\ \langle x, 1\rangle \in \bar{Q} .$$

(2.4.7) Proposition. (V, R) is a generating set of a finite convex set Q if and only if $(V\times\{1\} \cup (R\times\{0\})$ is a generating set of the finite cone $\bar{Q}$.

Proof: Let $(V\times\{1\}) \cup (R\times\{0\})$ be a generating set of $\bar{Q}$. Then $x \in Q$ iff $\langle x, 1\rangle \in \bar{Q}$ (by (2.4.6)) iff $\langle x, 1\rangle = \sum_{m\in M} \lambda_m \langle v^m, 1\rangle + \sum_{n\in N} \mu_n \langle r^n, 0\rangle$ for some λ_m, $\mu_n \geq 0$ iff $x = \sum_{m\in M} \lambda_m v^m + \sum_{n\in N} \mu_n r^n$ for some $\Sigma\lambda_m = 1$ and $\lambda_m, \mu_n \geq 0$. Therefore (V, R) is a generating set for Q.

Let (V, R) be a generating set for Q. Then $\bar{Q} = \text{CONE}((V\times\{1\}) \cup (R\times\{0\}))$ by definition, that is $(V\times\{1\}) \cup (R\times\{0\}))$ is a generating set for $\bar{Q}$. □

(2.4.8) With any polyhedron $P = P(A, b)$ in $\mathbb{R}^J$ we associate a polyhedral cone $\hat{P}$ in $\mathbb{R}^J \times \mathbb{R}$. $\hat{P}$ is the set of vectors $\langle x, t\rangle$ satisfying:

$$(2.4.9) \qquad \begin{cases} Ax - bt \geq 0 , \\ \qquad t \geq 0 . \end{cases}$$

Clearly $x \in P(A, b)$ if and only if $\langle x, 1\rangle$ satisfies (2.4.9).

(2.4.10) Theorem. The set P in $\mathbb{R}^J$ is a polyhedron if and only if P is a finite convex set. Moreover if (V, R) is a generating set for P then R is a generating set for RAY(P).

Proof: Let P be the polyhedron $P(A, b)$. Let $\hat{P}$ be defined as in (2.4.8), since $\hat{P}$ is a polyhedral cone, it is also a finite cone by Theorem (2.3.23). Let $\{\langle x^k, t_k\rangle : k \in K\}$ be a basis for $\hat{P}$, we have $t_k \geq 0$ for all $k \in K$ since $\langle x^k, t_k\rangle$ satisifes (2.4.9).

Let $M = \{k \in K: t_k > 0\}$ and $v^m = \frac{x^m}{t_m}$ for $m \in M$, let $N = K - M$ and $r^n = x^n$ for $n \in N$. Then $(V\times\{1\}) \cup (R\times\{0\})$ is a basis for $\hat{P}$ because the rays $[\langle v^m, 1\rangle]$ and $[\langle x^m, t_m\rangle]$ are equal.

Now we have $x \in P$ iff $\langle x, 1\rangle \in \hat{P}$ (by definition of $\hat{P}$) iff $\langle x, 1\rangle \in \text{CONE}((V\times\{1\}) \cup (R\times\{0\}))$ (because $(V\times\{1\}) \cup (R\times\{0\})$ is a basis of $\hat{P}$ iff $x \in \text{CONV}(V) + \text{CONE}(R)$ (by lemma (2.4.6)).

Hence P being equal to $\text{CONV}(V) + \text{CONE}(R)$ is a finite convex set. Let Q be a finite convex set and (V, R) be a basis of Q. Let $\bar{Q}$ be defined as in (2.4.5), since $\bar{Q}$ is a finite cone, it is also a polyhedral cone by Theorem (2.3.23). Let $\bar{Q}$ be the solutions $\langle x, t\rangle$ of the finite system: $\langle a^i, -b_i\rangle\langle x, t\rangle \geq 0$ for all $i \in I$.

If $\langle x, t\rangle \in \bar{Q}$ then $\langle x, t\rangle = \sum_{m\in M} \lambda_m \langle v^m, 1\rangle + \sum_{n\in N} \mu_n \langle r^n, 0\rangle$, $\lambda_m, \mu_n \geq 0$. Therefore $t = \Sigma\lambda_m \geq 0$. We obtain

(2.4.11) $\langle x, t\rangle \in \bar{Q}$ if and only if $Ax - bt \geq 0$ and $t \geq 0$.

Now we have $x \in Q$ _iff_ $\langle x, 1\rangle \in \bar{Q}$ (by lemma 2.4.6) _iff_ $Ax \geq b$ (by (2.4.11)) iff $x \in P(A, b) = \{x \in \mathbb{R}^J: Ax \geq b\}$. Let $x \in RAY(P)$. Then $Ax \geq 0$, i.e. $\langle x, 0\rangle \in \hat{P}$. Hence $\langle x, 0\rangle = \sum_{m\in M} \lambda_m \langle v^m, 1\rangle + \sum_{n\in N} \mu_n \langle r^n, 0\rangle$ for some $\lambda_m, \mu_n \geq 0$. Thus $\lambda_m = 0$: $m \in M$ since $\sum_{m\in M} \lambda_m = 0$. That is $x \in CONE(R)$. Let $x = \sum_{n\in N} \mu_n r^n$ for some $\mu_n \geq 0$. Clearly $\langle x, 0\rangle \in \hat{P}$. Thus $Ax \geq 0$, i.e. $x \in RAY(P)$.□

(2.4.12) _Lemma_. Let P be a polyhedron in $\mathbb{R}^J$ and $\hat{P}$ be defined as in (2.4.8). The following relation holds.

(2.4.13) x is an extreme point of P if and only if $[\langle x, 1\rangle]$ is an extreme ray of $\hat{P}$.

(2.4.14) $[x]$ is an extreme ray of P if and only if $[\langle x, 0\rangle]$ is an extreme ray of $\hat{P}$.

Proof: Let x not be an extreme point of P. Then $x = \lambda x^1 + (1 - \lambda)x^2$ where $0 < \lambda < 1$, $x^1 \neq x^2$ and x^1, $x^2 \in P$. Hence $\langle x^1, 1\rangle$, $\langle x^2, 1\rangle$, $\langle x, 1\rangle \in \hat{P}$ and $\langle x, 1\rangle = \lambda\langle x^1, 1\rangle + (1 - \lambda)\langle x^2, 1\rangle$, therefore we need only to prove that the rays $[\lambda\langle x^1, 1\rangle]$ and $[(1 - \lambda)\langle x^2, 1\rangle]$ are differents. Assume they are equal, that is, there exists $\theta \geq 0$ such that $\theta\lambda x^1 = (1 - \lambda)x^2$ and $\theta\lambda = (1 - \lambda)$. Then $\theta = \frac{1-\lambda}{\lambda}$ and $\frac{(1-\lambda)}{\lambda}\lambda x^1 = (1 - \lambda)x^2$. Therefore $x^1 = x^2$, absurd. If $[\langle x, 1\rangle]$ is not an extreme ray then $\langle x, 1\rangle = \langle x^1, t_1\rangle + \langle x^2, t_2\rangle$. We have $x^1 \neq x^2$, otherwise $x = 2x^1$ and $x^1 = x^2 = 0$ but then $[\langle 0, t_1\rangle] = [\langle 0, t_2\rangle]$. If $t_1, t_2 \neq 0$ then $\langle \frac{x^1}{t_1}, 1\rangle$, $\langle \frac{x^2}{t_2}, 1\rangle \in \hat{P}$ and $\frac{x^1}{t_1}, \frac{x^2}{t_2} \in P$. Therefore $x = t_1(\frac{x^1}{t_1}) + t_2(\frac{x^2}{t_2})$ and $t_1 + t_2 = 1$. If $t_1 = 0$ then $t_2 = 1$, $x^2 \in P$ and $x^1 \in RAY(P)$ since $Ax^1 - b0 \geq 0$, therefore $x^2 + 2x^1 \in P$ and $x = \frac{1}{2}x^2 + \frac{1}{2}(x^2 + 2x^1)$.

Let $[x]$ be not an extreme ray of P. Then $x = x^1 + x^2$, $[x^1] \neq [x^2]$ and $x^1, x^2 \in RAY(P)$ then $\langle x, 0\rangle = \langle x^1, 0\rangle + \langle x^2, 0\rangle$ and all belong to $\hat{P}$ since $Ax \geq 0$ implies $Ax - b0 \geq 0$.

If $\langle x, 0\rangle = \langle x^1, t_1\rangle + \langle x^2, t_2\rangle$ then $t_1, t_2 = 0$ because they are not negative, hence $Ax^1 \geq 0$ and $Ax^2 \geq 0$ hence $[x]$ is not an extreme ray of P. □

(2.4.15) <u>Theorem</u>. Let $P = P(A, b)$ be a polyhedron. Then x is an extreme point of P if and only if x is a basic solution of $P(A, b)$.

Proof: Let x be an extreme solution of P. Then $\langle x, 1\rangle$ is an extreme ray of $\hat{P}$ by (2.4.13). The set $\{\langle a^i, -b_i\rangle : a^i x - b_i = 0, i \in I\}$ has rank $|J| + 1 - 1 = |J|$ (by theorem (2.3.24)). Let $K = \{i \in I: a^i x = b_i\}$ $A^K = (a^i \in R^J: i \in K)$. Then $b_K = A^K x$ is linearly dependent of the columns of A^K, therefore $|J| = \text{rank}\{\langle a^i, -b_i\rangle: i \in K\} = \text{rank } A^K$. Being K the equality index set of x and having rank $|J|$, x is a basic solution.

Let x be a basic solution hence rank of $\{a^i\colon a^i x = b_i\}$ is $|J|$. The rank of $\{\langle a^i, -b_i\rangle\colon a^i x - b_i = 0\}$ is also $|J|$, because this set is the equality set of $\langle x, 1\rangle$ in $\hat{P}$ and $\langle x, 1\rangle \neq 0$. By Theorem (2.3.24) $\langle x, 1\rangle$ is an extreme ray of $\hat{P}$ and by (2.4.13) x is an extreme point of P. □

(2.4.16) <u>Theorem</u>. Let $P(A, b)$ be not empty and rank of A equals $|J|$. Call V the set of extreme points of $P(A, b)$ and R the set of extreme rays of $P(A, b)$. Then (V, R) is the unique basis of $P(A, b)$ and V is not empty.

Proof: Let $\hat{P}$ be the cone defined in (2.4.8), since rank of A is $|J|$ we obtain rank of $\{\langle a^i, -b_i\rangle\colon i \in I\} \cup \{\langle 0, 1\rangle\}$ is $|J| + 1$. By Theorem (2.3.25) and its corollary (2.3.26) the set E of extreme rays of $\hat{P}$ is the unique basis of $\hat{P}$. Since $t \geq 0$ for any $\langle x, t\rangle \in \hat{P}$ we can take the rays of $\hat{P}$ to be of the form $[\langle x, 1\rangle]$ or $[\langle x, 0\rangle]$ as in the proof of theorem (2.4.10). Let $V = \{x\colon \langle x, 1\rangle \in E\}$ and $R = \{x\colon \langle x, 0\rangle \in E\}$. Hence $E = (V\times\{1\}) \cup (R\times\{0\})$ and (V, R) is the unique basis of $P(A, b)$ by corollary (2.4.7) and E being unique basis of $\hat{P}$. By lemma (2.4.12) V is the set of extreme points of $P(A, b)$ and R is the set of extreme rays of $P(A, b)$.

If V is empty then $t = 0$ for any $\langle x, t\rangle \in \hat{P}$. But since $P(A, b)$ is not empty, there is an $x \in P(A, b)$ and $\langle x, 1\rangle \in \hat{P}$. □

(2.4.17) *Corollary*. $P(A, b)$ is pointed if and only if rank of A equals $|J|$ and $P(A, b)$ is not empty.

Proof: Let $P(A, b)$ be pointed. Then there is an extreme point x of $P(A, b)$ and $P(A, b)$ is not empty, moreover rank of $A = |J|$ since x is a basic solution (by theorem 2.4.15).

Let $P(A, b) \neq \emptyset$ and rank of A equals $|J|$, by the theorem above V is not empty. Therefore $P(A, b)$ is pointed. □

2.5. Commutative Semigroups

(2.5.1) A (finite commutative) *semigroup* is the ordered pair $(S, \underset{\sim}{+})$ where S is a non-empty finite set and $\underset{\sim}{+}$ is a function from S^2 in S (we denote $s \underset{\sim}{+} p \equiv \underset{\sim}{+}(s, p)$ for all $s, p \in S$), with $\underset{\sim}{+}$ satisfying:

associativity: $(s \underset{\sim}{+} p) \underset{\sim}{+} q = s \underset{\sim}{+}(p \underset{\sim}{+} q)$ for all $s, p, q \in S$.

commutativity: $s \underset{\sim}{+} p = p \underset{\sim}{+} s$ for all $s, p \in S$.

(2.5.2) The *order* of the semigroup $(S, \underset{\sim}{+})$ is the cardinality of S, when S is finite we call the semigroup *finite*.

(2.5.3) The *identity or zero* o of the semigroup is an element satisfying

$$s \underset{\sim}{+} o = s \quad \text{for all } s \in S .$$

Clearly such element, if there is one, is unique. When $(S, \underset{\sim}{+})$ doesn't have an identity we can always adjoin one element $o \notin S$ defining $s \underset{\sim}{+} o = s$ for all $s \in S$ and $o \underset{\sim}{+} o = o$ and the pair $(S \cup \{o\}, \underset{\sim}{+})$ will be a semigroup with identity o. Therefore we will denote by o the identity of $(S, \underset{\sim}{+})$ when it has one or this new element added in the way explained above.

(2.5.4) The <u>inverse</u> of an element s in S is another element p in S, when there is one, such that $s \underset{\sim}{+} p = o$, clearly s can have at most one inverse. When all the elements have inverses, the semigroup is called a <u>group</u>.

(2.5.5) For $s, p \in S$ we define $s \sim p$ to be the set of solutions x to the equation $p \underset{\sim}{+} x = s$, i.e. $s \sim p \equiv \{x \in S: p \underset{\sim}{+} x = s\}$.

(2.5.6) For any non-negative integer k and any s in S we define <u>$k \cdot s$</u> by the recursion:

$$k \cdot s = \begin{cases} o & \text{when } k = 0 \\ (k-1) \cdot s \underset{\sim}{+} s & \text{when } k > 0 \end{cases}$$

(2.5.7) Since we are considering only finite semigroups the sequence $0 \cdot s, 1 \cdot s, 2 \cdot s, \ldots$ has only finitely many different elements for any s in S, the <u>order of s</u>, written <u>$\circ(s)$</u>, is the minimum integer which repeats one element in the sequence, i.e.

$$\circ(s) = \min_{k \geq 0}\{k: \text{ there is } e < k \text{ such that } e \cdot s = k \cdot s\} .$$

The <u>loop of s</u> is the set $\{p \in S: p = k\cdot s,\ k \geq \circ(s)\}$, when s belongs to its loop we call s a <u>loop element</u>. It is easy to see that s is a loop element if and only if there is a $k > 0$ such that $s = s \underset{\sim}{+} k\cdot s$.

(2.5.8) We denote by $\underset{\sim}{\Sigma}$ the iteration of the operation $\underset{\sim}{+}$, that is for any succession $s^1, s^2, \ldots, s^k$ of elements from S,

$$\underset{j\in\{1,\ldots,k\}}{\underset{\sim}{\Sigma}} s^j \text{ denotes } s^1 \underset{\sim}{+} s^2 \underset{\sim}{+} \cdots \underset{\sim}{+} s^k .$$

(2.5.9) Let $H \subseteq S$ and $t \in \mathbb{N}_+^H$. We say that <u>t represents</u> s if $\underset{h\in H}{\underset{\sim}{\Sigma}} t_h \cdot h = s$. The <u>representation function</u> <u>θ_H</u>: $\mathbb{N}_+^H \to S$ is defined as $\theta_H(t) = \underset{h\in H}{\underset{\sim}{\Sigma}} t_h \cdot h$.

(2.5.10) Since $(S, \underset{\sim}{+})$ is associative and commutative $\underset{h\in H}{\underset{\sim}{\Sigma}} t_h \cdot h + \underset{h\in H}{\Sigma} t_h^1 \cdot h$ equals $\underset{h\in H}{\underset{\sim}{\Sigma}} (t_h + t_h^1)\cdot h$, that is <u>$\theta_H(t + t^1) = \theta_H(t) \underset{\sim}{+} \theta_H(t^1)$</u>.

(2.5.11) <u>Substitution Lemma</u>. Let $t, t', t'' \in \mathbb{N}_+^H$ be such that $t' \leq t$ and $\theta_H(t') = \theta_H(t'')$. Then $\theta_H(t - t' + t'') = \theta_H(t)$.

Proof: Since $t - t' \in \mathbb{N}_+^H$ we have

$$\theta_H(t - t' + t'') = \theta_H(t - t') \underset{\sim}{+} \theta_H(t'') = \theta_H(t - t') \underset{\sim}{+} \theta_H(t') = \theta_H(t)$$

by (2.5.10), the hypothesis and (2.5.10) again. □

The substitution lemma is useful because it allows us to replace a portion of a vector t by another representing the same element, in particular if $t_s > 0$ we can subtract δ^s from t and add to the vector t any representation of s without changing the semigroup element represented.

(2.5.12) If a subset R of S is closed under $\underset{\sim}{+}$, that is $r \underset{\sim}{+} h \in R$ for all $r, h \in R$, then $(R, \underset{\sim}{+})$ is a semigroup called <u>a subsemigroup of $(S, \underset{\sim}{+})$.</u>

(2.5.13) <u>Lemma</u>. Let $(S, \underset{\sim}{+})$ be a semigroup with identity o and H be a non-empty subset of S. Then the pair $(R, \underset{\sim}{+})$, where $R = \theta_H(\mathbb{N}_+^H)$, is a subsemigroup of $(S, \underset{\sim}{+})$ and o is represented by $\theta_H(0)$. Moreover if $(H, \underset{\sim}{+})$ is a subsemigroup of $(S, \underset{\sim}{+})$ then $R = H \cup \{o\}$.

The proof is trivial from the definitions of θ_H and $k \cdot s$. □

(2.5.14) Let $(S, \underset{\sim}{+})$ be a semigroup and H a subset of S. Then the <u>semigroup generated by H</u> is the semigroup $(\theta_H(\mathbb{N}_+^H), \underset{\sim}{+})$. That it is a semigroup follows from lemma (2.5.13). H is a <u>generating set</u> for $(S, \underset{\sim}{+})$ when the semigroup generated by H is $(S, \underset{\sim}{+})$, a minimal generating set is a <u>basis</u>. A <u>cyclic semigroup</u> is one having a basis of cardinality one.

Chapter 3

POLARITY OF POLYHEDRA

Given any symmetric relation Ω on a set X, the polar of any set P in X is defined as the set $P^{\Omega} = \{y \in X: x\Omega y \text{ for all } x \in P\}$. Several polarities are defined in polyhedral theory and the main properties studied are the closed polyhedra (P is closed when $P = P^{\Omega\Omega}$) and the relation between the defining irredundant systems and the basis for a closed polyhedron and the basis and the defining irredundant systems for its polar.

These properties are well-known for the polarity given by the relation $xy \leq 0$. It is used in Gale [G1] to obtain the theorems cited in chapter 2.

Another polarity is given by $xy \leq 1$, called "polarity of convex sets" (see Rockafellar [R1] or Stoer and Witzgall [S2]). A natural question is what happens with the polarity given by the relation $xy \geq 1$? Although some special cases have been treated elsewhere, as in Gomory [G5] and Fulkerson [F1], we have not seen a general solution elsewhere.

In section 3.3 we solve the main questions about this polarity. In section 3.1 we give simple properties of any polarity (we use symmetric relations because it is the only type we will consider). Section 3.2 is devoted to the polarity given by the relation $xy \geq 0$. In section 3.4 we study the polarity given by the relation $xy \leq 1$ using the same approach as in section 3.3.

3.1 Polarity Defined by a Relation

Let X be a given set and a symmetric relation $\Omega \subseteq X \times X$. We denote $(x, y) \in \Omega$ by $x\Omega y$.

For any set $T \subseteq X$ the polar (respect to Ω) of T is the set T^Ω given by

(3.1.1) $T^\Omega = \{y \in X: \ x\Omega y \ \text{ for all } \ x \in T\}$.

Let T and T' be both subsets of X. Then we have

(3.1.2) If $T \subseteq T'$ then $T^\Omega \supseteq T'^\Omega$. Clearly since if $y \in T'^\Omega$ then $x\Omega y$ for all $x \in T'$, in particular $x\Omega y$ for all $x \in T \subseteq T'$, hence $y \in T^\Omega$.

(3.1.3) $T^{\Omega\Omega}$ always contains T , because for any $x \in T$ and all $y \in T^\Omega$ we have $x\Omega y = y\Omega x$ (Ω is symmetric) and therefore $x \in (T^\Omega)^\Omega$.

(3.1.4) Lemma. We always have $T^{\Omega\Omega\Omega}$ equal to T^Ω.

Proof: We have $(T^\Omega)^{\Omega\Omega} \supseteq T^\Omega$ by (3.1.3).

Let $T' = T^{\Omega\Omega}$, hence $T \subseteq T'$ by (3.1.3), using (3.1.2) we obtain $T^\Omega \supseteq T'^\Omega = T^{\Omega\Omega\Omega}$. Therefore $T^\Omega = T^{\Omega\Omega\Omega}$. □

(3.1.5) Let $C \subseteq X$, we say that C is Ω-closed if $C = C^{\Omega\Omega}$.

(3.1.6) Lemma. $C \subseteq X$ is Ω-closed if and only if there exists $T \subseteq X$ such that $C = T^\Omega$;

Proof: If $C = T^\Omega$ then $C^{\Omega\Omega} = T^{\Omega\Omega\Omega} = T^\Omega = C$ by (3.1.4).

If $C = C^{\Omega\Omega}$ let $T = C^\Omega$. Hence $C = C^{\Omega\Omega} = T^\Omega$. □

3.2 Cone Polarity

(3.2.1) Consider the polarity given by the relation $x\gamma y \equiv xy \geq 0$. For any set Q in $\mathbb{R}^J$, the polar Q^γ is a convex cone, we call Q^γ the polar cone of Q.

(3.2.2) Lemma. Let C be the polyhedral cone, say $C = P(A, 0)$. The polar cone of C is CONE($\{a^i: a^i$ is a row of $A\}$).

Proof: Let $A = (a^i \in \mathbb{R}^J: i \in J)$. We have

$y \in C^\gamma$ iff $yx \geq 0$ for all $x \in C$ iff $\langle y, 0\rangle$ is a valid inequality for $P(A, 0)$

(by definition of valid inequality (2.2.15)) iff there exists $\lambda \in \mathbb{R}^I_+$ such that $y = \sum_{i \in I} \lambda_i a^i$ (lemma 2.2.18)

iff $y \in$ CONE($\{a^i: i \in I\}$) (by definition of CONE(2.3.19)). □

(3.2.3) Lemma: Let $A = (a^i \in \mathbb{R}^J: i \in I)$ where I is finite. Let C be the finite cone CONE($\{a^i: i \in I\}$). The polar cone of C is $P(A, 0)$.

Proof: Let $y \in P(A, 0)$. Then $y(\sum_{i \in I} \lambda_i a^i) = \sum_{i \in I} \lambda_i a^i y \geq \sum_{i \in I} \lambda_i 0 = 0$ if $\lambda \in \mathbb{R}^I_+$, hence $y \in C^\gamma$.

Let $y \notin P(A, 0)$. Then there is $i \in I$ such that $a^i y < 0$, but $a^i \in C$, hence $y \notin C^\gamma$. □

(3.2.4) Corollary. Polyhedral cones are γ-closed. □

(3.2.5) Corollary. If C is a polyhedral cone then its polar cone C^γ is also a polyhedral cone. Because C is also a finite cone by theorem (2.3.23), therefore C^γ is a polyhedral cone by the lemma above. □

(3.2.6) Theorem. Let C be a polyhedral cone. Then we have $\langle (a^i: i \in I), 0 \rangle$ is a defining irredundant system for C if and only if $\{a^i: i \in I\}$ is a cone-basis for C^γ.

Proof: $\langle (a^i: i \in I), 0 \rangle$ is a defining system for C if and only if $\{a^i: i \in I\}$ is a generating set for C^γ by lemmas (3.2.2) and (3.2.3). Hence $\langle (a^i: i \in I), 0 \rangle$ is a defining irredundant system for C if and only if for all $e \in I$ there is no $(\lambda_i \geq 0: i \in I - \{e\})$ such that $a^e = \sum_{i \in I-\{e\}} \lambda_i a^i$ (by lemma (2.2.18)) if and only if $\{a^i: i \in I\}$ is a minimal generating set for C^γ, i.e. $\{a^i: i \in I\}$ is a cone-basis for C^γ .□

The next lemma is used in section 3.3. We recall that dimension of a polyhedron is the maximum number of affinely independent points in the polyhedron minus one (see (2.4.3)).

(3.2.7) Lemma. Let C be the polyhedral cone $P(A, 0)$. Then rank of A equals dimension of C^γ.

Proof: Since $0 \in C^\gamma$ then dimension of C^γ equals rank of C^γ by (2.4.4). But rank of A equals rank of C^γ by (4.2.2). □

3.3 β-Polarity of Polyhedra

(3.3.1) Consider the polarity given by the relation $x\beta y \equiv xy \geq 1$.

(3.3.2) Clearly, for any set Q in $\mathbb{R}^J$ containing 0 the β-polar $Q^\beta = \emptyset$.

Also the β-polar of the empty set is $\mathbb{R}^J$. Hence $\emptyset$ and $\mathbb{R}^J$ are β-closed, and no set different from $\mathbb{R}^J$ and containing 0 is β-closed. We call $\emptyset$ and $\mathbb{R}^J$ the <u>trivial polyhedra</u>.

(3.3.3) The statements "$y \in P^\beta$" and "$\langle y, 1\rangle$ is a valid inequality for P" are equivalent. Because $y \in P^\beta$ iff $xy \geq 1$ for all $x \in P$ iff $\langle y, 1\rangle$ is a valid inequality for P by definition of valid inequality (2.2.15).

Our intention is to characterize the β-closed polyhedra and give the relation between defining systems and representing sets for these polyhedra and representing sets and defining systems for their β-polars.

(3.3.4) We define δ^K for any $K \subseteq I$ as the vector of R^I with components $\delta_i^K = 1$ if $i \in K$ and $\delta_i^K = 0$ if $i \in I - K$. We denote $\delta^{\{i\}}$ by δ^i for any $i \in I$.

Through the rest of this chapter we will need several times to consider two finite sets of vectors and a matrix formed using as rows the vectors of those sets. To avoid repetitions we will reserve the letters V and R to indicate the finite sets and consider always $V = \{v^m : m \in M\}$ and $R = \{r^n : n \in N\}$ where M and N are finite disjoint index sets. VR will denote the matrix $(a^i : i \in M \cup N;\ a^i = v^i,\ i \in M;\ a^i = r^i,\ i \in N)$.

(3.3.5) <u>Lemma</u>. For any non-empty polyhedron P if (V, R) is a representing set for P then
$P^\beta = P(VR, \delta^M) \equiv \{x:\ VRx \geq \delta^M\}$.

Proof: Let $x \in P$ and $y \in P(VR, \delta^M)$. There exist $(\lambda_m : m \in M)$ and $(\mu_n : n \in N)$ both non-negative and $\sum_{m\in M} \lambda_m = 1$ such that

$$x = \sum_{m\in M} \lambda_m v^m + \sum_{n\in N} \mu_n r^n .$$

Hence

$$xy = \sum_{m\in M} \lambda_m v^m y + \sum_{n\in N} \mu_n r^n y \geq \Sigma\lambda_m 1 + \Sigma\mu_n 0 = 1 \text{ (since } y \in P(VR,\delta^M)).$$

Therefore we have

$$P(VR, \delta^M) \subseteq P^\beta .$$

Let $y \notin P(VR, \delta^M)$. Then there is $e \in M \cup N$ such that $a^e y < \delta^M_e$. If $e \in M$ then $a^e = v^e \in P$ and $\delta^M_e = 1$, therefore $v^e y < 1$ and $y \notin P^\beta$. Otherwise $e \in N$, $a^e = r^e$ and $\delta^M_e = 0$, that is $r^e y < 0$. Let $x \in P$ (there is one since $P \neq \emptyset$). $x + \alpha r^e \in P$ for all $\alpha \in \mathbb{R}_+$. We have $xy + \alpha r^e y < 1$ where $\alpha > \frac{1-xy}{r^e y}$. Hence $y \notin P^\beta$. From this and $P(VR, \delta^M) \subseteq P^\beta$ we obtain $P^\beta = P(VR, \delta^M)$.□

(3.3.6) <u>Corollary</u>. If P is a polyhedron so is P^β. Because P is also a finite convex set by Theorem (2.4.10), therefore P^β is a polyhedron by the lemma above.□

(3.3.7) Let **P** be the family of non-trivial β-closed polyhedra in $\mathbb{R}^J$.

(3.3.8) <u>Lemma</u>. Let P be a non-trivial polyhedron. Then P is β-closed if and only if there is a system $\langle A, \delta^M \rangle$ where $M \neq \emptyset$ and such that $P = P(A, \delta^M) \equiv \{x: Ax \geq \delta^M\}$.

Proof: Let $P = P((a^i: i \in I), \delta^M)$ and $M \neq \emptyset$. Let $Q = CONV(\{a^i: i \in M\}) + CONE(\{a^i: i \in I-M\})$, Q is not empty because M is not empty. By lemma (3.3.5) we have $P = Q^\beta$, hence by lemma (3.1.6) P is β-closed.

Let P be β-closed, that is $P = (P^\beta)^\beta$. P^β is a polyhedron(by (3.3.6)), moreover P^β is non-trivial because P is non-trivial (see (3.3.2)). Let (V, R) be a basis of P^β and $A = VR$, by lemma (3.3.5) $P = P(A, \delta^M)$.□

(3.3.9) Lemma. Let $P = P(VR, \delta^M)$ be a non-trivial β-closed polyhedron. Then $P^\beta = CONV(V) + CONE(V \cup R)$.

Proof: Let $y \in P^\beta$, then $\langle y, 1\rangle$ is a valid inequality for $P(VR, \delta^M)$. By lemma (2.2.18) there are vectors $(\lambda_m \in \mathbb{R}_+: m \in M)$ and $(\mu_n \in \mathbb{R}_+: n \in N)$ such that

$$y = \sum_{m \in M} \lambda_m v^m + \sum_{n \in N} \mu_n r^n$$

and $$\alpha \equiv \sum \lambda_m \geq 1 .$$

We can write y as

$$y = \sum_{m \in M} \frac{\lambda_m}{\alpha} v^m + \sum_{m \in M} (\lambda_m - \frac{\lambda_m}{\alpha}) v^m + \sum_{n \in N} \mu_n r^n ,$$

here $\sum_{m \in M} \frac{\lambda_m}{\alpha} = 1$ and $\lambda_m - \frac{\lambda_m}{\alpha} = \lambda_m (1 - \frac{1}{\alpha}) \geq 0$ because $\lambda_m \geq 0$ and $\alpha \geq 1$. Therefore $y \in CONV(V) + CONE(V \cup R)$.

In the other way if

$$y = \sum_{m \in M} \lambda_m v^m + \sum_{m \in M} \mu_m v^m + \sum_{n \in N} \mu_n r^n , \text{ where } \lambda_m, \mu_m \geq 0: m \in M,$$

$\mu_n \geq 0$: $n \in N$ and $\sum_{m \in M} \lambda_m = 1$ then

$y = \sum_{m \in M} (\lambda_m + \mu_m) v^m + \sum_{n \in N} \mu_n r^n$, where $\lambda_m + \mu_m \geq 0 : m \in M$,

$\mu_n \geq 0$: $n \in N$ and $\sum_{m \in M} (\lambda_m + \mu_m) \geq 1$. By Lemma (2.2.18)

$\langle y, 1\rangle$ is a valid inequality for $P(VR, \delta^M)$, hence

$y \in P^\beta$. □

We recall that when $P = P(A, b)$, RAY(P) is $P(A, 0)$ by definition and when (V, R) is a generating set for P , RAY(P) is CONE(R) by theorem (2.4.10).

(3.3.10) <u>Corollary</u>. Let $P = P(VR, \delta^M)$ belong to **P** . The following equality holds, $(RAY(P))^\gamma = RAY(P^\beta)$.

Proof: We have

$(RAY(P))^\gamma = (P(VR, 0))^\gamma = CONE(V \cup R)$ by (3.2.2)

and $RAY(P^\beta) = RAY(CONV(V) + CONE(V \cup R)) = CONE(V \cup R)$ by (3.3.9) and (3.4.10).□

Using (3.3.5) to (3.3.9) we obtain the following theorem which characterizes the non-trivial β-closed polyhedra.

(3.3.11) <u>Theorem</u>. Let P be a non-trivial polyhedron in $\mathbb{R}^J$. Then the following statements are equivalent,

(3.3.12) P is β-closed.

(3.3.13) P is equal to some $P(A, \delta^M)$ where M is not empty.

(3.3.14) P is contained in RAY(P) and $0 \notin P$.

Proof: The equivalence of (3.3.13) and (3.3.12) is lemma

(3.3.8).

(3.3.13) implies (3.3.14) because if $x \in P(A, \delta^M)$ then $Ax \geq 0$ and $x \in RAY(P(A, \delta^M))$. Moreover $0 \notin P$ since P is β-closed and non-trivial (see (3.3.2)).

We will prove that (3.3.14) implies (3.3.12).

Let (V, R) be a basis for P and $P \subseteq RAY(P)$, $0 \notin P$. Since P is non-trivial and $0 \notin P$ we have P^{β} is non-trivial (see (3.3.2)).

By lemma (3.3.5) $P^{\beta} = P(VR, \delta^M)$ and M is not empty because P is non-empty and (V, R) is a basis of P.

By lemma (3.3.9), since P^{β} is β-closed, we have

$$P^{\beta\beta} = CONV(V) + CONE(V \cup R).$$

We only need to show that $CONE(V \cup R) = CONE(R)$ to prove that $P^{\beta\beta} = P$, because (V, R) is a basis for P, that is $P = CONV(V) + CONE(R)$. But by theorem (2.4.10) $RAY(P) = CONE(R)$, hence since $V \subseteq P \subseteq RAY(P) = CONE(R)$ we obtain $CONE(R) = CONE(R \cup V)$.□

(3.3.16) <u>Corollary</u>. Let $P \in \mathbf{P}$ and (V, R) be a basis for P. Then $\langle v^m, 1\rangle: m \in M$ are irredundant in $\langle A, \delta^M\rangle$, where $A = VR$.

Proof: Let $\langle v^e, 1\rangle$ for some $e \in M$ be redundant in $\langle A, \delta^M\rangle$. Hence $P^{\beta} = P(A, \delta^M) = P((a^i: i \in M \cup N - \{e\}), \delta^{M-\{e\}})$ by (3.3.5) and (2.2.16). Therefore $P = CONV(V - \{v^e\}) + CONE((a^i: i \in M \cup N - \{e\}))$. But $V \subseteq CONE(R)$ implies $CONE((\mathbf{a}^i: i \in M \cup N - \{e\})) = CONE(R)$

and (V, R) is not a basis for P. □

(3.3.17) <u>Corollary</u>: Let $P = P(A, b)$ belongs to **P** . Then dimension of P^{β} equals rank of A.

Proof: Clearly P^{β} is non-trivial because P is non-trivial. Hence

(3.3.18) dimension of P^{β} equals dimension of $RAY(P^{\beta})$,

because $P^{\beta} \subseteq RAY(P^{\beta})$ (by theorem (3.3.11)) and for any $x \in P^{\beta}$, $\{x\} + RAY(P^{\beta}) \subseteq P^{\beta}$, therefore if $\{x^k \in RAY(P^{\beta}): k \in K\}$ is affinely independent then $x + x^k \in P^{\beta}$ for all $k \in K$ and $\{x + x^k: k \in K\}$ is affinely independent.

Also we have $RAY(P^{\beta}) = (P(A, 0))^{\gamma}$ (by (3.3.10)) and dimension of $P(A, 0))^{\gamma}$ equals rank of A (by Lemma (3.2.7)). □

From (2.4.17) and (3.3.17) we obtain:

(3.3.19) Let P be a non-trivial β-closed polyhedron. Then P is pointed if and only if P^{β} is full dimension.

(3.3.20) <u>Lemma</u>: Let $P \in$ **P** be pointed and V, R be the sets of extreme points and extreme rays of P respectively. For any $r^n \in R$, $<r^n, 0>$ is redundant in $<VR, \delta^M>$ if and only if $v^m \in [r^n]$ for some $m \in M$.

Proof: The "if" part is trivial.

Let $e \in N$ and $<r^e, 0>$ be redundant in $<VR, \delta^M>$. Let $K = N - \{e\}$.

By lemma (2.2.18) there exist vectors $(\lambda_m \in \mathbb{R}_+: m \in M)$

and $(\mu_k \in \mathbb{R}_+ : k \in K)$ such that

$$(3.3.21) \qquad r^e = \sum_{m \in M} \lambda_m v^m + \sum_{k \in K} \mu_k r^k .$$

Since $V \subseteq P \subseteq \text{RAY}(P) = \text{CONE}(R)$ (by theorems (3.3.11) and (2.4.10)) there exists vectors $(\gamma^m \in \mathbb{R}_+^N : m \in M)$ such that

$$(3.3.22) \qquad v^m = \sum_{n \in N} \gamma_n^m r^n .$$

Reemplacing (3.3.22) in (3.3.21) and regrouping we obtain

$$r^e = \sum_{k \in K} (\mu_k + \sum_{m \in M} \lambda_m \gamma_k^m) r^k + (\sum_{m \in M} \lambda_m \gamma_e^m) r^e .$$

Let $\beta_k = \mu_k + \sum_{m \in M} \lambda_m \gamma_k^m : k \in K$ and $\beta_e = -1 + \sum_{m \in M} \lambda_m \gamma_e^m$.

Then

$$(3.3.23) \qquad \sum_{n \in N} \beta_n r^n = 0 \quad , \text{ where } \beta_n \geq 0 \text{ for } n \in K.$$

β_e cannot be less than zero; otherwise $r^e = \sum_{k \in K} \frac{\beta_k}{-\beta_e} r^k$, $\frac{\beta_k}{-\beta_e} \geq 0 : k \in K$ and r^e is not an extreme ray.

$\beta_n : n \in N$ cannot be greater than zero because then $r = \sum_{k \in N - \{n\}} \frac{\beta_k}{\beta_n} r^k \in \text{RAY}(P)$ and $r^n = -r$. Therefore for any $x \in P$ we have $x + r^n, x + r \in P$, hence $\frac{1}{2}(x + r^n) + \frac{1}{2}(x + r) = x + \frac{1}{2} r^n + \frac{1}{2} r = x$ and $x + r^n \neq x + r$ ($r^n \neq 0$ since r^n is an extreme ray of P), that is P is not pointed.

Then β_n is equal to zero for all $n \in N$.

Since $\beta_k = \mu_k + \sum_{m \in M} \lambda_m \gamma_k^m = 0 : k \in K$ and all the values are non-negative we obtain $\mu_k = 0 : k \in K$ and

$\gamma_k^m = 0$: $k \in K$ for all $\lambda_m > 0$. Combining this with (3.3.21) we have $r^e = \sum_{m\in M} \lambda_m v^m$ and at least for one $m \in M$, $\lambda_m > 0$ because $r^e \neq 0$. Let $\lambda_m > 0$, then $v^m = \sum_{n\in N} \gamma_n^m r^n = \gamma_e^m r^e$ since $\gamma_n^m = 0$: $n \in N - \{e\}$. Therefore $v^m \in [r^e]$. □

We have now all the elements to prove the main theorem of this chapter.

(3.3.24) <u>Theorem</u>. Let $P \in \mathbf{P}$ be full dimension. Then $<(a^i: i \in I), \delta^M>$ is an irredundant system for P if and only if

(3.3.25) $V = \{a^i: i \in M\}$ is the set of extreme points of P^β,

and

(3.3.26) $\{a^i: i \in I - M\} = R - T$ where R is the set of extreme rays of P^β and $T = \{r \in R: a^i \in [r]$ for some $i \in M\}$.

Proof: By (3.3.19) P^β is pointed since $P = P^{\beta\beta}$ is full dimension. (V, R) is the unique basis of P^β (by (2.4.16)).

"if Part": any $<a^i, 1>: i \in M$ is irredundant (by (3.3.16)) in $<VR, \delta^M>$. Hence it is irredundant in $<(a^i: i \in I), \delta^M>$ which has less rows than $<VR, \delta^M>$.

That $<a^i, 0>: a^i \in R - T$ is irredundant is given by lemma (3.3.20). By (3.3.5) $P = P(VR, \delta^M)$ and by definition of redundant inequality (see (2.2.16)) $P(VR, \delta^M) = P((a^i: i \in I), \delta^M)$.

"only if" part: Let $\langle A, \delta^M\rangle$ be an irredundant system for P. It exists by theorem (3.3.11). Let $X = \{a^i: i \in M\}$ and $Y = \{a^i: i \in I - M\}$. $(X, X \cup Y)$ is a generating set for P^β (Lemma (3.3.9)).

Since (V, R) is the unique basis of P^β, we have $V \subseteq X$ and $R \subseteq X \cup Y$. If $y \in X - V$ then $\langle y, 1\rangle$ is redundant by lemma (2.2.18) because $y \in \text{CONV}(V) + \text{CONE}(R)$. Therefore $V = X$ and X satisfies (3.3.25).

Now if $r \in Y - R$ then $\langle r, 0\rangle$ is redundant because $r \in \text{CONE}(R)$ (apply Lemma (2.2.18)). Hence $R \supseteq Y$. By Lemma (3.3.20) $\langle r, 0\rangle: r \in R$ is redundant only if $r \in T$ hence $Y = R - T$ and Y satisfies (3.3.26). □

(3.3.27) Let's call $\underline{\mathbf{P}(C)}$, where C is a cone in $\mathbb{R}^J$, the family $\{P \in \mathbf{P} : \text{RAY}(P) = C\}$.

We have shown that with each $P \in \mathbf{P}(C)$ the β-polarity associates a unique $P^\beta \in \mathbf{P}(C^\gamma)$. In particular if C is pointed and full dimension, so are P, P^β and C^γ. For these families we have characterized their defining irredundant systems in terms of extreme points and extreme rays of the polars.

Of special interest is the family $\mathbf{P}(\mathbb{R}^J_+)$. $\mathbb{R}^J_+$ has the properties: it is pointed, full dimension and $(\mathbb{R}^J_+)^\gamma = \mathbb{R}^J_+$. Hence $P^\beta \in \mathbf{P}(\mathbb{R}^J_+)$ when $P \in \mathbf{P}(\mathbb{R}^J_+)$. The pairs $(P, P^\beta): P \in \mathbf{P}(\mathbb{R}^J_+)$ are what Fulkerson ([F1]) calls blocking pairs.

For any given cone $C \subseteq \mathbb{R}^J$, define $x \leq^c y$ for $x, y \in C$, to mean $y - x \in C$. Clearly "$\leq^c$" is a partial order relation. Notice that "$x \leq^c y$" is equivalent to "$x \leq y$" when C is $\mathbb{R}^J_+$.

(3.3.28) <u>Lemma</u>. Let $P \in \mathbf{P}(C)$. If x is an extreme point of P, then x is a minimal member of P relative to the order $\leq^c$.

Proof: Let $x, x' \in P$, $x' \leq^c x$ and $x' \neq x$. Then $r = x - x' \in C$ and $r \neq 0$. Therefore $x + r \in P$ since $RAY(P) = C$. But then $x = \frac{1}{2}x' + \frac{1}{2}(x + r)$ and x is not an extreme point of P. □

(3.3.29) <u>Theorem</u>. Let P be a polyhedron in $\mathbf{P}(C)$, where C is a full dimension polyhedral cone.

Let Q be a polyhedron contained in P^β and let the extreme points of P^β be contained in Q.

Then the system of inequalities

(3.3.30) $\begin{cases} vx \geq 1 & \text{for all extreme point } v \text{ of } Q, \\ rx \geq 0 & \text{for all extreme ray of } C^\gamma. \end{cases}$

is a defining system for P.

Moreover, when v' is an extreme point of Q, the inequality $v'x \geq 1$ is redundant in (3.3.30) if and only if there is $y \in Q$ such that $y \leq^{c^\gamma} v'$ and $y \neq v'$.

Proof: Let V be the set of extreme points of P^β.

Let V' denote the set of extreme points of Q.

Then we have

(3.3.31) $$V \subseteq V',$$

because V is contained in Q and for any $v \in V$ if there are $y^1, y^2 \in Q$ such that $v = \lambda y^1 + (1 - \lambda)y^2$ for some λ satisfying $0 < \lambda < 1$ then $y^1 = y^2 = v$ since y^1, $y^2 \in Q \subseteq P^\beta$ and v is an extreme point of P^β. Therefore v is an extreme point of Q, that is $v \in V'$.

Let R be the set of extreme rays of P^β. Then R is the set of extreme rays of C^γ because $RAY(P^\beta) = (RAY(P))^\gamma$ by (3.3.10) and (RAY(P) = C (since $P \in \mathbf{P}(C)$).

Therefore (V', R) is a generating set of P^β because (V, R) is a basis for P^β and $V \subseteq V' \subseteq P^\beta$. Then (3.3.30) is a defining system for $P = P^{\beta\beta}$ (by lemma (3.3.5)).

If there is v' in V' and y in Q such that $y \leq^{C^\gamma} v'$ and $y \neq v'$ then v' is not an extreme point of P^β because $y \in P^\beta$ and using lemma (3.3.28). Therefore the inequality $v'x \geq 1$ is redundant in (3.3.30) by theorem (3.3.24).

Let $v' \in V'$ and let the inequality $v'x \geq 1$ be redundant in (3.3.30).

Let $V'' = V' - \{v'\}$.

By Lemma (2.2.18) we have

$$v' = \sum_{v \in V''} \lambda_v v + \sum_{r \in R} \mu_r r \quad \text{and} \quad \alpha \equiv \sum_{v \in V''} \lambda_v \geq 1$$

for some $\lambda_v, \mu_r \geq 0$.

Let $y = \sum_{v \in V''} \frac{\lambda_v}{\alpha} v$, clearly $y \in Q$ since y is a convex combination of points in Q.

y is different from v', otherwise v' would not be an extreme point of Q because y is a convex combination of points of $V'' = V' - \{v'\}$.

Therefore we only need to show that $y \leq^{C^{\gamma}} v'$ in order to complete the proof of the theorem.

That is, we need to show that $v' - y \in C^{\gamma}$.

Since $v' - y = \sum_{v \in V''} \lambda_v (1 - \frac{1}{\alpha}) v + \sum_{r \in R} \mu_r r$, $v' - y$ belongs to CONE($V'' \cup R$) because $\lambda_v \geq 0$ for all $v \in V''$, $\mu_r \geq 0$ for all $r \in R$ and $1 - \frac{1}{\alpha} \geq 0$ (being $\alpha \geq 1$). But $V'' \subseteq$ CONE(R) $= C^{\gamma}$ because $V'' \subseteq Q \subseteq P^{\beta} \subseteq C^{\gamma}$ (since $P^{\beta} \in$ **P** (C^{γ})). Therefore CONE($V'' \cup R$) = CONE(R) = C^{γ}. Thus $v' - y \in C^{\gamma}$. □

3.4. <u>α-Polarity of Polyhedra</u>

Let A be the matrix $(a^i \in \mathbb{R}^J : i \in I)$ and b be the vector $(b_i \in \mathbb{R} : i \in I)$.

(3.4.1) We define $Q(A, b)$ by

$$Q(A, b) \equiv \{x \in \mathbb{R}^J : Ax \leq b\} = P((-a^i : i \in I), (-b_i : i \in I)).$$

(3.4.2) By Lemma (2.2.18) $\langle -a, -\beta \rangle$ is a valid inequality for $Q(A, b)$ if and only if there exists $\lambda \in \mathbb{R}^I_+$ such that $a = \sum_{i \in I} \lambda_i a^i$ and $\beta \geq \sum_{i \in I} \lambda_i b_i$.

(3.4.3) We will now consider the polarity given by the relation $x\alpha y \equiv xy \leq 1$. Clearly for any Q in $\mathbb{R}^J$, 0 belongs to Q^{α}. Also the statements "$\langle -y, -1 \rangle$ is a valid inequality for Q" and "$y \in Q^{\alpha}$" are equivalent for any polyhedron Q.

(3.4.4) <u>Lemma</u>. For any non-empty polyhedron Q, if (V, R) is a representing set for Q then Q^{α} equals

$Q(VR, \delta^M)$.

The proof is similar to the proof of lemma (3.3.5).□

(3.4.5) Corollary. If Q is a polyhedron so is Q^α.

Proof: If Q is empty then Q^α equals $\mathbb{R}^J$ and $\mathbb{R}^J$ is a polyhedron. Let Q be non-empty. By theorem (2.4.10) Q is a finite convex set, hence Q^α is a polyhedron by the lemma above.□

(3.4.6) Lemma. Let Q be a polyhedron. Then Q is α-closed if and only if Q equals $Q(A, \delta^M)$ for some matrix $A = (a^i: i \in I)$ and some $M \subseteq I$.

Proof: Let $Q = Q(A, \delta^M)$, we can consider $M \neq \emptyset$ because $0x \leq 1$ is always a valid inequality for Q, hence we have

$Q = (\mathrm{CONV}(\{a^i: i \in M\}) + \mathrm{CONE}(\{a^i: i \in I - M\}))^\alpha$ (by (3.5.4)),

and Q is α-closed by lemma (3.1.6).

Let Q be α-closed, that is $Q = (Q^\alpha)^\alpha$, Q^α is a non-empty polyhedron by (3.4.3) and (3.4.5). Let (V, R) be a basis of Q^α and $A = VR$, by lemma (3.4.4) Q equals $Q(A, \delta^M)$.□

(3.4.7) Lemma. Let $Q = Q(VR, \delta^M)$ be an α-closed polyhedron. Then $Q^\alpha = \mathrm{CONV}(V \cup \{0\}) + \mathrm{CONE}(R)$.

Proof: By (3.4.2) and (3.4.3) $y \in Q^\alpha$ if and only if there exist $\lambda = (\lambda_m \geq 0: m \in M)$ and $\mu = (\mu_n \geq 0: n \in N)$ such that $y = \sum_{m\in M} \lambda_m v^m + \sum_{n\in N} \mu_n r^n$ and $\sum_{m\in M} \lambda_m \leq 1$ if and only if $y = \sum_{m\in M} \lambda_m v^m + (1 - \sum_{m\in M} \lambda_m)0 + \sum_{n\in N} \mu_n r^n$ and λ, $\mu \geq 0$ if and only if $y \in \text{CONV}(V \cup \{0\}) + \text{CONE}(R)$.□

(3.4.8) <u>Theorem</u>. Let Q be a polyhedron. Then Q is α-closed if and only if $0 \in Q$.

Proof: Clearly if Q is α-closed then $0 \in Q$. Let $0 \in Q$ and (V, R) be a basis for Q. We have $Q^\alpha = Q(VR, \delta^M)$ by lemma (3.4.4), therefore $Q^{\alpha\alpha} = \text{CONV}(V \cup \{0\}) + \text{CONE}(R)$ by lemma (3.4.7). We only need to show that $Q \supseteq Q^{\alpha\alpha}$ by (3.1.3).

Since $0 \in Q$ and (V, R) is a basis for Q there exists $(\lambda_m^0 \geq 0: m \in M)$, $(\mu_n^0 \geq 0: n \in N)$ such that $0 = \sum_{m\in M} \lambda_m^0 v^m + \sum_{n\in N} \mu_n^0 r^n$ and $\Sigma\lambda_m^0 = 1$. Let $x \in Q^{\alpha\alpha}$, hence there exist $(\lambda_m \geq 0: m \in M)$, $\lambda_0 \geq 0$, $(\mu_n \geq 0: n \in N)$ with $\sum_{m\in M} \lambda_m + \lambda_0 = 1$ and $x = \sum_{m\in M} \lambda_m v^m + \lambda_0 0 + \sum_{n\in N} \mu_n r^n$.

Therefore we have

$$x = \sum_{m\in M} (\lambda_m + \lambda_0\lambda_m^0)v^m + \Sigma(\mu_n + \lambda_0\mu_n^0)r^n \quad \text{where}$$

$(\lambda_m + \lambda_0\lambda_m^0 \geq 0:\ m \in M)$, $(\mu_n + \lambda_0\mu_n^0 \geq 0:\ n \in N)$ and $\sum_{m\in M} (\lambda_m + \lambda_0\lambda_m^0) = \sum_{m\in M} \lambda_m + \lambda_0 = 1$. Thereafter $x \in Q$.□

(3.4.9) <u>Corollary</u>. Let $Q = Q(A, b)$ be α-closed. Then dimension of Q^α equals rank of A. In particular we have Q is full dimension if and only if Q^α is pointed.

The proof is the same as in lemma (3.2.7) for cone polarity, notice in that proof that we only need that $0 \in Q$, Q^α and "if $y \in Q^\alpha$ then y is linearly dependent from the rows of A" (this last statement is provided by (3.4.2) and (3.4.3)). □

(3.4.10) <u>Theorem</u>. Let Q be a α-closed polyhedron. Then $\langle -VR, -\delta^M \rangle$ is an irredundant system for Q if and only if $0 \notin V$ and either (V, R) or $(V \cup \{0\}, R)$ is a basis for Q^α.

Proof: Clearly $0 \notin V$ whenever $\langle -VR, -\delta^M \rangle$ is an irredundant system for Q. Let $Q = Q(VR, \delta^M)$. Then we have $v^e x \le 1$ is redundant if and only if $Q^\alpha = \text{CONV}(V - \{v^e\} \cup \{0\}) + \text{CONE}(R)$ (by (3.4.7)) if and only if neither $(V \cup \{0\}, R)$ nor (V, R) is a basis of Q^α, here the only non-trivial part is to show that (V, R) is not a basis of Q^α, let then (V, R) be a basis of Q^α, $0 = \sum_{m \in M} \lambda_m^0 v^m + \sum_{n \in N} \mu_n^0 r^n$ and $v^e = \sum_{m \in M-\{e\}} \lambda_m v^m + \lambda_0 0 + \sum_{n \in N} \mu_n r^n$, where $\lambda_m^0, \mu_n^0, \lambda_m, \mu_n, \lambda_0 \ge 0$ and $\sum_{m \in M} \lambda_m^0 = 1$, $\sum_{m \in M-\{e\}} \lambda_m + \lambda_0 = 1$. Therefore we have

$$v^e = \sum_{m \in M-\{e\}} (\lambda_m + \lambda_0 \lambda_m^0) v^m + \lambda_0 \lambda_e^0 v^e + \sum_{n \in N} (\mu_n + \lambda_0 \mu_n^0) r^n.$$

If $\lambda_0 \lambda_e^0 = 0$ then (V, R) is not a basis, otherwise $\lambda_0 \lambda_e^0 < 1$ and

$$v^e = \sum_{m \in M-\{e\}} \frac{(\lambda_m + \lambda_0 \lambda_m^0)}{(1-\lambda_0 \lambda_e^0)} v^m + \sum_{n \in N} \frac{(\mu_n + \lambda_0 \mu_n^0) r^n}{(1-\lambda_0 \lambda_e^0)},$$

again (V, R) is not a basis.

Now consider the rays, $r^e x \le 0$ is redundant if and only if $Q^\alpha = \text{CONV}(V \cup \{0\}) + \text{CONE}(R - \{r^e\})$ (by (3.4.7)) if and only if neither (V, R) nor $(V \cup \{0\}, R)$ is a basis of Q^{α}, otherwise R is a basis for $RAY(Q^{\alpha}) = CONE(R - \{r^e\})$ (by theorem (2.4.10)).□

(3.4.11) We call $x \in Q(A, b)$ <u>interior</u> when $a^i x < b_i : i \in I$.

(3.4.12) <u>Lemma</u>. 0 is interior in $Q(A, \delta^M)$ if and only if $(Q(A, \delta^M))^{\alpha}$ is a polytope.

Proof: 0 is interior in $Q(A, \delta^M)$ if and only if $M = I$ if and only if $RAY((Q(A, \delta^M))^{\alpha}) = \{0\}$ (by (2.4.10) and (3.4.7)) if and only if $(Q(A, \delta^M))^{\alpha}$ is a polytope.□

Chapter 4

J-UNBOUNDED SETS

4.1. J-Unbounded Sets

Let J be a finite set, we denote by $\mathbb{N}^J$ the vectors of $\mathbb{R}^J$ that have integer components and by $\mathbb{N}_+^J$ the set $\mathbb{N}^J \cap \mathbb{R}_+^J$.

(4.1.1) Let $K \subseteq J$ and $K \neq \emptyset$. We call a set T in $\mathbb{N}_+^J$ K-unbounded whenever T satisfies:

(4.1.2) $0 \notin T$ and $T \neq \emptyset$,

(4.1.3) if $t \in T$ then $t_j = 0$ for $j \in J - K$,

(4.1.4) for all $t \in T$ and for all $j \in K$ there is $k \geq 1$ such that $t + k\delta^j \in T$. Clearly this implies that given any $\mu \geq 0$ there is $k > \mu$ such that $t + k\delta^j \in T$.

We will consider first J-unbounded sets. Our aim is to show that if T is J-unbounded then $\text{CONV}(T) \in \mathbf{P}(\mathbb{R}_+^J)$. We will show how K-unbounded sets arise when considering the sets of solutions of semigroup programs. We can then use polyhedral polarity to characterize irredundant linear systems for semigroup programs.

(4.1.5) Let T be a J-unbounded set, we denote by $\bar{T}$ the set of minimal points in T, i.e. $t \in \bar{T}$ when $t \in T$ and there is no $t' \in T$ such that $t' \neq t \geq t'$.

(4.1.6) Lemma. Let $\bar{T}$ be defined as in (4.1.5). Then $\bar{T}$ is finite.

Proof: Clearly the elements of $\bar{T}$ are pairwise incomparables in the sense that neither $t \le t'$ nor $t' \le t$ for any pair t, t' of different elements of T. We will prove the lemma by showing that any set A in $\mathbb{N}_+^J$ of pairwise incomparable elements is finite. We prove this last statement by induction on $|J|$. Notice that for $|J| = 1$ the set A can have at most one element. Suppose any set in $\mathbb{N}_+^I$, of pairwise incomparable elements be finite when $|I| = |J| - 1$.

Let $A \subseteq \mathbb{N}_+^J$ be a set of pairwise incomparable elements. Let $A_{(\alpha,j)} \equiv \{x \in A: x_j = \alpha\}$. The sets $A_{(\alpha,j)}$ have pairwise incomparable elements, therefore these sets are finite because deleting the j^{th} component these sets are still pairwise incomparable in $\mathbb{N}_+^{J-\{j\}}$. Let $a \in A$. Then for any $a' \in A$ and $a' \ne a$, there exists $j \in J$ such that $a'_j < a_j$ and $a' \in A_{(a'_j,j)}$. Therefore

$$A = \{a\} \cup \Big(\bigcup_{\substack{\alpha \in N_j \\ j \in J}} A_{(\alpha,j)} \Big), \text{ where } N_j = \{n \in \mathbb{N}_+ : n < a_j\}.$$

Hence $|A| \le 1 + \sum_{\substack{\alpha \in N_j \\ j \in J}} |A_{(\alpha,j)}|$. But each $A_{(\alpha,j)}$ is finite and the number of terms is $\sum_{j \in J} a_j$. Therefore $|A|$ is finite since $\sum_{j \in J} a_j$ is finite. □

(4.1.7) Lemma. Let T be J-unbounded and $\bar{T}$ defined as in (4.1.5). Then CONV(T) is the polyhedron $\text{CONV}(\bar{T}) + \mathbb{R}_+^J$.

Proof: Let $x = \sum_{i \in I} \lambda_i t^i$, where $\lambda_i \geq 0: i \in I$, $\sum_{i \in I} \lambda_i = 1$, $|J| < \infty$ and $t^i \in T: i \in I$. Let $S^i \in \bar{T}$: $s^i \leq t^i$, $i \in I$ (Clearly there exist such s^i's). Then $x = \sum_{i \in I} \lambda_i s^i + \sum_{i \in I} \lambda_i (t^i - s^i) = \sum_{i \in I} \lambda_i s^i + \sum_{j \in J} (\sum_{i \in I} \lambda (t^i_j - s^i_j))\delta^j \in$ $\mathrm{CONV}(\bar{T}) + \mathbb{R}^J_+$. Therefore $\mathrm{CONV}(T) \subseteq \mathrm{CONV}(\bar{T}) + \mathbb{R}^J_+$.

Let $s \in \bar{T}$ and $(\mu_j \geq 0: j \in J)$, there exist $k_j > (\sum_{e \in J} \mu_e): j \in J$ (by (3.4.4)) such that $s + k_j \delta^j \in T$ for all $j \in J$. Let $x = s + \sum_{j \in J} \mu_j \delta^j$, we want to show that $x \in \mathrm{CONV}(T)$ since this will imply $\{s\} + \mathbb{R}^J_+ \subseteq \mathrm{CONV}(T)$, for all $s \in \bar{T}$. Therefore $\mathrm{CONV}(\bar{T}) + \mathbb{R}^J_+ \subseteq \mathrm{CONV}(T)$ as we need.

We can write x as

$$x = s + \sum_{j \in J} \frac{\mu_j}{k_j} k_j \delta^j = (1 - \sum_{j \in J} \frac{\mu_j}{k_j}) s + \sum_{j \in J} \frac{\mu_j}{k_j} (s + k_j \delta^j)$$

where the sum of the coefficients equals 1. All the vectors belong to T and all the coefficients are greater than or equal to zero because $\mu_j, k_j \geq 0$ for all $j \in J$ and by the choice of k_j we have

$$\sum_{j \in J} \frac{\mu_j}{k_j} \leq \frac{\sum_{j \in J} \mu_j}{|J| \min_{j \in J}\{k_j\}} \leq \frac{\sum_{j \in J} \mu_j}{|J| \sum_{j \in J} \mu_j} \leq 1 \quad . \square$$

(4.1.8) <u>Theorem</u>. For any J-unbounded T in $\mathbb{R}^J$, $\mathrm{CONV}(T)$ is a pointed full dimension polyhedron in $\mathbf{P}(R^J_+)$ with the following irredundant defining system

(i) $\begin{cases} vx \geq 1, \text{ for all extreme points } v \text{ of } (\mathrm{CONV}(T))^{\beta}, \\ x_j \geq 0, \text{ for all } j \in J \text{ such that } \min_{t \in T}\{t_j\} = 0. \end{cases}$

Furthermore the set $(\mathrm{CONV}(T))^{\beta}$ is the set of solutions π to the system

(ii) $\begin{cases} t\pi \geq 1 \text{ , for any minimal element } t \text{ of } T, \\ \pi_j \geq 0 \text{ , for all } j \in J. \end{cases}$

Proof: From (4.1.6) and (4.1.7) CONV(T) is a polyhedron and $\mathrm{CONV}(T) \subseteq \mathbb{R}_+^J = \mathrm{RAY}(\mathrm{CONV}(T))$, since $0 \notin T$ we have $\mathrm{CONV}(T) \in \mathbf{P}(\mathbb{R}_+^J)$ by theorem (3.3.11). But all the sets in $\mathbf{P}(\mathbb{R}_+^J)$ are pointed and full dimension.

$(\mathrm{CONV}(T))^{\beta}$ also belongs to $\mathbf{P}(\mathbb{R}_+^J)$ therefore $\{\delta^j : j \in J\}$ is its set of extreme ray.

Let P denote the set CONV(T) and e be an element of J.

If we show that there is an extreme point v of P^{β} belonging to the ray $[\delta^e]$ if and only if $\min_{t \in T}\{t_e\} > 0$ then system (i) would correspond to the system of theorem (3.3.24) which is an irredundant defining system for P.

Let $\min_{t \in T}\{t_e\} = 0$. Then $\lambda\delta^e \notin P^{\beta}$ for any $\lambda \geq 0$. Therefore no extreme point of P^{β} belongs to the ray $[\delta^e]$.

Let $k = \min_{t \in T}\{t_e\} > 0$. Then $\frac{1}{k}\delta^e \in P^{\beta}$. Moreover $\frac{1}{k}\delta^e$ is an extreme point of P^{β}. Otherwise there are $x^1, x^2 \in P^{\beta}$ satisfying $x^1 \neq x^2$ and $\lambda x^1 + (1-\lambda)x^2 = \frac{1}{k}\delta^e$

where $0 < \lambda < 1$. Then $\lambda x_j^1 + (1-\lambda)x_j^2 = 0$ for all $j \in J - \{e\}$, that is $x_j^1 = x_j^2 = 0$ for all $j \in J - \{e\}$ because $P^\beta \subseteq \mathbb{R}_+^J (P^\beta \in \mathbf{P}(\mathbb{R}_+^J))$. Therefore $x^i = x_e^i \delta^e$ and $\min_{t \in T}\{x^i t\} = \min_{t \in T}\{x_e^i t_e\} = x_e^i k$ for $i = 1, 2$. But then $x_e^i \geq \frac{1}{k}$ because $x_e^i \delta^e \in P^\beta$ for $i = 1, 2$ and $x_e^1 + x_e^2 = \frac{1}{k}$, this contradicts $x^1 \neq x^2$.

By (4.1.7) and (3.3.5) (ii) is a defining system for $(\mathrm{CONV}(T))^\beta$. □

(4.1.9) We denote by $\underline{X^K}$ the set $\{x \in \mathbb{R}^J : x_j = 0, j \in J - K\}$.

(4.1.10) Let T be J-unbounded in $\mathbb{R}^J$ and K a not empty proper subset of J. We denote by $\underline{T^K}$ the set $T \cap X^K$.

(4.1.11) From the definition of K-unbounded T^K is K-unbounded.

The next theorem relates $\mathrm{CONV}(T^K)$ with $\mathrm{CONV}(T)$.

(4.1.12) Theorem. Let T and T^K be defined as in (4.1.10). Let $P = \mathrm{CONV}(T)$ and $P^K = \mathrm{CONV}(T^K)$. The following relations hold

(4.1.13) $$P^K = P \cap X^K.$$

(4.1.14) The set V^K of extreme points of P^K equals the set $V \cap X^K$ where V is the set of extreme points of P.

Proof: Since $T^K \subseteq X^K$ we have for any x equal to a convex combination of elements in T^K that $x_j = 0$ for all $j \in J - K$. Hence $P^K \subseteq X^K$.

Also $P^K \subseteq P$ because $T^K \subseteq T$. Therefore $P^K \subseteq P \cap X^K$.

Let L be a finite set and let $x = \sum_{e \in L} \lambda_e t^e$ where $t^e \in T$ and $\lambda_e > 0$ for all $e \in L$ and $\sum_{e \in L} \lambda_e = 1$. If $x \in X^K$ then $x_j = \sum_{e \in L} \lambda_e t_j^e = 0$ for all $j \in J - K$. Since $\lambda_e > 0$ and $t_j^e \geq 0$ for all $e \in L$, t_j^e must be zero for all $j \in J - K$ and all $e \in L$. Therefore $t^e \in T^K$ for all $e \in L$, this implies $x \in P^K$. Hence we have $P \cap X^K \subseteq P^K$. Therefore (4.1.13) holds.

Let $x \in V^K$. By (4.1.13) $x \in X^K$. If $x = \frac{1}{2}(x^1 + x^2)$, where $x^1, x^2 \in P$, then $\frac{1}{2}(x_j^1 + x_j^2) = 0$ for all $j \in J - K$, but since $x^1, x^2 \geq 0$ we have $x_j^1 = x_j^2 = 0$ for all $j \in J - K$. Hence $x^1, x^2 \in P^K$, this means that $x^1 = x^2 = x$ because x is an extreme point of P^K. Therefore $x \in V \cap X^K$.

If $x \in P^K - V^K$ then $x \in P - V$ because $P^K \subseteq P$. Hence if $x \in V \cap X^K$ then $x \in V^K$ since $x \in P^K$ by (4.1.13). □

(4.1.15) <u>Corollary</u>. If $\mathrm{CONV}(T) = P((a^i \in \mathbb{R}^J : i \in I), b)$, where T is a J-unbounded set in $\mathbb{R}^J$ then for any non-empty subset K of J we have

$$P((a_j^i : j \in K) : i \in I), b) = \mathrm{CONV}(\{t_K \in \mathbb{R}^K : t \in T \cap X^K\}). \square$$

4.2. H-Unbounded Semigroup Programs.

(4.2.1) A semigroup program (H, b, c) over $(S, \underset{\sim}{+})$ where $(S, \underset{\sim}{+})$ is a semigroup, H is a subset of S, b belongs to S and c is a vector in $\mathbb{R}_+^H$, is the program

(4.2.2) $$\text{minimize } \sum_{h \in H} c_h t_h$$
$$\text{over } t \in \mathbb{N}_+^H \text{ satisfying}$$

(4.2.3) $$\theta_H(t) = \underset{\sim}{\sum_{h \in H}} t_h \cdot h = b.$$

Through this chapter we will consider $(S, \underset{\sim}{+})$ fixed and understand that any semigroup program is over $(S, \underset{\sim}{+})$. Without loss of generality we can make some more assumptions:

(4.2.4) Clearly there is a $t \in N_+^H$ satisfying (4.2.3) if and only if b belongs to the semigroup generated by H, therefore we assume that b belongs to this semigroup. Furthermore, when b is the identity o of S one optimal solution is 0, therefore we assume b different from o.

(4.2.5) For any $h \in H$ if $b \sim h$ is empty then $t_h = 0$ for any solution t of (4.2.3) because $t_h > 0$ and $\theta_H(t) = b$ implies $h \underset{\sim}{+} \theta_H(t - \delta^h) = \theta_H(\delta^h) \underset{\sim}{+} \theta_H(t-\delta^h) = \theta_H(t) = b$, hence $\theta_H(t - \delta^h) \in b \sim h$. Therefore the programs (H, b, c) and (H - {h}, b, c) are equivalents.

Consequently we assume that $b \sim h \neq \emptyset$ for any $h \in H$.

(4.2.6) We also assume that H is a generator set for $(S, \underset{\sim}{+})$ because the elements in S and not in the semigroup generated by H cannot be represented by any $t \in \mathbb{N}_+^H$.

Henceforth we will assume that any semigroup program (H, b, c) satisfies (4.2.4) to (4.2.6) unless specifically omitted.

(4.2.7) We denote by <u>$T(H, b)$</u> the set $\{t \in \mathbb{N}_+^H : \theta_H(t) = b\}$ and by <u>$E(H, b)$</u> the set $\text{CONV}(T(H, b))$. When $T(H, b)$ is H-unbounded we call the semigroup program (H, b, c) <u>H-unbounded</u>.

(4.2.8) We will assume that <u>(H, b, c) is H-unbounded</u>.

In this case we can use the results of section 4.1 to obtain the next theorem.

(4.2.9) <u>Theorem</u>. Let (H, b, c) be an H-unbounded semigroup program over $(S, \underset{\sim}{+})$.

(4.2.10) $E(H, b)$ is a pointed full dimension polyhedron in $\mathbf{P}(\mathbb{R}_+^H)$.

(4.2.11) And the following is an irredundant defining system for $E(H, b)$

$$\begin{cases} vx \geq 1 \text{ , for all extreme points } v \text{ of } E^{\beta}(H, b) \\ x_h \geq 0 \text{ , for all } h \in H \text{ such that } \min\limits_{t \in T(H,b)} \{t_h\} = 0. \end{cases}$$

(4.2.12) Where the polar $E^{\beta}(H, b)$ of $E(H, b)$ is the set of solutions π of

$$\begin{cases} t\pi \geq 1 \text{ for any } t \in T(H,b) \cap \{t \in \mathbb{N}_+^H: \\ \qquad\qquad t_h \leq o(h),\ h \in H\}. \\ \pi_h \geq 0 \text{ for all } h \in H. \end{cases}$$

Proof: This is immediate from theorem (4.1.8) and we only need to point out that the minimal elements t of $T(H, b)$ satisfy $t_h \leq o(h)$: $h \in H$.□

Now we pause to characterize the H-unbounded semigroup programs. We recall the definition of H-unbounded from section 4.1. $T \subseteq \mathbb{N}_+^H$ is H-unbounded when

(4.2.13) $0 \notin T$ and $T \neq \emptyset$

(4.2.14) for all $t \in T$ and for all $h \in H$ there is $k \geq 1$ such that $t + k\delta^h \in T$.

(4.2.15) Condition (4.2.13) is valid for any $T(H, b)$ by assumption (4.2.4). Condition (4.2.14) means that for any t such that $\theta_H(t) = b$ and for any $h \in H$ there is $k_h \geq 1$ such that $\theta_H(t+k\delta^h) = b \pm k_h \cdot h = b$.

The next theorem characterizes H-unbounded semigroup programs

(4.2.16) <u>Theorem</u>. Let (H, b, c) be a semigroup program. Then (H, b, c) is H-unbounded if and only if b is a loop element (see (2.5.7)).

Proof: Let b be a loop element. Then there exists $k > 0$ such that $b = b \underset{\sim}{+} k \cdot b$. Let $h \in H$, by assumption (4.2.5) $b \sim h \neq \emptyset$ hence there is an element $s \in S$ such that $h \underset{\sim}{+} s = b$. We will show that there exists $k_h \geq 1$ such that $b \underset{\sim}{+} k_h \cdot h = b$, then (H, b, c) is H-unbounded by (4.2.15).

Let $p \cdot h = \circ(h) \cdot h$, $p < \circ(h)$ (such a p exists by the definition of $\circ(h)$) and let $q = \circ(h) - p > 0$, hence $p \cdot h = (p + q) \cdot h$.

Since $b \underset{\sim}{+} k \cdot b = b$ we have for any integer $p \geq 1$, $b \underset{\sim}{+} (pk) \cdot b = b \underset{\sim}{+} k \cdot b \underset{\sim}{+} ((p - 1)k) \cdot b = b \underset{\sim}{+} ((p - 1)k) \cdot b$. Therefore, using induction on p, we obtain $b \underset{\sim}{+} (pk) \cdot b = b$ for any $p \geq 1$.

Therefore $b = b \underset{\sim}{+} (pk) \cdot b = b \underset{\sim}{+} (pk) \cdot (h \underset{\sim}{+} s)$, since $h \underset{\sim}{+} s = b$,

$b = b \underset{\sim}{+} (pk) \cdot h \underset{\sim}{+} (pk) \cdot s = b \underset{\sim}{+} k \cdot (p \cdot h) \underset{\sim}{+} (pk) \cdot s =$

$b \underset{\sim}{+} k \cdot ((p + q) \cdot h) \underset{\sim}{+} (pk) \cdot s$, because $p \cdot h = (p + q) \cdot h$,

$b = b \underset{\sim}{+} (kq) \cdot h \underset{\sim}{+} (kp) \cdot h \underset{\sim}{+} (kp) \cdot s = b \underset{\sim}{+} (kq) \cdot h \underset{\sim}{+} (kp) \cdot b$, since $h \underset{\sim}{+} s = b$,

$b = b \underset{\sim}{+} (kq) \cdot h$, since $b \underset{\sim}{+} (kp) \cdot b = b$.

Moreover $kq \geq 1$ since $k, q \geq 1$. Let $k_h = kq$. Then k_h is as desired above.

Let (H, b, c) be H-unbounded, then by (4.2.15) there is $k_h > 0$ such that $b \underset{\sim}{+} k_h \cdot h = b$ for any $h \in H$.

For any $n \in \mathbb{N}_+^H$ the next relation holds

$$b \underset{\sim}{+} (\underset{h\in H}{\tilde{\Sigma}} (n_h k_h)\cdot h) = b.$$

Because it is clearly true whenever $n = 0$ or $n = \delta^h$ for all $h \in H$, i.e. whenever $\sum_{h\in H} n_h \le 1$. Let's assume it is true for any $n \in \mathbb{N}_+^H$ with $\sum_{h\in H} n_h \le p$. Let $m \in \mathbb{N}_+^H$ satisfies $\sum_{h\in H} m_h = p + 1$ and $m_e \ge 1$ for some $e \in H$. Denoting by n the vector $m - \delta^e$, we have

$$b \underset{\sim}{+} (\underset{h\in H}{\tilde{\Sigma}} (m_h k_h)\cdot h) = b \underset{\sim}{+} k_e \cdot e \underset{\sim}{+} (\underset{h\in H}{\tilde{\Sigma}} (n_h k_h)\cdot h) =$$

$$b \underset{\sim}{+} (\underset{h\in H}{\tilde{\Sigma}} (n_h k_h)\cdot h) = b$$

because $n = m - \delta^e \in \mathbb{N}_+^H$ and $\sum_{h\in H} n_h = \sum_{h\in H} m_h - 1 = p.$

We denote by $\lambda \equiv \prod_{h\in H} k_h > 0$, and by $\lambda_h \equiv \dfrac{\lambda}{k_h}$.

Since $T(H, b) \ne \emptyset$ there is $t \in \mathbb{N}_+^H$ such that $\theta_H(t) = b$. Let $n_h = \lambda_h t_h$, then $n_h k_h = \lambda_h t_h k_h = \dfrac{\lambda}{k_h} t_h k_h = \lambda t_h$.

Therefore we obtain

$$b = b \underset{\sim}{+} (\underset{h\in H}{\tilde{\Sigma}} (\lambda t_h)\cdot h) = b \underset{\sim}{+} \theta_H(\lambda t) = b \underset{\sim}{+} \lambda\cdot\theta_H(t) =$$

$$b \underset{\sim}{+} \lambda\cdot b .$$

Therefore b is a loop element. □

(4.2.17) Corollary. Let (S, b, c) be a semigroup program over $(S, \underset{\sim}{+})$ not satisfying (4.2.5) but b being a loop element. Then $(\{s \in S: b \sim s \ne \phi\}, \underset{\sim}{+})$ is a subsemigroup of $(S, \underset{\sim}{+})$.

Proof: Since b is a loop element, there is $k \geq 0$ such that

$$b = b \underset{\sim}{+} b \underset{\sim}{+} k \cdot b.$$

Let $s, h \in S$ satisfy $b \sim s, b \sim h \neq \emptyset$. Therefore there are $s', h' \in S$ such that $s \underset{\sim}{+} s' = h \underset{\sim}{+} h' = b$. Then we have

$$b = b \underset{\sim}{+} b \underset{\sim}{+} k \cdot b = (s \underset{\sim}{+} s') \underset{\sim}{+} (h \underset{\sim}{+} h') \underset{\sim}{+} k \cdot b =$$

$$(s \underset{\sim}{+} h) \underset{\sim}{+} (s' \underset{\sim}{+} h' \underset{\sim}{+} k \cdot b).$$

Therefore $b \sim (s \underset{\sim}{+} h) \neq \emptyset$ since $s' \underset{\sim}{+} h' \underset{\sim}{+} k \cdot b \in b \sim (s \underset{\sim}{+} h)$. We have proved that the set $\{s \in S : b \sim s \neq \emptyset\}$ is closed under $\underset{\sim}{+}$ satisfying the definition (2.5.12) of semigroup. □

Chapter 5

MASTER SEMIGROUP PROGRAMS

In this chapter we develop more conditions satisfied by the minimal elements of $E^{\beta}(H, b)$. These minimal elements give the inequalities with right hand side greater than zero in the defining systems for $E(H, b)$ (see (3.3.28)).

When (H, b, c) is an H-unbounded master semigroup program, we use these conditions to obtain polyhedra with more structure whose set of extreme points contains (and some times are equal to) the set of extreme points of $E^{\beta}(H, b)$.

5.1 Subadditivity and Complementarity

(5.1.1) Minimality Lemma. Let (H, b, c) be a semigroup program and $t \in T(H, b)$ be an optimal solution of (H, b, c). Then for any non-negative integer vector $r \le t$, r is an optimal solution of the semigroup program $(H, \theta_H(r), c)$, i.e. $cr = \min_{\theta_H(r')=\theta_H(r)} \{cr'\}$.

Proof: Let $\theta_H(r') = \theta_H(r)$, by the substitution lemma we have $t - r + r' \in T(H, b)$. Since t is optimal

$$c(t - r + r') = ct - cr + cr' \ge ct \; .$$

Thus $cr' \ge cr$. □

Notice that the minimality lemma is valid for any semigroup program.

Through the rest of this section we consider (H, b, c) to be a fixed H-unbounded semigroup program over $(S, \underset{\sim}{+})$

satisfying assumptions (4.2.4) to (4.2.6).

(5.1.2) We call π a <u>support</u> of $E(H, b)$ when π is a minimal element of the polar $E^{\beta}(H, b)$, therefore $\pi x \geq 1$ for all $x \in E(H, b)$ and no $\pi' \leq \pi$, $\pi' \neq \pi$ satisfies $\pi' x \geq 1$ for all $x \in E(H, b)$. By lemma (3.3.28) the extreme points of $E^{\beta}(H, b)$ are supports of $E(H, b)$ since $E^{\beta}(H, b) \in \mathbf{P}(\mathbb{R}_{+}^{H})$.

(5.1.3) <u>Lemma</u>. Let π be a support of $E(H, b)$ and let t^{0} be an optimal solution of (H, b, π). Then πt^{0} equals 1.

Proof: Let t^{0} be an optimal solution of (H, b, π). Since $\pi \in E^{\beta}(H, b)$ $\pi t^{0} \geq 1$. Moreover $\pi t \geq \pi t^{0}$ for all $t \in T(H, b)$ because t^{0} is optimal. Thus $(\frac{\pi}{\pi t^{0}}) t \geq 1$ for all $t \in T(H, b)$, that is $\frac{\pi}{\pi t^{0}} \in E^{\beta}(H, b)$. But π is minimal in $E^{\beta}(H, b)$ and $\frac{\pi}{\pi t^{0}} \leq \pi$. Thereafter $\frac{\pi}{\pi t^{0}} = \pi$, this is possible only if $\pi t^{0} = 1$ because $\pi \neq 0$ ($\pi \in E^{\beta}(H, b)$ and $0 \notin E^{\beta}(H, b)$. □

(5.1.4) <u>Lemma</u>. Let π be a support of $P(H, b)$. For any $e \in H$ there is $r^{0} \in \mathbb{N}_{+}^{H}$ such that $r^{0} + \delta^{e} \in T(H, b)$ and $\pi(r^{0} + \delta^{e}) = 1$.

Proof: Let π be a support of $E(H, b)$ and let e belong to H. Then we have $\pi_{e} \geq 0$ since $\pi \in E^{\beta}(H, b) \subseteq \mathbb{R}_{+}^{H}$.

Case 1: Let $\pi_{e} = 0$.

Let t^{0} be an optimal solution of (H, b, π). By lemma (5.1.3) $\pi t^0 = 1$. Moreover there is $k_e \geq 1$ such that $t^0 + k_e\delta^e \in T(H, b)$ since (H, b, π) is H-unbounded. Let $r^0 = t^0 + (k_e - 1)\delta^e$. Then $r^0 \in \mathbb{N}_+^H$, $r^0 + \delta^e = t^0 + k_e\delta^e \in T(H, b)$ and $\pi(r^0 + \delta^e) = \pi(t^0 + k_e\delta^e) = \pi t^0 + \pi_e k_e = \pi t^0 = 1$ because $\pi_e = 0$ and choice of t^0.

Case 2: Let $\pi_e > 0$.

Since $b \sim e$ is non-empty (by (4.2.5)) there is $s \in S$ such that $s \underset{\sim}{+} e = b$. Moreover there is $r \in \mathbb{N}_+^N$ such that $\theta_H(r) = s$ because H generates S (by (4.2.6)). Then we have $\theta_H(r + \delta^e) = \theta_H(r) \underset{\sim}{+} \theta_H(\delta^e) = s \underset{\sim}{+} e = b$, that is $r + \delta^e \in T(H, b)$. Therefore the set $R = \{r \in \mathbb{N}_+^H : r + \delta^e \in T(H, b)\}$ is non-empty.

Consequently $\pi_0 = \min_{r \in R}\{\pi(r + \delta^e)\}$ is defined.

If $\pi_0 = 1$ then the lemma follows taken as r^0 any $r \in R$ which satisfies the minimum $\pi(r^0 + \delta^e) = \pi_0 = 1$. We will prove that $\pi_0 = 1$.

Since $\pi \in E^\beta(H, b)$ and $r + \delta^e \in T(H, b)$ for all $r \in R$ we have $\pi(r + \delta^e) \geq 1$ for all $r \in R$. Thus $\pi_0 \geq 1$.

Let $\alpha = \min\{\frac{\pi_0 - 1}{\circ(e)}, \pi_e\}$ and let $\pi' = \pi - \alpha\,\delta^e$. We claim that $\pi' \in E^\beta(H, b)$, because:

For all $t \in T \equiv \{t \in T(H, b) : t_n \leq \circ(n) \text{ for all } h \in H\}$;

If $t_e = 0$ then $\pi' t = (\pi - \alpha\delta^e)t = \pi t - \alpha t_e = \pi t \geq 1$ since $t \in T \subseteq T(H, b)$ and $\pi \in E^\beta(H, b)$.

If $t_e > 0$ then $r = t - \delta^e \in R$. Hence $\pi t = \pi(r + \delta^e) \geq \pi_0$ by definition of π_0. Therefore $\pi' t = (\pi - \alpha\delta^e)t =$

$\pi t - \alpha t^e \geq \pi t - \frac{\pi_0 - 1}{\circ(e)} t_e \geq \pi t - (\pi_0 - 1)$ by definition of α and because $t_e \leq \circ(e)$, that is $\pi' t \geq (\pi t - \pi_0) + 1 \geq 1$ because $\pi t \geq \pi_0$.

Thus $\pi' t \geq 1$ for all $t \in T$. Also $\pi' \geq 0$ because $\pi \in E^\beta(H, b) \subseteq \mathbb{R}_+^H$ and $\alpha \leq \pi_e$. By (4.2.12) we have $\pi' \in E^\beta(H, b)$.

Therefore $\pi' \in E^\beta(H, b)$ and $\pi' = \pi - \alpha\delta^e \leq \pi$ because $\alpha \geq 0$ ($\pi_0 - 1$, $\circ(e)$ and π_e are all non-negative). Since π is a support of $E(H, b)$, it is minimal on $E^\beta(H, b)$. Therefore $\pi' = \pi - \alpha\delta^e = \pi$, that is $\alpha = 0$. Thus $\pi_0 = 1$ since $\pi_e > 0$. □

(5.1.5) Lemma. Let (H, b, c) be an H-unbounded semigroup program over $(S, \underset{\sim}{+})$ and π be a support of $E(H, b)$. Then $\pi_e = \min\limits_{\theta_H(r)=e} \{\pi r\}$ for all $e \in H$.

Proof: Let $e \in H$. By lemma (5.1.4) there is $r^0 \in N_+^H$ satisfying $r^0 + \delta^e \in T(H, b)$ and $\pi(r^0 + \delta^e) = 1$. Let $t^0 = r^0 + \delta^e$, hence t^0 is an optimal solution of (H, b, π) and $\delta^e \leq t^0$. Therefore $\pi\delta^e = \min\limits_{\theta_H(r)=\theta_H(\delta^e)} \{\pi r\}$ by the Minimality Lemma.

But $\pi\delta^e = \pi_e$ and $\theta_H(\delta^e) = e$. Thus $\pi_e = \min\limits_{\theta_H(r)=e} \{\pi r\}$. □

(5.1.6) Let $(S, \underset{\sim}{+})$ be a semigroup and b a fixed element of S. Let $s, \bar{s} \in S$. We call $\bar{s}$ a <u>b-complementor</u> of s when $s + \bar{s} = b$ and for all $p, q \in S$ we have $s \underset{\sim}{+} p = q \underset{\sim}{+} \bar{s} = b$ implies $p \underset{\sim}{+} q = b$. Clearly s may have no b-complementor or several.

(5.1.7) When $\bar{s}$ is a b-complementor of s, s is a b-complementor of $\bar{s}$ because the definition is symmetric.

When $(S, \underset{\sim}{+})$ is a group, clearly for any s in S there is a unique solution x to $s \underset{\sim}{+} x = b$. This is $b \underset{\sim}{+} (-s)$ where $-s$ is the inverse of s (hence $b \underset{\sim}{+} (-s)$ is the b-complementor of s). For other semigroups we lose this property and we need something to replace it. The b-complementors will keep the properties we need.

We close this section with a theorem that gives several properties of supports of $E(H, b)$, and thereafter of extreme points of $E^{\beta}(H, b)$ since there are supports of $E(H, b)$.

(5.1.8) <u>Theorem</u>. Let π be a support of $E(H, b)$. Then π satisfies the following conditions.

(5.1.9) When $b \in H$ we have $\pi_b = 1$.

(5.1.10) When $o \in H$ we have $\pi_o = 0$.

(5.1.11) Subadditivity: If $h, e, s \in H$ and $h = e \underset{\sim}{+} s$ then $\pi_e + \pi_s \geq \pi_h$.

(5.1.12) Monotony: When $e, s \in H$ and $b \sim e \subseteq b \sim s$ we have $\pi_e \leq \pi_s$.

(5.1.13) Complementarity: When $e, \bar{e} \in H$ and $\bar{e}$ is a b-complementor of e we have $\pi_e + \pi_{\bar{e}} = 1$.

Proof: By lemma (5.1.3), $\min_{\theta_H(t)=b} \{\pi t\} = 1$ and by lemma (5.1.5) $\pi_b = \min_{\theta_H(t)=b} \{\pi t\} = 1$ when $b \in H$. Hence (5.1.9) holds.

If $o \in H$ then $\pi_o = \min_{\theta_H(t)=o} \{\pi t\} = 0$ by lemma (5.1.5) and because $\pi \geq 0$ and $\theta_H(0) = o$. Hence (5.1.10) holds.

Let $h, e, s \in H$ and $h = e \underset{\sim}{\pm} s$. Then $\theta_H(\delta^e + \delta^s)=h$. By lemma (5.1.5) $\pi_h \leq \pi(\delta^e + \delta^s) = \pi_e + \pi_s$. Hence (5.1.11) holds.

Let $e, s \in H$ and $b \sim e \subseteq b \sim s$. By (5.1.4) there is $r^0 \in \mathbb{N}_+^H$ satisfying $r^0 + \delta^e \in T(H, b)$ and $\pi(r^0 + \delta^e) = \pi r^0 + \pi_e = 1$. Therefore $\theta_H(r^0 + \delta^e) = \theta_H(r^0) \underset{\sim}{\pm} e = b$ and $\theta_H(r^0) \in b \sim e \subseteq b \sim s$. Then $\theta_H(r^0) \underset{\sim}{\pm} s = b$, that is $r^0 + \delta^s \in T(H, b)$. Therefore $\pi(r^0 + \delta^s) = \pi r^0 + \pi_s \geq 1 = \pi r^0 + \pi_e$, i.e. $\pi_s \geq \pi_e$. Hence (5.1.12) holds.

Let $e, \bar{e} \in H$ and $\bar{e}$ be a b-complementor of e. Since $e \underset{\sim}{\pm} \bar{e} = b$ we have $\pi_e + \pi_{\bar{e}} \geq \pi_b = 1$ by (5.1.11) and (5.1.9). In order to show that $\pi_e + \pi_{\bar{e}} \leq 1$, let $r^0, r^1 \in \mathbb{N}_+^H$ satisfy $r^0 + \delta^e, r^1 + \delta^{\bar{e}} \in T(H, b)$ and $\pi r^0 + \pi_e = \pi r^1 + \pi_{\bar{e}} = 1$ (they exist by lemma (5.1.4)). Since $\bar{e}$ is a b-complementor of e and $\theta_H(r^0) \underset{\sim}{\pm} e = \theta_H(r^1) + \bar{e} = b$ we have $\theta_H(r^0) \underset{\sim}{\pm} \theta_H(r^1) = \theta_H(r^0 + r^1) = b$, that is $r^0 + r^1 \in T(H, b)$. Therefore $\pi(r^0 + r^1) \geq 1$ because $\pi \in E^\beta(H, b)$. Therefore $\pi r^0 + \pi r^1 \geq 1 = \pi r^0 + \pi_e$, that is $\pi r^1 \geq \pi_e$. Then $1 = \pi r^1 + \pi_{\bar{e}} \geq \pi_e + \pi_{\bar{e}}$. Thus (5.1.13) holds.□

5.2. Master Semigroup Programs

(5.2.1) A Master Semigroup Program (H, b, c) is a semigroup program (H, b, c) over $(S, \underset{\sim}{+})$ where H is the whole S. Thereafter we will denote a master semigroup program by (S, b, c) understanding that it is over $(S, \underset{\sim}{+})$.

Assumptions (4.2.4) and (4.2.6) can be kept without loss of generality. When b is a loop element the master semigroup program (S^1, b, c) is equivalent to the master semigroup program (S, b, c) where $S = \{s \in S^1 : b \sim s \neq \emptyset\}$, that $(S, \underset{\sim}{+})$ is a semigroup was proved in corollary (4.2.17). Moreover (S, b, c) satisfies assumption (4.2.5). Since (S, b, c) is S-unbounded if and only if b is a loop element (by theorem (4.2.16)), we can assume (4.2.4) to (4.2.6) to be satisfied by S-unbounded master semigroup programs without loss of generality.

Let us consider the relation between the S-unbounded master semigroup program (S, b, c) and the semigroup program $(H, b, (c_h : h \in H))$ where H is a subset of S (we do not require H to be a generator of S , we only require that the semigroup generated by H to be a subsemigroup of S). It is easy to see that T(H, b) is equivalent to $T(S, b) \cap X^H$ where $X^H = \{x \in \mathbb{R}^S : x_s = 0$ for every $s \in S - H\}$. By theorem (4.1.12) E(H, b) is equivalent to $E(S, b) \cap X^H$. Therefore knowing a defining system for E(S, b) we know a defining system for E(H, b).

With this introduction we pass to obtain more stronger characterizations for defining systems of master semigroup programs.

For the rest of this section we will consider (S, b, c) to be a fixed S-unbounded master semigroup program satisfying assumptions (4.2.4) to (4.2.6) unless specifically omitted.

(5.2.2) In order for the inequality $x_s \geq 0$ to be redundant in the defining system for $E(S, b)$ given in (4.2.11) it is necessary that $\min_{t \in T(S,b)} \{t_s\} > 0$. Hence the only inequality that may be redundant is $x_b \geq 0$ because $\delta^b \in T(S, b)$ and then $\min_{t \in T(S,b)} \{t_s\} \leq \delta^b_s = 0$ for all $s \in S - \{b\}$.

(5.2.3) An S-unbounded master semigroup program (S, b, c) is <u>trivial</u> when $b \sim \{s\} = \{b\}$ for all $s \in S - \{b\}$.

(5.2.4) <u>Lemma</u>. Let (S, b, c) be trivial. Then the system

$$\begin{cases} x_b \geq 1 \\ x_s \geq 0 \quad \text{for all} \quad s \in S - \{b\}. \end{cases}$$

is an irredundant defining system for $E(S, b)$.

Proof: If we prove that $t \in T(S, b)$ implies $t_b \geq 1$ then the only minimal element in $T(S, b)$ is δ^b. By theorem (4.1.8) we have $E^\beta(S, b)$ is the solution set of

$$\begin{cases} \pi_b \geq 1, \\ \pi_s \geq 0 \quad \text{for all} \quad s \in S . \end{cases}$$

It is easy to see that the only extreme point of $E^{\beta}(S, b)$ is δ^b. Therefore the system of the lemma is an irredundant defining system for $E(S, b)$ by theorem (4.1.8) and (5.2.2).

Proof that $t \in T(S, b)$ implies $t_b \geq 1$.

Let T be the set $\{t \in T(S, b): t_b = 0\}$.

Let t^0 satisfied $\sum_{s \in S} t^0_s = \min_{t \in T}\{\sum_{s \in S} t_s\}$. t^0 cannot be 0, otherwise $\theta_s(t^0) = \theta_s(0) = o \neq b$ by assumption (4.2.4).

Hence there is $e \in S - \{b\}$ such that $t^0_e > 0$. Therefore $\theta_s(t^0) = \theta_s(t^0 - \delta^e) \underset{\sim}{+} \theta_s(\delta^e) = \theta_s(t^0 - \delta^e) \underset{\sim}{+} e = b$. Then $\theta_s(t^0 - \delta^e) \in b \sim e = \{b\}$. This is absurd because $\theta_s(t^0 - \delta^e) = b$ implies $t^0 - \delta^e \in T$ and $\sum_{s \in S} (t^0 - \delta^e)_s = \sum_{s \in S} t^0_s - 1 < \sum_{s \in S} t^0_s$ contradicting the choice of t^0. Thus T is empty, that is if $t \in T(S, b)$ then $t_b > 0$.□

(5.2.5) <u>Lemma</u>. For any non-trivial S-unbounded master semigroup program $(\tilde{S}, b, c)$ the system

$$(5.2.6)\begin{cases} vx \geq 1, \text{ for all extreme point } v \text{ of } E^{\beta}(S, b), \\ x_s \geq 0 \text{ , for all } s \in S. \end{cases}$$

Is an irredundant defining system for $E(S, b)$.

Proof: By (5.2.2) we only need to prove that $\min_{t \in T(S,b)} \{t_b\} = 0$

because then system (5.2.6) will be the same as (4.2.11).

Since (S, b, c) is non-trivial there are $s, h \neq b$ satisfying $s \underset{\sim}{+} h = b$. Then $\delta^s + \delta^h \in T(S, b)$ and $\min_{t \in T(S,b)} \{t_b\} \leq (\delta^s + \delta^h)_b = 0$. Since $T(S, b) \subseteq \mathbb{R}_+^S$ we also have $\min_{t \in T(S,b)} (t_b) \geq 0$. □

Let (S, b, c) be an S-unbounded master semigroup program. Then theorem (5.1.6) reads: When π is a support of $E(S, b)$, π satisfies:

(5.2.7) $\pi_b = 1$.

(5.2.8) Subadditivity: For all $r, s \in S$ we have $\pi_r + \pi_s \geq \pi_{r \underset{\sim}{+} s}$.

(5.2.9) Monotony: For all $s, r \in S$ if $b \sim s \subseteq b \sim r$ then $\pi_s \leq \pi_r$.

(5.2.10) Complementarity: For any $s \in S$ if $\bar{s}$ is a b-complementor of s then $\pi_s + \pi_{\bar{s}} = 1$.

Notice that (5.1.10) is a consequence of (5.2.7) and (5.2.10) because o is always a b-complementor of b since $b \underset{\sim}{+} o = b$ and $r \underset{\sim}{+} b = s \underset{\sim}{+} o = b$ implies $s = b$, hence $r \underset{\sim}{+} s = r \underset{\sim}{+} b = b$. Therefore $\pi_b + \pi_o = 1 + \pi_o = 1$.

(5.2.11) <u>Theorem</u>. Let (S, b, c) be an S-unbounded master semigroup program. If $\pi \in \mathbb{R}_+^S$ satisfies (5.2.7) and (5.2.8) then π belongs to $E^\beta(S, b)$.

Proof: Let $\pi \in \mathbb{R}_+^S$ satisfied (5.2.7) and (5.2.8).

Let $T = \{t \in T(S, b): \pi t < 1\}$.

Let t^0 be such that $\sum_{s \in S} t^0_s = \min_{t \in T}\{\sum_{s \in S} t_s\}$. If $\sum_{s \in S} t^0_s = 1$ then $t^0 = \delta^e$ for some $e \in S$ because $t^0 \in \mathbb{N}^S_+$. Moreover $b = \theta_S(t^0) = \theta_S(\delta^e) = e$ because $t^0 \in T(S, b)$. Hence $t^0 = \delta^b$ and $\pi t^0 = \pi \delta^b = \pi_b = 1$ by (5.2.7), contradicting that $\pi t^0 < 1$. Then we have $\sum_{s \in S} t^0_s \geq 2$. Therefore there are $r, h \in S$ satisfying $\delta^r + \delta^h \leq t^0$. By the substitution lemma (2.5.11) we have $t^1 = t^0 - \delta^r - \delta^h + \delta^{r \underset{\sim}{+} h} \in T(S, b)$. But now $\sum_{s \in S} t^1_s = \sum_{s \in S} t^0_s - 2 + 1 < \sum_{s \in S} t^0_s$, and $\pi t^1 = \pi t^0 - \pi_r - \pi_h + \pi_{r \underset{\sim}{+} h} \leq \pi t^0 < 1$ (by subadditivity of π), and this contradicts the choice of t^0.

Thus $\pi t \geq 1$ for all $t \in T(S, b)$ and $\pi \in \mathbb{R}^S_+$. By theorem (4.2.9) $\pi \in E^\beta(S, b)$. □

The next four theorems characterizing defining systems for $E(S, b)$ are a consequence of the one above and (5.2.7) to (5.2.10). They are the main results of this chapter.

(5.2.12) <u>Subadditivity Theorem</u>. Let (S, b, c) be an S-unbounded master semigroup program. Let <u>PS(S, b)</u> be the polyhedron of solutions π of the system

$$(5.2.13) \begin{cases} \pi_b = 1, & \\ \pi_r + \pi_s \geq \pi_{r \underset{\sim}{+} s} & \text{for all } r, s \in S, \\ \pi_s \geq 0 & \text{for all } s \in S. \end{cases}$$

Then the system

$$(5.2.14)\quad \begin{cases} vx \geq 1 & \text{for all extreme points } v \text{ of } PS(S,b), \\ x_s \geq 0 & \text{for all } s \in S. \end{cases}$$

is a defining system for $E(S, b)$.

Proof: Since $E(S, b) \in \mathbf{P}(\mathbb{R}_+^S)$ we have $E^\beta(S, b) \in \mathbf{P}(\mathbb{R}_+^S)$. Therefore $E^\beta(S, b) \subseteq \mathbb{R}_+^S$.

Let π be an extreme point of $E^\beta(S, b)$. Then π is a support of $E(S, b)$ (by definition (5.1.2) and lemma (3.3.28)).

(5.2.15) Therefore π satisfies: $\pi_b = 1$ (by (5.2.7)), $\pi_r + \pi_s \geq \pi_{r \tilde{+} s}$ for all $r, s \in S$ (by (5.2.8)); $\pi_s \geq 0$ for all $s \in S$ (because $\pi \in E^\beta(S, b) \subseteq \mathbb{R}_+^S$). Hence $\pi \in PS(S, b)$.

Thus the set of extreme points of $E^\beta(S, b)$ is contained in $PS(S, b)$.

That $PS(S, b) \subseteq E^\beta(S, b)$ is a consequence of theorem (5.2.12).

Hence the theorem is a special case of theorem (3.3.29) when $P = E(S, b)$ and $Q = PS(S, b)$. □

(5.2.16) Monotony Theorem. Let (S, b, c) be an S-unbounded master semigroup program. Let PM(S, b) be the polyhedron of solutions π of the system

$$(5.2.17)\quad \begin{cases} \pi_b = 1, & \\ \pi_s + \pi_h \geq \pi_{s \tilde{+} h} & \text{for all } s, h \in S, \\ \pi_s \leq \pi_r & \text{for all } s, r \in S \text{ satisfying } b \sim s \subseteq b \sim r, \\ \pi_s \geq 0 & \text{for all } s \in S. \end{cases}$$

Then the system

$$(5.2.18)\quad \begin{cases} vx \geq 1 & \text{for all extreme points } v \text{ of } PM(S,b), \\ x_s \geq 0 & \text{for all } s \in S. \end{cases}$$

is a defining system for E(S, b).

(Same proof that (5.2.12) using PM(S, b) instead PS(S,b) and adding "π satisfies $\pi_s \leq \pi_r$ for all $s, r \in S$ satisfying $b \sim s \subseteq b \sim r$ (by (5.2.9))" in (5.2.15)). □

(5.2.19) Complementarity Theorem. Let (S, b, c) be an S-unbounded master semigroup program. Let PC(S, b) be the polyhedron of solutions π of the system,

$$(5.2.20)\quad \begin{cases} \pi_b = 1, \\ \pi_s + \pi_h \geq \pi_{s \underset{\sim}{+} h} & \text{for all } s, h \in S, \\ \pi_s + \pi_{\bar{s}} = 1 & \text{for all } s \in S \text{ when } \bar{s} \text{ is a b-complementor of } s, \\ \pi_s \geq 0 & \text{for all } s \in S. \end{cases}$$

Then the system

$$(5.2.21)\quad \begin{cases} vx \geq 1 & \text{for all extreme points } v \text{ of } PC(S,b), \\ x_s \geq 0 & \text{for all } s \in S. \end{cases}$$

is a defining system for E(S, b).

(Same proof as (5.2.12) using PC(S, b) instead of PS(S, b) and adding "π satisfies $\pi_s + \pi_{\bar{s}} = 1$ for all $s \in S$ when $\bar{s}$ is a b-complementor of s (by (5.2.10))" in (5.2.15)). □

(5.2.22) Let $(S, \underset{\sim}{+})$ be a semigroup and b an element of S . The semigroup is called b-complementary when every element in S has a b-complementor and <u>unique b-complementary</u> when it is b-complementary and there is only one complementor for each element in S.

(5.2.23) <u>Lemma</u>. Let (H, b, c) be a semigroup program over the b-complementary semigroup $(S, \underset{\sim}{\pm})$. Then (H, b, c) is H-unbounded.

Proof: We will show a stronger result:

(5.2.24) b is a loop element if and only if $b \sim (\circ(b) \cdot b) \neq \emptyset$.

If (5.2.24) holds then the lemma is proved because when the semigroup is b-complementary we have $b \sim s \neq \emptyset$ for all $s \in S$ (since the b-complementor of s belongs to $b \sim s$) in particular $b \sim (\circ(b) \cdot b) \neq \emptyset$ and by (5.2.24) b is a loop element. Now (H, b, c) is H-unbounded by theorem (4.2.16).

Proof of (5.2.24): Let $p \cdot b = \circ(b) \cdot b$, $p < \circ(b)$ (such a p exists by definition (2.5.7) of $\circ(b)$). Denoting by $q \equiv \circ(b) \cdot p$ we have $p \cdot b = (p + q) \cdot b$.

Let b be a loop element. Then there is $k \geq \circ(b) > p$ such that $k \cdot b = b$, therefore $b = (k - p) \cdot b \underset{\sim}{\pm} p \cdot b = (k - p) \cdot b \underset{\sim}{\pm} \circ(b) \cdot b$, hence $b \sim (\circ(b) \cdot b)$ is not empty because it contains $(k - p) \cdot b$.

Let $s \in b \sim (\circ(b) \cdot b) \neq \emptyset$. Since $\circ(b) \cdot b = p \cdot b$ we have $s \underset{\sim}{\pm} p \cdot b = b$. Hence we have
$b = s \underset{\sim}{\pm} p \cdot b = s \underset{\sim}{\pm} (p + q) \cdot b = (s \underset{\sim}{\pm} p \cdot b) \underset{\sim}{\pm} q \cdot b = b \underset{\sim}{\pm} q \cdot b$.

Thereafter b is a loop element. □

(5.2.25) <u>Strong Complementarity Theorem</u>: Let (S, b, c) be a non-trivial master semigroup program with (S, $\underset{\sim}{+}$) b-complementary. Then the system (5.2.21) is an irredundant defining system for E(S, b).

Proof: By lemma (5.2.23) we have (S, b, c) is S-unbounded. Hence (5.2.21) is a defining system for E(S, b) by the Complementary Theorem (5.2.19).

By lemma (5.2.5) we have for all $s \in S$ the inequality $x_s \geq 0$ is irredundant in (5.2.21) because (S, b, c) is non-trivial.

In order to show that for all extreme points v of PC(S, b), the inequality $vx \geq 1$ is irredundant in (5.2.21) would be enough to show that:

(5.2.26) All the points of PC(S, b) are minimal in PC(S, b), and apply theorem (3.3.29) with P = E(S, b) and Q = PC(S, b).

Proof of (5.2.26): Let π and π^1 in PC(S, b) satisfy $\pi_s < \pi_s^1$. Let $\bar{s}$ be a b-complementor of s (there is such an $\bar{s}$ because (S, $\underset{\sim}{+}$) is b-complementary).

Then we have $\pi_s + \pi_{\bar{s}} = \pi_s^1 + \pi_{\bar{s}}^1 = 1$. Since $\pi_s < \pi_s^1$ we obtain $\pi_{\bar{s}} > \pi_{\bar{s}}^1$. Therefore all the points in PC(S, b) are minimal in PC(S, b). □

(5.2.27) Let (S, $\underset{\sim}{+}$) be a semigroup. We denote by $\underline{m_s}$ the number of b-complementors of $s \in S$ in (S, $\underset{\sim}{+}$).

(5.2.28) Lemma. Let (S, b, c) be an S-unbounded master semigroup program with $(S, \underset{\sim}{\pm})$ being b-complementary. Let π be a support of P(S, b). Then $\sum_{s \in S} m_s \pi_s = \frac{1}{2} \sum_{s \in S} m_s$.

Proof: By (5.2.10) $\pi_s + \pi_{\bar{s}} = 1$ for each $\bar{s}$ which is a b-complementor of s. If we add all these equations π_s will appear m_s times as first term and m_s times as second term (by (5.1.7)). Hence the sum of all these equations would add to $\sum_{s \in S} 2m_s \pi_s$. Clearly there are $\sum_{s \in S} m_s$ of these equations. Therefore $\sum_{s \in S} 2m_s \pi_s = \sum_{s \in S} m_s$. □

(5.2.29) Corollary. In the conditions of the lemma above, if $(S, \underset{\sim}{\pm})$ is unique b-complementary then $\sum_{s \in S} \pi_s = \frac{|S|}{2}$. □

(5.2.30) Constant Sum Theorem. Let (S, b, c) be a non-trivial master semigroup program with $(S, \underset{\sim}{\pm})$ b-complementary. Let PCS(S, b) be the polyhedron of solutions π to the system,

$$(5.2.31) \begin{cases} \pi_b = 1, \\ \pi_s + \pi_h \geq \pi_{s \underset{\sim}{\pm} h} & \text{for all } s, h \in S \\ \sum_{s \in S} m_s \pi_s = \frac{1}{2} \sum_{s \in S} m_s , \\ \pi_s \geq 0 & \text{for all } s \in S . \end{cases}$$

Then the system

$$(5.2.32) \begin{cases} vx \geq 1 & \text{for all extreme points v of PCS(S,b),} \\ x_s \geq 0 & \text{for all } s \in S. \end{cases}$$

is an irredundant defining system for E(S, b).

Proof: The proof that (5.2.32) is a defining system for $E(S, b)$ is the same as (5.2.12) using also (5.2.28) in (5.2.15).

That the system is irredundant follows from theorem (3.3.29) because the equation $\sum_{s \in S} m_s \pi_s = \frac{1}{2} \sum_{s \in S} m_s$ insures minimality of the solutions of (5.2.31) since all the coefficients are greater than zero. □

Notice that this theorem is tighter than (4.2.25) since any solution to (5.2.21) satisfies (5.2.31).

The examples below show that systems (5.2.14) and (5.2.18) can be redundant:

(5.2.33) Example of $PS(S, b)$ with vertex not minimal.

The semigroup is over $\{1, 2, 3, 4\}$ with $\underset{\sim}{+}$ defined by $s \underset{\sim}{+} h = \min\{s + h, 4\}$ and $b = 4$.

We claim that $v^0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$ and $v^1 = (\frac{1}{2}, 1, \frac{1}{2}, 1)$ are vertices of (5.2.14). Therefore v^1 is not minimal.

It is easy to verify that v^0 and v^1 satisfy (5.2.14), moreover v^0 satisfies as equalities:

$$\begin{cases} \pi_4 = 1 \\ \pi_1 + \pi_3 = \pi_4 \\ \pi_2 + \pi_2 = \pi_4 \\ \pi_3 + \pi_3 = \pi_4 \end{cases} \quad \text{i.e.} \quad \begin{cases} \pi_4 = 1 \\ \pi_1 + \pi_3 - \pi_4 = 0 \\ 2\pi_3 - \pi_4 = 0 \\ 3\pi_3 - \pi_4 = 0 \end{cases}$$

since $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \end{pmatrix}$ is not singular v^0 is a vertex.

v^1 satisfies as equalities

$$\begin{cases} \pi_4 = 1 \\ 2\pi_1 - \pi_2 = 0 \\ \pi_1 + \pi_3 - \pi_4 = 0 \\ 2\pi_3 - \pi_4 = 0 \end{cases} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 2 & -3 \end{pmatrix} \text{ is not singular.}$$

(5.2.34) Example of PM(S, b) with vertex not minimal. Consider the same program as in (5.2.33). We have $4 \sim 1 = \{3, 4\}$, $4 \sim 2 = \{2, 3, 4\}$, $4 \sim 3 = \{1, 2, 3, 4\}$ and $4 \sim 4 = \{o, 1, 2, 3, 4\}$. Therefore $\pi_1 \leq \pi_2 \leq \pi_3 \leq \pi_4$ for any π in PM(S, b).

Clearly $v^0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$ is an extreme point of PM(S, b) and $v^1 = (1, 1, 1, 1) \in$ PM(S, b) is not minimal.

v^1 satisfies as equalities

$$\begin{cases} \pi_1 - \pi_2 = 0 \\ \pi_2 - \pi_3 = 0 \\ \pi_3 - \pi_4 = 0 \\ \pi_4 = 1 \end{cases} \quad \text{and} \quad \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ is not singular.}$$

This leaves us only system (5.2.20) in the Complementarity Theorem. We have not been able to find an example with redundant inequalities in (5.2.20), furthermore trying to find each example we arrived at the following conjecture:

(5.2.35) <u>Conjecture</u>. Let (S, b, c) be a non-trivial S-unbounded master semigroup program and PC(S, b) be defined as in (5.2.19). Then the system

$$\begin{cases} vx \geq 1 \text{ for all extreme points } v \text{ of } PC(S, b), \\ x_s \geq 0 \text{ for all } s \in S . \end{cases}$$

is an irredundant defining system for (S, b).

Notice that conjecture (5.2.35) and theorem (5.2.25) are equivalent when the semigroup is b-complementary. Since a group is always b-complementary for any b in the group, the conjecture doesn't have any effect in group programs. The situation changes when we consider covering programs because, although a covering semigroup $(A(b), \underset{\sim}{+})$ is b-complementary, a subsemigroup of $(A(b), \underset{\sim}{+})$ containing b may be not.

The next example shows a subsemigroup of a covering semigroup which contains b and is not b-complementary.

(5.2.36) Let $b = (1, 1, 1)$, $s_3 = (0, 1, 1)$, $s_5 = (1, 0, 1)$, $s_6 = (1, 1, 0)$, $0 = (0, 0, 0)$ and let $H = \{b, s_3, s_5, s_6, 0\}$. It is easy to check that $(H, \underset{\sim}{+})$ is a subsemigroup of the covering semigroup $(A((1, 1, 1)), \underset{\sim}{+})$.

s_3 has no b-complementor because

$b \sim s_3 = \{b , s_5, s_6\}$ and

b is not a b-complementor of s_3 since $s_3 \underset{\sim}{+} s_5 = 0 \underset{\sim}{+} b = b$ and $s_5 \underset{\sim}{+} 0 = s_5 \neq b$.

s_5 is not a b-complementor of s_3 since $s_3 \underset{\sim}{+} s_6 = s_6 \underset{\sim}{+} s_5 = b$ and $s_6 \underset{\sim}{+} s_6 = s_6 \neq b$.

s_6 is not a b-complementor of s_3 since $s_3 \underset{\sim}{+} s_5 = s_5 \underset{\sim}{+} s_6 = b$ and $s_5 \underset{\sim}{+} s_5 = s_5 \neq b$.

Thus $(H, \underset{\sim}{+})$ is not b-complementary.

Chapter 6

b-COMPLEMENTARY SEMIGROUPS

We have shown in the introduction chapter that groups are uniquely b-complementary for any element b in the group and that any covering semigroup $(A(b), \underset{\sim}{+})$ is uniquely b-complementary.

We will present in this chapter a large family of uniquely b-complementary semigroups. This family includes groups and covering semigroups.

We also show that given any unbounded semigroup program there is an equivalent semigroup program over a semigroup with unique complementors.

6.1. Sum of Semigroups

(6.1.1) Let K be a finite set and for each $k \in K$, let $(S^k, \underset{\sim}{+})$ be a semigroup. The semigroup $(S, \underset{\sim}{+})$, where $S = \underset{k \in K}{X} S^k$ and where for any $s, h \in S$, $(s \underset{\sim}{+} h)_k = (s_k \underset{\sim}{+} h_k)$, is called the <u>sum of $(S^k, \underset{\sim}{+}): k \in K$</u>.

(6.1.2) <u>Lemma</u>. Let $(S, \underset{\sim}{+})$ be the sum of $(S^k, \underset{\sim}{+}): k \in K$ and $b \in S$. Then b is a loop element in $(S, \underset{\sim}{+})$ if and only if b_k is a loop element in $(S^k, \underset{\sim}{+})$ for all $k \in K$.

Proof: From definition (6.1.1) if $b = q \cdot b$ then $b_k = q \cdot b_k: k \in K$. Therefore b_k is a loop element in $(S^k, \underset{\sim}{+})$ for all $k \in K$ when b is a loop element in $(S, \underset{\sim}{+})$.

Let b_k be a loop element in $(S^k, \underset{\sim}{+})$ for all

$k \in K$. Then there is a vector $(q_k \geq 1 : k \in K)$ such that $b_k = (1 + q_k) \cdot b_k : k \in K$. Clearly $b_k = (1 + \lambda_k q_k) b_k$ for all $\lambda_k \in \mathbb{N}_+ : k \in K$. Let $q = \prod_{k \in K} q_k$ and $\lambda_k = \frac{q}{q_k}$. Therefore $b_k = (1 + q) \cdot b_k : k \in K$, that is $b = (1 + q) \cdot b$, since $q \geq 1$ $(q_k \geq 1 : k \in K)$ b is a loop element in $(S, \underset{\sim}{+})$. □

We recall that the semigroup program (H, b, c) over $(S, \underset{\sim}{+})$ is H-unbounded if and only if b is a loop element in $(S, \underset{\sim}{+})$, by theorem (4.2.16).

(6.1.3) <u>Lemma</u>. Let $(S, \underset{\sim}{+})$ be the sum of $(S^k, \underset{\sim}{+}) : k \in K$ and let $b \in S$. For any $s \in S$, $\bar{s}$ is a b-complementor of s in $(S, \underset{\sim}{+})$ if and only if $\bar{s}_k$ is a b_k-complementor of s_k for all $k \in K$.

Proof: Let $\bar{s}$ be a b-complementor of s in $(S, \underset{\sim}{+})$.

Let $e \in K$ and let $h_e, r_e \in S^e$. Consider the vectors $h, r \in S$ defined by

$$(6.1.4)\quad h_k = \begin{cases} \bar{s}_k & \text{if } k \neq e \\ h_e & \text{if } k = e \end{cases} \quad \text{and} \quad r_k = \begin{cases} s_k & \text{if } k \neq e \\ r_e & \text{if } k = e. \end{cases}$$

Since $s \underset{\sim}{+} \bar{s} = b$ we have $s_e \underset{\sim}{+} \bar{s}_e = b_e$. Let $s_e \underset{\sim}{+} h_e = r_e \underset{\sim}{+} \bar{s}_e = b_e$ then $s \underset{\sim}{+} h = r \underset{\sim}{+} \bar{s} = b$. Therefore $h \underset{\sim}{+} r = b$ because $\bar{s}$ is a b-complementor of s. Hence $h_e \underset{\sim}{+} r_e = b_e$. Therefore $\bar{s}_e$ is a b_e-complementor of s_e in $(S^e, \underset{\sim}{+})$. Thus $\bar{s}_k$ is a b_k-complementor of s_k in $(S^k, \underset{\sim}{+})$ for all $k \in K$

since the choice of e was arbitrary.

Let $\bar{s}_k$ be a b_k-complementor of s_k for all $k \in K$. Then $s \underset{\sim}{+} \bar{s} = b$ and if $s \underset{\sim}{+} h = r \underset{\sim}{+} \bar{s} = b$ then we have $s_k \underset{\sim}{+} h_k = r_k \underset{\sim}{+} \bar{s}_k = b_k$ for all $k \in K$. Therefore for all $k \in K$ we have $h_k \underset{\sim}{+} r_k = b_k$ because $\bar{s}_k$ is a b_k-complementor of s_k, that is $h \underset{\sim}{+} r = b$. Thus $\bar{s}$ is a b-complementor of s in $(S, \underset{\sim}{+})$. □

(6.1.5) <u>Corollary</u>. Let $(S, \underset{\sim}{+})$ be the sum of $(S^k, \underset{\sim}{+}): k \in K$ and let $b \in S$. Then $(S, \underset{\sim}{+})$ is b-complementary if and only if $(S^k, \underset{\sim}{+})$ is b_k-complementary for all $k \in K$. □

(6.1.6) <u>Corollary</u>. Let $(S, \underset{\sim}{+})$ be the sum of $(S^k, \underset{\sim}{+}): k \in K$ and let $b \in S$. Then $(S, \underset{\sim}{+})$ is uniquely b-complementary if and only if $(S^k, \underset{\sim}{+})$ is uniquely b_k-complementary for all $k \in K$.

Proof: Let $(S, \underset{\sim}{+})$ be b-complementary.

Let $s \in S$. If r, h are b-complementors of s and $r \neq h$ then there is $k \in K$ such that $r_k \neq h_k$. By (6.1.3) r_k and h_k are b_k-complementors of s_k. Hence $(S^k, \underset{\sim}{+})$ is not uniquely b_k-complementary.

Let $s_e \in S^e$, let r_e, h_e be b_e-complementors of s_e and let $r_e \neq h_e$. Let $s_k \in S^k$ for all $k \in K - \{e\}$. Since $(S, \underset{\sim}{+})$ is b-complementary, there is a b-complementor $\bar{s}$ of $s = (s_k: k \in K)$. By (6.1.3) we have $r = (r_e; \bar{s}_k: k \in K - \{e\})$ and $h = (h_e; \bar{s}_k: k \in K - \{e\})$ are b-complementors of s. Thus $(S, \underset{\sim}{+})$ is not uniquely b-complementary because $r \neq h$. □

6.2. Cyclic Semigroups

(6.2.1) A cyclic semigroup is a semigroup generated by one element. If x is the generator then its elements are

$$0 \cdot x,\ 1 \cdot x, \ldots, c \cdot x,\ (c+1) \cdot x, \ldots, (c+d) \cdot x = c \cdot x, \ldots$$

Where $c \cdot x$ is the first element in the loop of x and d is the number of elements in the loop of x. This semigroup is denoted by $<c;\ d>$ and the loop of x (see (2.5.7)) is called the loop of $<c;\ d>$.

(6.2.2) It is easy to see that $<c;\ d>$ is a group if and only if $c = 0$. And that $<c;\ d>$ is a covering semigroup if and only if $d = 1$.

(6.2.3) Clearly any cyclic semigroup $<c;\ d>$ can be represented by the semigroup $(\{0, 1, \ldots, c, \ldots, c+d-1\}, \underset{\sim}{+})$ where for all non-negative integers s, h we define

$$s \underset{\sim}{+} h = \begin{cases} s + h & \text{if } s + h < c \\ r \text{ (where } c \le r < c + d \text{ and } r - c \text{ is} \\ \text{congruent to } s + h - c \text{ module } d) \\ \text{if } s + h \ge c. \end{cases}$$

Thereafter we will consider this representation for $<c;\ d>$.

(6.2.4) Proposition. Let b be in the loop of $<c;\ d>$ and let s, h be two elements of $<c;\ d>$. Then we have $s \underset{\sim}{+} h = b$ if and only if there is a non-negative integer k such that $s + h = b + kd$.

Proof: Let $s \underset{\sim}{+} h = b$. Since $b \geq c$ (because b is a loop element) then $s + h - c$ is congruent to $b - c$ module d (by (6.2.3)). Therefore there is a non-negative integer k such that $b - c + kd = s + h - c$, that is $b + kd = s + h$.

Let k be a non-negative integer and let $b + kd = s + h$. Then $b - c$ is congruent to $s + h - c$ module d and $s + h \geq b \geq c$. Thus we have $s \underset{\sim}{+} h = b$ by (6.2.3). □

(6.2.5) Lemma. Let $(S, \underset{\sim}{+})$ be a subsemigroup of $\langle c; d \rangle$ and let $b \in S$ belong to the loop of $\langle c; d \rangle$. For any $s \in S$ the element $\bar{s} = \min_{h \in b \sim s} \{h\}$ is a b-complementor of s in $(S, \underset{\sim}{+})$.

Proof: Let $s \in S$.

$b \sim s$ is non-empty because $b \underset{\sim}{+} d \cdot s = b$, by (6.2.4), and $b \underset{\sim}{+} (d - 1) \cdot s \in S$ since $b, s \in S$. Hence $b \underset{\sim}{+} (d - 1)s \in b \sim s$.

Let $\bar{s} = \min_{h \in b \sim s} \{h\}$. Therefore $s \underset{\sim}{+} \bar{s} = b$. To prove that $\bar{s}$ is a b-complementor of s in $(S, \underset{\sim}{+})$ we only need to show that for any $h, r \in S$ satisfying $s \underset{\sim}{+} h = r \underset{\sim}{+} \bar{s} = b$ we have $h \underset{\sim}{+} r = b$.

By (6.2.4) we have:

$s \underset{\sim}{+} \bar{s} = b$ iff $s + \bar{s} = b + k_s d$ where $k_s \in \mathbb{N}_+$.

$s \underset{\sim}{+} h = b$ iff $s + h = b + k_n d$ where $k_n \in \mathbb{N}_+$.

$r \underset{\sim}{+} \bar{s} = b$ iff $r + \bar{s} = b + k_r d$ where $k_r \in \mathbb{N}_+$.

Hence $s + h + r + \bar{s} = 2b + (k_h + k_r)d$, that is

$$h + r = 2b + (k_h + k_r)d - (s + \bar{s}) = b + (k_h + k_r - k_s)d.$$

By (6.2.4) we have $h \underset{\sim}{+} r = b$ because $k_s \le k_h$ by choice of $\bar{s}$. □

(6.2.6) Corollary. Let $(S, \underset{\sim}{+})$ be a subsemigroup of $<c; d>$ and let $b \in S$ belong to the loop of $<c; d>$. Then $(S, \underset{\sim}{+})$ is b-complementary. □

Combining corollaries (6.1.5) and (6.2.6) we obtain the following family of b-complementary semigroups.

(6.2.7) Let $(S, \underset{\sim}{+})$ be a sum of subsemigroups of $<c_k; d_k>: k \in K$ and let $b = (b_k \in \mathbb{N}_+: k \in K)$ satisfy $c_k \le b_k < c_k + d_k$ for all $k \in K$. Then $(S, \underset{\sim}{+})$ is b-complementary.

In the next section we will extend this family.

6.3. Reduction of Semigroups.

(6.3.1) Let $(S, \underset{\sim}{+})$ be a semigroup and let $b \in S$ be a loop element. Let the relation ω in S^2 be defined by $s\omega h$ if $b \sim s = b \sim h$. Let R be the family of equivalence classes of ω. We denote by R(S, b) the family $R - \{\{s \in S: b \sim s = \emptyset\}\}$.

(6.3.2) Proposition. $\{b\}$ belongs to R(S, b), because $o \in b \sim b$ and $o \notin b \sim s$ for any $s \ne b$. Hence the only element in the equivalence class of b is b itself. □

(6.3.3) For any two elements r^1, r^2 in $R(S, b)$ we define $r^1 \underset{\sim}{+} r^2$ to be r if there are $s^1, s^2 \in S$ such that $s^1 \in r^1$, $s^2 \in r^2$ and $s^1 \underset{\sim}{+} s^2 \in r$.

(6.3.4) <u>Lemma</u>. Let $(S, \underset{\sim}{+})$ be a semigroup and let $b \in S$ be a loop element. Then $(R(S, b), \underset{\sim}{+})$ is a semigroup.

Proof: To prove the lemma we only need to show that for any r^1, r^2 in $R(S, b)$ there is one and only one $r \in R(S, b)$ satisfying $r = r^1 \underset{\sim}{+} r^2$. Because the other conditions of a semigroup are trivially satisfied.

Let $s^1 \in r^1$ and $s^2 \in r^2$. Then $b \sim s^1$ and $b \sim s^2$ are non-empty. Hence (by (4.2.17)) $b \sim (s^1+s^2)$ is non-empty. Therefore there is $r \in R(S, b)$ such that $s^1 \underset{\sim}{+} s^2 \in r$. Thus $r = r^1 \underset{\sim}{+} r^2$.

To show that r is unique is enough to show that $b \sim s = b \sim s^1$ and $b \sim h = b \sim h^1$ imply $b \sim (s \underset{\sim}{+} h) = b \sim (s^1 \underset{\sim}{+} h^1)$.

Let $b \sim s = b \sim s^1$ and let $b \sim h = b \sim h^1$. Then $x \in b \sim (s \underset{\sim}{+} h)$ iff $s \underset{\sim}{+} h \underset{\sim}{+} x = b$ iff $h \underset{\sim}{+} x \in b \sim s$ iff $h \underset{\sim}{+} x \in b \sim s^1$ iff $h \underset{\sim}{+} x \underset{\sim}{+} s^1 = b$ iff $s^1 \underset{\sim}{+} x \in b \sim h$ iff $s^1 \underset{\sim}{+} x \in b \sim h^1$ iff $s^1 \underset{\sim}{+} h^1 \underset{\sim}{+} x = b$ iff $x \in b \sim (s^1 \underset{\sim}{+} h^1)$.

Thus $b \sim (s \underset{\sim}{+} h)$ equals $b \sim (s^1 \underset{\sim}{+} h^1)$. □

(6.3.5) We call $(R(S, b), \underset{\sim}{+})$ the <u>b-reduction</u> of

$(S, \underset{\sim}{+})$. We say that $(S, \underset{\sim}{+})$ is <u>b-reduced</u> if $(S, \underset{\sim}{+})$ is isomorphic to $(R(S, b), \underset{\sim}{+})$.

(6.3.6) <u>Proposition</u>. Let $(S, \underset{\sim}{+})$ be a semigroup and let $b \in S$. Let s, e, h belong to S. If e, h are b-complementors of s then $b \sim e = b \sim h$.

Proof: Let e, h be b-complementors of s. Let $x \in b \sim e$. Then we have $s \underset{\sim}{+} h = x \underset{\sim}{+} e = b$. Since e is a b-complementor of s we have $h \underset{\sim}{+} x = b$. Therefore $x \in b \sim h$. Similarly we obtain if $x \in b \sim h$ then $x \in b \sim e$. Thus $b \sim e = b \sim h$. □

The next two statements are an immediate consequence of (6.3.6).

(6.3.7) If $(S, \underset{\sim}{+})$ is b-reduced then the b-complementors are unique.

(6.3.8) "$(S, \underset{\sim}{+})$ is b-complementary and b-reduced" is equivalent to "$(S, \underset{\sim}{+})$ is uniquely complementary".

(6.3.9) <u>Lemma</u>. Let $(S, \underset{\sim}{+})$ be a b-complementary semigroup. Then $(R(S, b), \underset{\sim}{+})$ is a uniquely $\{b\}$-complementary semigroup.

Proof: Let r, r^1 belong to $R(S, b)$ and let $s \in r$ and $s^1 \in r^1$.

First we will prove,

(6.3.10) r^1 is a $\{b\}$-complementor of r in

$(R(S, b), \underset{\sim}{+})$ if and only if s^1 is a b-complementor of s in $(S, \underset{\sim}{+})$.

Let $s \underset{\sim}{+} s^1 \neq b$. Then $r \underset{\sim}{+} r^1 \neq \{b\}$, that is r^1 is not a b-complementor of r.

Let $s \underset{\sim}{+} s^1 = b$. Then $r \underset{\sim}{+} r^1 = \{b\}$.

Let $p, p^1 \in R(S, b)$ and let $h \in p$, $h^1 \in p^1$.

Let $s \underset{\sim}{+} h = s^1 \underset{\sim}{+} h^1 = b$. Then $r \underset{\sim}{+} p = r^1 \underset{\sim}{+} p^1 = \{b\}$.

If s^1 is a b-complementor of s then we have $h \underset{\sim}{+} h^1 = b$. Hence $p \underset{\sim}{+} p^1 = \{b\}$. Therefore r^1 is a $\{b\}$-complementor of r.

If s^1 is not a b-complementor of s then there are h, h^1, as defined above, satisfying $h \underset{\sim}{+} h^1 \neq b$. Then $p \underset{\sim}{+} p^1 \neq \{b\}$. Thus r^1 is not a $\{b\}$-complementor of r. This completes the proof of (6.3.10).

Let $(S, \underset{\sim}{+})$ be b-complementary. Then $(R(S, b), \underset{\sim}{+})$ is $\{b\}$-complementary by (6.3.10). Therefore $(R(S, b), \underset{\sim}{+})$ is uniquely $\{b\}$-complementary by (6.3.6). □

(6.3.11) Let (S, b, c) be an S-unbounded master semigroup program. For any $r \in R(S, b)$ we denote by $z_r = \min_{s \in r}\{c_s\}$ and we denote by $\bar{r}$ a fixed element in r satisfying $c_{\bar{r}} = z_r$. We say that the vector t in $\mathbb{N}_+^S$ is equivalent to the vector $n = (n_r \in \mathbb{N}_+ : r \in R(S, b))$ if t satisfies

$$\text{for all } s \in S:\ t_s = \begin{cases} n_r & \text{if } s = \bar{r} \text{ for some } r \in R(S, b) \\ 0 & \text{otherwise.} \end{cases}$$

(6.3.12) <u>Theorem</u>. Let (S, b, c) be an S-unbounded master semigroup program. If n^0 is an optimal solution of $(R(S, b), \{b\}, z)$ then the vector t^0 equivalent to n^0 is an optimal solution of (S, b, c).

Proof: Let n^0 be an optimal solution of $(R(S, b), \{b\}, z)$ and let t^0 be equivalent to n^0. Let $R = \{\bar{r}: r \in R(S, b)\}$.

Then $\underset{s \in S}{\tilde{\Sigma}} t^0_s \cdot s = \underset{\bar{r} \in R}{\tilde{\Sigma}} t^0_{\bar{r}} \cdot \bar{r} = \underset{\bar{r} \in R}{\tilde{\Sigma}} n^0_r \cdot \bar{r} = b$ because $\bar{r} \in r$ for all $r \in R(S, b)$ and $\underset{r \in \tilde{R}(S,b)}{\Sigma} n^0_r \cdot r = \{b\}$ (by choice of n^0). Therefore $t^0 \in T(S, b)$.

Clearly we have $zn^0 = \sum_{\bar{r} \in R} c_{\bar{r}} n^0_r = \sum_{\bar{r} \in R} c_{\bar{r}} t^0_{\bar{r}} = \sum_{s \in S} c_s t^0_s = ct^0$.

(6.3.13) Let $t^1 + \delta^h \in T(S, b)$ and let $h \in r \in R(S, b)$. Then $\underset{s \in S}{\tilde{\Sigma}} t^1_s \cdot s \,\tilde{+}\, h = b$. Therefore $\underset{s \in S}{\tilde{\Sigma}} t^1_s \cdot s \in b \sim h = b \sim \bar{r}$. Thus $t^1 + \delta^{\bar{r}} \in T(S, b)$.

For any $t \in T(S, b)$ we define

$$\bar{t}_s = \begin{cases} \sum_{h \in r} t_h & \text{if } s = \bar{r}, \\ 0 & \text{otherwise.} \end{cases}$$

Iterating (6.3.13) we obtain $\bar{t} \in T(S, b)$.
Moreover we obtain $c\bar{t} \le ct$ by iterating

$$c(t^1 + \delta^{\bar{r}}) = ct^1 + c_{\bar{r}} \le ct^1 + c_h = c(t^1 + \delta^h).$$

Let $n = (n_r: n_r = \bar{t}_{\bar{r}}$ for all $r \in R(S, b))$. Clearly $n \in T(R(S, b), \{b\})$ and $ct^0 = zn^0 \le zn = c\bar{t} \le ct$. Hence t^0 is an optimal solution of (S, b, c).□

Theorem (6.3.12) is the main motivation to study reductions of semigroup. Now we pause to consider some more properties of reductions.

(6.3.14) <u>Proposition</u>. Let $(S, \underset{\sim}{+})$ be a subsemigroup of the cyclic semigroup $\langle c; d \rangle$ and let b be a loop element in $(S, \underset{\sim}{+})$. Then the b-reduction of $(S, \underset{\sim}{+})$ is uniquely $\{b\}$-complementary.

Proof: If b belongs to the loop of $\langle c; d \rangle$ then $(S, \underset{\sim}{+})$ is b-complementary by (6.2.6). Therefore the b-reduction of $(S, \underset{\sim}{+})$ is uniquely b-complementary by (6.3.9).

If b is a loop element of $(S, \underset{\sim}{+})$ and b does not belong to the loop of $\langle c; d \rangle$ then the only possibility is $b = o$ and $c > 0$.

In this case we have $o \sim o = \{o\}$ and $o \sim s = \emptyset$ for all $s \in S - \{o\}$. Hence $R(S, o) = \{\{o\}\}$. Trivially $(R(S, o), \underset{\sim}{+})$ is uniquely $\{o\}$-complementary.□

(6.3.15) <u>Theorem</u>. Let $(S, \underset{\sim}{+})$ be a semigroup and $b \in S$ be a loop element. Let $(S, \underset{\sim}{+})$ be a sum of b_k-reduced subsemigroups of $\langle c_k; d_k \rangle$: $k \in K$. Then $(S, \underset{\sim}{+})$ is uniquely b-complementary.

Proof: Immediate from (6.1.6) and (6.3.14). □

Theorem (6.3.15) provides a large family of uniquely b-complementary semigroups. That this family is larger than the one obtained using only sums of uniquely b_k-complementary subsemigroup of cyclic groups is shown by the next example.

Consider the cyclic semigroup $<6; 2>$ and the **sub-semigroup generated by $\{2, 3\}$ and its 7-reduction,** call $(S, \underset{\sim}{+})$ this 7-reduced semigroup. It is easy to check that $S = \{\{0\}, \{2\}, \{3\}, \{5\}, \{4, 6\}, \{7\}\}$ and for all $s \in S$, $s \neq \{0\}$ we have $2 \cdot s = \{4, 6\}$.

(6.3.16) We will show that $(S, \underset{\sim}{+})$ is not a sub-semigroup of a sum of cyclic semigroups. We know $(S, \underset{\sim}{+})$ is 7-complementary by (6.3.14).

Assume $(S, \underset{\sim}{+}) = (\{x^s \in \mathbb{N}_+^K: s \in S\}, \underset{\sim}{+})$ is a sub-semigroup of the sum of $<c_k; d_k>: k \in K$.

Notice that $\{5\}$ is not a loop element of $(S, \underset{\sim}{+})$ but $\{4, 6\}$ is one. By lemma (6.1.2) we have

(6.3.17) There is $e \in K$ such that $x_e^{\{2\}} + x_e^{\{3\}} < c_e$, and

(6.3.18) For all $k \in K$ $2x_k^{\{2\}} \geq c_k$ and $2x_k^{\{3\}} \geq c_k$.

From (6.3.18) we obtain $x_k^{\{2\}} + x_k^{\{3\}} \geq \frac{c_k}{2} + \frac{c_k}{2} = c_k$ for all $k \in K$ contradicting (6.3.17).

A well known theorem of algebra says that any finite commutative group is a sum of cyclic groups. This is not true for b-complementary semigroups. We will show in the next example that also reductions of subsemigroups of cyclic semigroups are not enough to describe b-complementary semigroups.

(6.3.19) There are unique b-complementary semigroups that are no sum of b_k-reductions of subsemigroups of cyclic semigroups.

Consider the sum of $\langle 1; 2\rangle$ and $\langle 1; 1\rangle$ and the elements $o = (0, 0)$, $s^1 = (0, 1)$, $s^2 = (1, 1)$, $s^3 = (2, 0)$, $b = (2, 1)$. It is easy to see that $(H = \{o, s^1, s^2, s^3, b\}, \underset{\sim}{\pm})$ is a subsemigroup of the sum of $\langle 1; 2\rangle$ and $\langle 1; 1\rangle$.

We have in $(H, \underset{\sim}{\pm})$

$b \sim o = \{b\}$, b is b-complementor of o,

$b \sim s^1 = \{s^3, b\}$, s^3 is b-complementor of s^1,

$b \sim s^2 = \{s^2\}$, s^2 is b-complementor of s^2,

$b \sim s^3 = \{s^1, b\}$, s^1 is b-complementor of s^3,

$b \sim b = \{o, s^1, s^3, b\}$, o is b-complementor of b.

Hence $(H, \underset{\sim}{\pm})$ is uniquely b-complementary.

If $(H, \underset{\sim}{\pm})$ is a sum of b_k-reduction of subsemigroups of cyclic the number of summand has to be 1 because the order of the sume is the product of the orders of the summands and $|H| = 5$ is prime. But this is impossible because in any linear group $\langle c; d\rangle$, for any two elements $s, h \neq o$ in $\langle c; d\rangle$ the semigroups generated by s' and h meet in a non-zero element (since $h \cdot s = s \cdot h$ and $h \cdot s$ is generated by s and $s \cdot h$ is generated by h) but the semigroups generated by s^1 and s^3 in $(H, \underset{\sim}{\pm})$ are $(\{o, s^1\}, \underset{\sim}{\pm})$ and $(\{o, s^3\}, \underset{\sim}{\pm})$.

Chapter 7

MASTER PACKING PROGRAMS

In this chapter we develop results for packing programs similar to that obtained for covering programs. The extension of this results to semigroups is still in the research stage although we have some definitions of packing over semigroups which are compatible with packing programs. These results will appear elsewhere when completed.

7.1. Packing Programs

(7.1.1) We call Packing Program (A, b, c) the program given by

$$\text{maximize} \quad ct$$

over $t \in \mathbb{N}_+^A$ satisfying

$$\sum_{a \in A} t_a a \le b ,$$

with parameters $b \in \mathbb{N}_+^I$, $A \subseteq \mathbb{N}_+^I$ and $c \in \mathbb{R}_-^A$. Moreover we can restrict ourselves to consider $b > 0$, $0 \ne a \le b$ for all $a \in A$ and $c \in \mathbb{R}_+^A$ without loss of generality.

(7.1.2) We denote by F(A, b) the set $\{t \in \mathbb{N}_+^A : \sum_{a \in A} t_a a \le b\}$ and by C(A, b) the set CONV(F(A, b)).

Some trivial properties of F(A, b) and C(A, b) are:

(7.1.3) $0 \le t_a \le \min_{\substack{0 \ne a_i \\ i \in I}} \{\frac{b_i}{a_i}\}$ for all $t \in F(A, b)$,

therefore F(A, b) is finite and C(A, b) is a polytope.

(7.1.4) We have that $0 \in F(A, b)$ and $\delta^a \in F(A, b)$ for all $a \in A$. Therefore $C(A, b)$ is α-closed (theorem (3.5.8)) and it is full-dimension.

(7.1.5) We have that 0 is an extreme point of $C(A, b)$, hence $C(A, b)$ is pointed.

(7.1.6) Let us consider the polar $C^\alpha(A, b)$. By lemma (3.5.4) $C^\alpha(A, b) = \{\pi\colon \pi t \le 1 \text{ for all } t \in F(A,b)\}$, moreover $C^\alpha(A, b)$ is pointed and full dimension since so is $C(A, b)$ (by (3.5.9)), therefore the unique basis for $C^\alpha(A, b)$ is (V, R) where $V = \{v^m\colon m \in M\}$ is the set of extreme points of $C^\alpha(A, b)$ and $R = \{r^n\colon n \in N\}$ is the set of extreme rays of $C^\alpha(A, b)$ (by (2.4.16)).

(7.1.7) <u>Lemma</u>. The cone $RAY(C^\alpha(A, b))$ is the set $\mathbb{R}^A_- \equiv \{r\colon -r \in \mathbb{R}^A_+\}$.

Proof: By (7.1.6) and the definition of RAY we have that $RAY(C^\alpha(A, b)) = \{r\colon rt \le 0 \text{ for all } t \in F(A, b)\}$. Clearly $\mathbb{R}^A_- \subseteq RAY(C^\alpha(A, b))$.

Let $rt \le 0$ for all $t \in F(A, b)$. Hence $r\delta^a = r_a \le 0$ for all $a \in A$ since $\delta^a \in T(A, b)$ for all $a \in A$ (by 7.1.4). Therefore $r \in \mathbb{R}^A_-$. □

(7.1.8) <u>Corollary</u>. The set of extreme rays of $C^\alpha(A, b)$ is $\{-\delta^a\colon a \in A\}$. □

(7.1.9) <u>Theorem</u>: Let $V \equiv \{v^m\colon m \in M\}$ be the set of extreme points of $C^\alpha(A, b)$. Then $C(A, b)$ is the set of solutions x of the system

$$(7.1.10)\qquad \begin{cases} v^m x \le 1 , & m \in M \\ x_a \ge 0 , & a \in A . \end{cases}$$

Moreover this system is irredundant.

Proof: Clearly $x_a \ge 0$ is equivalent to $-\delta^a x \le 0$, but $\{-\delta^a: a \in A\}$ is the set of extreme rays of $C^\alpha(A, b)$ by (7.1.8). Hence the theorem will be a consequence of theorem (3.4.10) if we prove that $0 \notin V$. But $0 \in V$ would imply that 0 is a basic solution of $\{\pi: \pi t \le 1 ; t \in F(A, b)\}$ but this is impossible because $0t < 1$ for all $t \in F(A, b)$.□

(7.1.11) We call a vector π proper whenever $\pi \in C^\alpha(A, b)$ and π is maximal in $C^\alpha(A, b)$.

(7.1.12) Lemma. The extreme points of $C^\alpha(A, b)$ are proper.

Proof: Let $\pi \in C^\alpha(A, b)$ and π be not maximal, hence there is a $v \in C^\alpha(A, b)$ such that $v \ne \pi \le v$. Then $[\pi - v]$ is a ray of $C^\alpha(A, b)$ by (7.1.8). Therefore $\pi + \pi - v \in C^\alpha(A, b)$. Therefore π is not an extreme point of $C^\alpha(A, b)$ because $\pi = \frac{1}{2}(2\pi - v) + \frac{1}{2}v$. □

(7.1.13) We call the packing program (A, b, c) a Master Packing Program when A is the set

$$\{a \in \mathbb{N}_+^I: 0 \ne a \le b\}.$$

For master packing programs we can obtain stronger characterizations for irredundant defining systems as we will show in the next section.

7.2. Super-additivity and Complementarity

We will give here a characterization of irredundant systems for Master Packing Programs similar to that of Master Covering Programs.

We keep in this section the notation and definitions of the preceding section.

(7.2.1) We define the function ϕ from $\mathbb{N}_+^A$ into $\mathbb{N}_+^I$ by

$$\phi(t) = \sum_{a \in A} t_a a .$$

(7.2.2) Lemma. Let $r, t \in \mathbb{N}_+^A$ satisfy $r \le t$ and t is an optimal solution to the packing program (A, b, c). Then cr equals maximum of ch for all $h \in \mathbb{N}_+^A$ such that $\phi(h) \le \phi(r)$.

Proof: Let $h \in \mathbb{N}_+^A$ satisfy $\phi(h) \le \phi(r)$.

Since $t - r + h \ge 0$ and $\phi(t - r + h) = \phi(t) - \phi(r) + \phi(h) \le \phi(t) \le b$, we have $t - r + h \in F(A, b)$. Hence $c(t - r + h) = ct - cr + ch \le ct$ because t is optimal. Therefore we obtain $cr \ge ch$. □

(7.2.3) Lemma. Let (A, b, c) be a packing program and let π be proper. Then for all $a \in A$ there exists $t^0 \in F(A, b)$ satisfying $\pi t^0 = 1$ and $t_a^0 > 0$.

Proof: Let $a \in A$ and let π be proper.

We denote by F the set $F(A, b) \cap \{t \in \mathbb{N}_+^A : t_a > 0\}$. F is non-empty because $\delta^a \in F(A, b)$ by (7.1.4).

Let k denote the maximum integer less than or equal to $\min\limits_{\substack{0 \neq a_i \\ i \in I}} \{\frac{b_i}{a_i}\}$. Clearly we have $k \geq 1$ (since $a \leq b$).

Let $\xi = \min\limits_{t \in F}\{\frac{1-\pi t}{k}\}$.

If we show that $(\pi + \xi\delta^a)r \leq 1$ for all $r \in F(A, b)$ then $\pi + \xi\delta^a \in C^\alpha(A, b)$. Thus ξ has to be 0 because $\pi + \xi\delta^a \geq \pi$ (by choice of ξ) and π is proper. Hence there is $t^0 \in F$ such that $\pi t^0 = 1$ and the lemma is proved.

Let $r \in F(A, b)$. Then $(\pi + \xi\delta^a)r = \pi r + \xi r_a$. If $r_a = 0$ then $(\pi + \xi\delta^a)r = \pi r \leq 1$ because $\pi \in C^\alpha(A, b)$. If $r_a > 0$ then $\pi r + \xi r_a \leq \pi r + \frac{1-\pi r}{k} r_a \leq \pi r + (1-\pi r) = 1$ because $r \in F$ and then we have $\xi \leq \frac{1-\pi r}{k}$ and $r_a \leq k$ by choice of ξ and k.

In either case $(\pi + \xi\delta^a)r \leq 1$. □

(7.2.4) <u>Lemma</u>. Let (A, b, c) be a packing program and let π be proper. Then for all $a \in A$, π_a is the maximum of πh over $h \in \mathbb{N}_+^A$ satisfying $\phi(h) \leq a$.

Proof; Let $a \in A$ and π be proper.

By (7.2.3) there exists $t^0 \in F(A, b)$ such that $\pi t^0 = 1$ and $t_a^0 > 0$. Hence πt^0 is the optimum of πt over $F(A, b)$. Since $0 \leq \delta^a \leq t^0$, we have

$$\pi_a = \pi\delta^a = \max_{\substack{h \geq 0 \\ \phi(h) \leq \phi(\delta^a)}} \{\pi h\}$$

by lemma (7.2.2). Thus the lemma is proved because $\phi(\delta^a) = a$. □

(7.2.5) <u>Theorem</u>. Let (A, b, c) be a packing program and let π be proper. Then π satisfies the following conditions.

(7.2.6) If $b \in A$ then $\pi_b = 1$.

(7.2.7) Monotonic: For all $a, a' \in A$, if $a \geq a'$ then $\pi_a \geq \pi_{a'}$.

(7.2.8) Superadditivity: If $a, a', a'' \in A$ and $a''=a+a'$ then $\pi_a + \pi_{a'} \leq \pi_{a+a'}$.

(7.2.9) Complementarity: If $a, b - a \in A$ then $\pi_a + \pi_{b-a} = 1$.

Proof:

Of (7.2.6): The optimum of πt over $t \in \mathbb{N}_+^A$ satisfying $\phi(t) \leq b$ is 1 (by (7.2.3)). Then, by (7.2.4), we have $\pi_b = 1$.

Of (7.2.7): by (7.2.3) there is $t^0 \in F(A, b)$ such that $\pi t^0 = 1$ and $t_a^0 \geq 1$. Let $t' = t^0 - \delta^a + \delta^{a'}$. We obtain $\phi(t') = \phi(t^0) - a + a' \leq \phi(t^0) \leq b$ and $t' \in \mathbb{N}_+^A$. Hence $t' \in F(A, b)$. Therefore we have $\pi t' = \pi t^0 - \pi_a + \pi_{a'} = 1 - \pi_a + \pi_{a'} \leq 1$, i.e. $\pi_{a'} \leq \pi_a$.

Of (7.2.8): By (7.2.4) we have $\pi_{a''} = \max\{\pi h\}$ over $h \in \mathbb{N}_+^A$ satisfying $\phi(h) \leq a''$. Therefore $\pi_{a''} \geq \pi(\delta^a + \delta^{a'}) = \pi_a + \pi_{a'}$ because $\phi(\delta^a + \delta^{a'}) = a + a' = a''$.

Of (7.2.9): Since $a + (b - a) = b$ we have $\delta^a + \delta^{b-a} \in F(A, b)$. Thus $\pi(\delta^a + \delta^{b-a}) = \pi_a + \pi_{b-a} \leq 1$.

Moreover by (7.2.3), there exists $t^0 \in F(A, b)$ satisfying $\pi t^0 = 1$ and $t^0_a \geq 1$.

By lemma (7.2.4) we have $\pi(t^0 - \delta^a) \leq \pi_{b-a}$ because $\phi(t^0 - \delta^a) = \phi(t^0) - a \leq b - a$.

Thus $\pi_a + \pi_{b-a} \geq \pi_a + \pi(t^0 - \delta^a) = \pi\delta^a + \pi(t^0 - \delta^a) = \pi t^0 = 1.$□

7.3. Master Packing Programs

Through this section we will consider (A, b, c) to be a master packing program.

(7.3.1) Lemma. Let (A, b, c) be a master packing program. If $\pi \in \mathbb{R}^A_+$ satisfies $\pi_b = 1$ and $\pi_a + \pi_{a'} \leq \pi_{a+a'}$ for all $a, a' \in A$ such that $a + a' \leq b$ then $\pi \in C^\alpha(A, b)$.

Proof: Let π satisfy the conditions of the lemma and $\pi \notin C^\alpha(A, b)$. That is, there is $t \in F(A, b)$ such that $\pi t > 1$. Take t^0 satisfying

$$\sum_{a \in A} t^0_a = \min_{t \in F}\{ \sum_{a \in A} t_a\}, \text{ where } F = \{t \in F(A, b): \pi t > 1\}.$$

Case 1: Suppose $\sum_{a \in A} t^0_a = 1$, i.e. $t^0 = \delta^a$ for some $a \in A$. But $\pi_a, \pi_{b-a} \geq 0$ and $\pi_a + \pi_{b-a} \leq \pi_b = 1$. Hence $\pi_a = \pi\delta^a = \pi t^0 \leq 1$ contradicting the choice of t^0.

Case 2: Let $\sum_{a \in A} t^0_a \geq 2$. In this case there exist $a, a' \in A$ such that $\delta^a + \delta^{a'} \leq t^0$. Hence $a + a' = \phi(\delta^a + \delta^{a'}) \leq \phi(t^0) \leq b$.

Therefore $\pi_a + \pi_{a'} \leq \pi_{a+a'}$. Let $a'' = a + a'$ and consider the vector $t' = t^0 + \delta^{a''} - \delta^a - \delta^{a'}$.

We have $\phi(t') = \phi(t^0) + \phi(\delta^{a''}) - \phi(\delta^a) - \phi(\delta^{a'}) = \phi(t^0) + a'' - a - a' = \phi(t^0) \leq b$. Hence $t' \in F(A, b)$ and $\pi t' = \pi t^0 + \pi_{a''} - \pi_a - \pi_{a'} \geq \pi t^0 > 1$, again this is absurd because $\sum_{a \in A} t'_a = \sum_{a \in A} t^0_a + 1 - 2 < \sum_{a \in A} t^0_a$. □

We have now all the elements to prove the next three theorems, they are the main results of this chapter.

(7.3.2) <u>Theorem</u>. Let $V \equiv \{v^m \in \mathbb{R}^A : m \in M\}$ be the set of extreme points of the polyhedron

$$P = \{\pi \in \mathbb{R}^A_+ : \pi_b = 1;\ \pi_a + \pi_{a'} \leq \pi_{a+a'} \text{ when } a+a' \leq b\}.$$

Then $C(A, b)$ is the set of solutions x of the system

$$\begin{cases} v^m x \leq 1\ , & m \in M \\ x_a \geq 0, & a \in A. \end{cases}$$

Proof: By lemma (7.3.1) P is contained in $C^\alpha(A, b)$.

Hence it is enough to show that the extreme points of $C^\alpha(A, b)$ belong to V because then the theorem will be a consequence of theorem (7.1.9).

(7.3.3) Let v be an extreme point of $C^\alpha(A, b)$ since v is proper (Lemma (7.1.12)) v belongs to P because by Theorem (7.2.5) $v_b = 1$ (by (7.2.6)) and $v_a + v_{a'} \leq v_{a+a'}$ when $a + a' \leq b$ (by (7.2.8)). Therefore $v \in V$ otherwise v will be a convex combination of other points of P and these other points belong also to $C^\alpha(A, b)$. □

(7.3.4) <u>Theorem</u>. Let $V = \{v^m \in \mathbb{R}^A: m \in M\}$ be the set of extreme points of the polyhedron

$P = \{\pi \in \mathbb{R}_+^A: \pi_b = 1, \pi_a + \pi_{a'} \le \pi_{a+a'}$ when $a + a' \le b, \pi_a \ge \pi_{a'}$ when $a \ge a'\}$. Then $C(A, b)$ is the set of solutions x of the system

$$\begin{cases} v^m x \le 1, & m \in M \\ x_a \ge 0, & a \in A . \end{cases}$$

The proof is the same as in theorem (7.3.2) using in (7.3.3) theorem (7.2.5) and conditions (7.2.6), (7.2.7) and (7.2.8). □

(7.3.5) <u>Theorem</u>. Let $V = \{v^m \in \mathbb{R}^A: m \in M\}$ be the set of extreme points of the polyhedron

$P = \{\pi \in \mathbb{R}_+^A: \pi_b = 1, \pi_a + \pi_{a'} \le \pi_{a+a'}$ when $a + a' \le b$, $\pi_a + \pi_{b-a} = 1$ for all $a \in A - \{b\}\}$. Then $C(A, b)$ is the set of solutions x of the system

$$\begin{cases} v^m x \le 1, & m \in M \\ x_a \ge 0, & a \in A . \end{cases}$$

Moreover this system is irredundant.

Proof: If we prove that V is the set of extreme points of $C^\alpha(A, b)$ then this theorem is a consequence of theorem (7.1.9).

The proof that the extreme points of $C^\alpha(A, b)$ belong to V is the same as in theorem (7.3.2) using in (7.3.3): Theorem (7.2.5) and conditions (7.2.6), (7.2.8) and (7.2.9). It is easy to see that the elements of V

are proper because if π is proper and $\pi \geq v^e$ for some $e \in M$ then $\pi \in P$ and $\pi_a \geq v_a^e$, $\pi_{b-a} \geq v_{b-a}^e$ for all $a \in A - \{b\}$, but $\pi_a + \pi_{b-a} = v_a^e + v_{b-a}^e = 1$ hence $\pi_a = v_a^e$ for all $a \in A - \{b\}$. Also we have $\pi_b = v_b^e = 1$.

Since $v^e \in C^\alpha(A, b)$, by (7.1.8) we have there are vectors $(\lambda_k \geq 0: k \in K)$ and $(\mu_a \geq 0: a \in A)$ such that $\sum_{k\in K} \lambda_k = 1$ and $v^e = \sum_{k\in K} \lambda_k \pi^k - \sum_{a\in A} \mu_a \delta^a$, where $\{\pi^k: k \in K\}$ is the set of extreme points of $C^\alpha(A, b)$.

Now $\sum_{a\in A} \mu_a$ equals zero, otherwise v^e will not be maximal, moreover $\pi^k \in P$ (by theorem (7.2.5)) for all $k \in K$, hence the only possibility is that $v^e = \pi^k$ for some $k \in K$ because v^e is an extreme point of P.□

Chapter 8

SEMIGROUP PROGRAM ALGORITHMS

We do not have good results in algorithms for semigroup programs. But even trivial properties have to be stated because semigroup programs have not been considered before. This chapter only intends to give a bootstrap to the study of algorithms for semigroup programs.

Algorithms to solve group programs have been given in Gomory [G5], Shapiro [S1], Hu [S2], Yoseloff [Y1] and Johnson [J1] as well as in other papers.

Although it would seem easier to solve group programs than integer programs, the group programs have shown the same order of difficulty as integer programs.

The group program algorithms referenced above can be modified to solve semigroup programs. The inefficiency is not avoided by these modifications, but it is not increased.

Shapiro's Algorithm is based in dynamic programming. The modifications for semigroup programs consist mainly of replacing the use of the inverse element of an element s in the group by all the elements in $b \sim s$. The validity of the modification is provided by the Minimality Lemma.

Hu's algorithm only uses subadditivity. Then the validation of the modification for semigroup programs may be provided by Theorem (5.2.11). The modification consists in keeping a record of how the elements are formed. That is, when an element s is obtained by adding p to q (i.e. $s = p \underset{\sim}{+} q$) it is necessary to keep record of p and q because there may be more than one solution to $p \tilde{+} x = s$. For a group program we need to keep only one, say p, and then we recover q as $s - p$.

E. Johnson has modified his algorithm to solve semigroup programs derived from covering programs [J2].

Gomory [G5] shows how to transform a group program to a shortest path problem. Hence any shortest path algorithm, like Dijkstra [D2], may be used.

In Section 8.1 we show how to transform a semigroup program into a shortest path problem. Since Yoseloff's algorithm only uses the Minimality Lemma and the transformation into a shortest path problem, the modification to a semigroup algorithm is straightforward.

8.1 Shortest Paths

(8.1.1) A directed graph G is a pair (V, E) of finite sets together with two functions tail and head from E in V. The elements of V are called vertices and the elements of E are called edges. An edge e is from v toward w when v is the tail of e and w is the head of e.

(8.1.2) A path is a sequence $(e_1,\ldots,e_n)$ of edges satisfying head of e_i equals tail of e_{i+1} for $i = 1$ to $n - 1$. A path from v to w is a path $(e_1,\ldots,e_n)$ satisfying tail of e_1 equals v and head of e_n is w.

(8.1.3) A length function is a real function over

E. Given a length function z, we define the <u>length of a path</u> $p = (e_1,\dots,e_n)$ as $\sum_{i=1}^{n} z(e_i)$. A <u>shortest path</u> from v to w is a path from v to w with minimum length.

(8.1.4) Let (H, b, c) be a semigroup program over $(S, \underset{\sim}{+})$. We associate with (H, b, c) the directed graph $G(S, E)$, where $E = \{e(s, h): s \in S, h \in H\}$ and for any $e(s, h) \in E$ we have tail of $e(s, h)$ is s and head of $e(s, h)$ is $s \underset{\sim}{+} h$. We also associate the length function $z(e(s, h)) = c_h$ for all $h \in H$ and for all $s \in S$. That is, we have an edge from s toward $s \underset{\sim}{+} h$, for each s in S and each h in H, with length c_h. We call $e(s, h)$ an <u>h-edge.</u>

(8.1.5) Let p be a path from o to b. We denote by t^p the vector in $\mathbb{N}_+^H$ whose components t_h^p are the number of h-edges in p. Clearly $t^p \in T(H, b)$ and the length of p equals ct^p.

(8.1.6) Given a path $p = (e_1,\dots,e_n)$ from u to v and a path $q = (f_1,\dots,f_m)$ from v to w, we denote by $p \vee q$ the path $(e_1,\dots,e_n, f_1,\dots,f_m)$ from u to w.

(8.1.7) Let k be a non-negative integer. Let $s \in S$ and $h \in H$. We denote by $p(s, h, k)$ the path $(e(s, h), e(s \underset{\sim}{+} h, h),\dots,e(s \underset{\sim}{+} (k - 1)\cdot h, h))$ from s to $s \underset{\sim}{+} k\cdot h$.

(8.1.8) Let $(h_1,\dots,h_m)$ be any indexing of H.

Let $t \in T(H, b)$ and let p be the path

$$p(o, h_1, t_{h_1}) \vee p(t_{h_1} \cdot h_1, h_2, t_{h_2}) \vee \ldots \vee p(\sum_{i=1}^{m-1} t_{h_i} \cdot h_i, h_m, t_{h_m}).$$

Then p is a path from o to b and length of p equals ct.

A consequence of (8.1.5) and (8.1.8) is the following

(8.1.9) Theorem. Let (H, b, c) be a semigroup program over $(S, \underset{\sim}{+})$. Let the directed graph $G = (S, E)$ and the length function z be defined as in (8.1.4). Let p be a shortest path in G with respect to z. Then t^p is an optimal solution of (H, b, c), where t^p is defined as in (8.1.5).□

Notice that Theorem (8.1.9) is valid for any semigroup program because it does not depend on (H, b, c) to be H-unbounded.

8.2. Group vs. Covering Approach

Given a covering program we had pointed, in the Introduction chapter, two ways to solve it.

(8.2.1) The group approach is to solve the linear programming obtained by releasing the integrality condition. Then obtain the group program associated to the optimal solution of the linear program and solve this group program.

(8.2.2) The covering semigroup approach is to solve the semigroup program obtained directly from the covering program.

Clearly, the efficiency of any shortest path algorithm used to solve a semigroup program depends on the order of the group or semigroup (and this is true for all the known group program algorithms or its extension to a semigroup algorithm).

Since the order of the covering semigroup is always not smaller than the order of the group obtained from the group approach, it could appear that this would be true also for the order of the semigroup generated by the columns of the matrix of the covering program. We can argue that the covering approach gives always a solution to the covering program and the group approach may give none. However, the order of the semigroup can be significantly smaller than the order of the group.

We will show a family of covering programs whose generated semigroups have order 5 and the order of the generated group grows exponentially. Moreover the group approach gives no solution to the covering program.

Let us denote by

$$s = 2^n - 1,\ h = 2^n + 2^{n-1} - 1,\ b = 2^n + 1$$

The family of covering program is defined for any $n \geq 3$ as minimize $st_s + ht_h$, over $st_s + ht_h \geq b$ where $t_s, t_h \in \mathbb{N}_+$.

Covering approach: It is easy to see that the semigroup has the five elements $0, s, h, 2 \cdot s = 2^{n+1}-2, b$ with the following addition table

$\underset{\sim}{+}$	0	s	h	2·s	b
0	0	s	h	2·s	b
s	s	2·s	b	b	b
h	h	b	b	b	b
2·s	2·s	b	b	b	b
b	b	b	b	b	b

It is also b-complementary since it is a subsemigroup of a cyclic one and b is in the loop of the cyclic semigroup.

Group approach: We first solve the linear program

$$\min \left\{ st_s + ht_h \right\}$$

over $st_s + ht_h - x = b$ where $t_s, t_h, x \in \mathbb{R}_+$.

This program has two optimal solutions.

Solution 1: $t_s = \frac{b}{s}$; $t_h, x = 0$.

(8.2.3) Hence $t_s = \frac{b}{s} - \frac{h}{s}t_h + \frac{1}{s}x$.

The group program is then: minimize x over

$$\frac{2^{n-1}}{2^n-1} t_h - \frac{1}{2^n-1}x \equiv \frac{1}{2^n-1}(\text{mod } 1).$$

We have that 2^{n-1} and 2^n-1 are relative primes since any common divisor being a divisor of 2^{n-1} also divides 2^n and therefore -1.

Hence the order of the subgroup generated by $\frac{2^{n-1}}{2^n-1}$ is 2^n-1. This shows that the order of the group grows exponentially and also that there exists $k \in \mathbb{N}_+$ such that $k\cdot\frac{2^{n-1}}{2^n-1} \equiv \frac{2}{2^n-1}$ mod 1. Since the cost of t_h is zero, ($t_h = k$, $x = 0$) is an optimal solution. To obtain a solution to the original problem we need

$$t_s = \frac{b}{s} - \frac{h}{s}\cdot k \text{ be greater than or equal to zero.}$$

That is $b = 2^{n+1} \geq hk = (2^n + 2^{n-1}-1)k$, hence $k \leq 1$ because $2(2^n + 2^{n-1}-1) = 2^{n+1} + 2^n - 2 > 2^{n+1}$ for $n \geq 2$. But if $\frac{2^{n-1}}{2^n-1} \equiv \frac{2}{2^n-1}$ (mod 1) n is equal to 2 and we have assumed $n \geq 3$. Therefore the group program gives us no solution.

Solution 2: $t_h = \frac{b}{h}$; t_s, $x = 0$.

(8.2.4) Hence $t_h = \frac{b}{h} - \frac{s}{h}t_s + \frac{1}{h}x$,

and the group program is minimize x over

$$\frac{2^n-1}{2^n+2^{n-1}-1}\, t_s - \frac{1}{2^n+2^{n-1}-1}\, x \equiv \frac{2^{n-1}+1}{2^n+2^{n-1}-1} \text{ (mod 1)}$$

Again 2^n-1 and $2^n+2^{n-1}-1$ are relative primes because a common divisor being a divisor of 2^n-1 has to divide 2^{n-1} and we have shown that these are relative primes.

By the same argument as in solution 1, there exists an optimal solution to the group program with $t_s = k$, $x = 0$. If $\frac{b}{h} - \frac{s}{h}k \geq 0$ we obtain $2^{n+1} \geq (2^n-1)k$, and k has to be less than 3 because $3(2^n-1) = (2 + 1)2^n - 3 = 2^{n+1} + 2^n - 3 > 2^{n+1}$ for $n \geq 2$. But neither $\frac{2^n-1}{2^n+2^{n-1}-1}$ nor $\frac{2^{n+1}-2}{2^n+2^{n-1}-1}$ are congruent mod 1 to $\frac{2^{n-1}+1}{2^n+2^{n-1}-1}$, because the first would imply

$2^n - 1 = 2^{n-1} + 1$, that is $2^n - 2^{n-1} = 2^{n-1}(2 - 1) = 2^{n-1} = 2$.

Hence $n = 2$ and we assume $n \geq 3$.

The second would imply

$$\frac{2^{n+1}-2}{2^n+2^{n-1}-1} = 1 + \frac{2^{n-1}+1-2}{2^n+2^{n-1}-1} = 1 + \frac{2^{n-1}-1}{2^n+2^{n-1}-1} \equiv \frac{2^{n-1}+1}{2^n+2^{n-1}-1}$$

(mod 1). Therefore $2^{n-1}-1 = 2^{n-1}+1$ absurd.

Clearly, there are covering programs for which the order of the semigroup grows exponentially while the order of the group is bounded.

For example: minimize t over $t \geq 2^n$ and $t \in \mathbb{N}_+$. Here the semigroup generated by the covering approach has order 2^n+1. However the group generated by the group approach has order 1 and gives the optimal solution since the linear programming optimal solution is integer.

Thus it may be useful to use a combination of the two approaches in heuristic algorithms.

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