A HIGH-LEVEL, VARIABLE-FREE CALCULUS FOR RECURSIVE PROGRAMMING

by

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0. ABSTRACT AND INTRODUCTION

A typeless calculus together with its model is presented which we claim gives us the ability:

i) to specify the semantics of Algol-like languages in a way that unifies Scott's method and interpreter methods.

ii) to formalize proofs of assertions about programs and make such proofs manageable, not only for machine checking but also for humans to understand and to devise them.

iii) to make algebraic correctness proofs of compilers for languages allowing recursive procedures and features such as call by name. Until now, rigorous correctness proofs for compilers are available only for simple languages (F. Lockwood Morris).

The calculus originates from the recent analysis of derived operations of universal algebras by T. S. E. Maibaum, but is reformulated to take advantage of Scott's theory of computation. The result is a calculus whose well formed expressions (w.f.e.) can behave both as a function and as a scalar. In the formulation of functions no use is made of the usual notion of variables. Hence there is no need for the machinery of λ-abstraction, variable-binding, renaming and argument substitution as happens in Church's λ-calculus. Yet, unlike Curry's "low level" combinatory logic, the w.f.e.'s of our "high level" calculus look like direct transliterations of Algol-like programs. The calculus appears to combine the advantages of λ-calculus and combinatory logic.

A model for the calculus is exhibited in which all well formed expressions have meaning and all reduction rules preserve the meaning of the w.f.e.'s on which they operate. The calculus is then

proved to have the so-called Church-Rosser property, thus its consistency as a symbol-manipulation system is ensured. Also exhibited is a "safest" reduction method which corresponds intuitively to passing parameters by name.

I. THE CALCULUS

Introduction:

Our calculus can be viewed as an intermediate language between machines languages and Algol-like ones. Compared to the lasts, it has the same ability to express recursive programs, yet can also be used to formulate proofs of assertions about programs. Compared to machines languages, our calculus is more precise (i.e. more mathematical), yet still is mechanically interpretable.

Basic symbols:

1) The integer numerals: 0, 1, -1, 2, -2, ...
2) Truth value symbols: true, false
3) Arithmetic operators and predicates: add, sub, mult, div, eq, greater, less, ...
4) Selector operators: s_1, s_2, s_3, ...
   Conditional operator: if
   Fix point operator: Y
   Composition operator: C
   Application operator: eval
   Constant operator: K.

Well formed expressions:

1) Each basic symbol is a w.f.e.
2) If $A_1, A_2, \ldots, A_n$ $(n \geq 2)$ are w.f.e.'s then $A_1[\![A_2, \ldots, A_n]\!]$ is a w.f.e.
3) There are no other kinds of w.f.e.'s.

Axiom schemas:

Let $A_1, A_2, \ldots, B_1, B_2, \ldots$ be w.f.e.'s and

$N_1, N_2, \ldots$ be integer numerals.

1) For $i = 1,2,3,\ldots$ and $p \geq i$

$s_i[A_1,\ldots,A_p] = A_i$

2) $C[A_1,A_2,\ldots,A_p][B_1,B_2,\ldots,B_q]$

$= A_1[A_2[B_1,B_2,\ldots,B_q],\ldots,A_p[B_1,B_2,\ldots,B_q]]$ $\quad p=2,3,\ldots;\ q=1,2,\ldots$

3) $Y[A_1] = A_1[Y[A_1]]$

4) if [true, A_1, A_2] = A_1

if [false, A_1, A_2] = A_2

5) eval $[A_1,A_2,\ldots,A_p] = A_1[A_2,\ldots,A_p]$ $\quad p = 2,3,\ldots$

6) $K[A_1][B_1,B_2,\ldots,B_q] = A_1$ $\quad q = 1,2,\ldots$

7) add $[N_1,N_2] = N_3$

whenever N_3 is the sum numeral of the numerals N_1 and N_2.

(plus similar axiom schemas for sub, mult, div).

8) eq $[N_1,N_2]$ = true whenever N_1 is the same numeral as N_2.

eq $[N_1,N_2]$ = false whenever N_1 is not the same numeral as N_2.

(plus similar axiom schemas for greater, less, ...).

9) $N_1[\![A_1, A_2, \ldots, A_p]\!] = N_1$ $\quad p = 1, 2, \ldots$

$\text{true}\ [\![A_1, A_2, \ldots, A_p]\!] = \text{true}$

$\text{false}\ [\![A_1, A_2, \ldots, A_p]\!] = \text{false}$

Abbreviation:

For any w.f.e. A

$\overline{A}$ stands for $K[\![A]\!]$.

Inference rules:

Let A_1, A_2, ..., B_1, B_2, ... be w.f.e.'s.

1) From $A_1 = A_2$ infer

$$A_1[\![B_1, B_2, \ldots, B_p]\!] = A_2[\![B_1, B_2, \ldots, B_p]\!]. \quad r \geq 1$$

2) From $A_1 = A_2$ infer

$$B_1[\![\ldots, A_1, \ldots]\!] = B_1[\![\ldots, A_2, \ldots]\!].$$

(same place)

3) From $A_1 = A_2$ and $A_2 = A_3$ infer $A_1 = A_3$.

From $A_1 = A_2$ infer $A_2 = A_1$.

Reduction rules:

1) A reduction rule is obtained by replacing the equality sign "=" in each axiom by an arrow "→".

2) There are no other kinds of reduction rules.

Examples of w.f.e.'s and reductions:

1) The function f of two variables, where

$$f(x, y) = (x * y)/(x - y)$$

can be denoted by the w.f.e.:

$C[\![div, mult, sub]\!]$.

Let's compute f(5,3):

$$C[\![div,mult,sub]\!][\![5,3]\!] = div[\![mult[\![5,3]\!],sub[\![5,3]\!]]\!] \quad \text{- A2}$$
$$= div[\![15,2]\!] \quad \text{- A7}$$
$$= 7 . \quad \square \quad \text{- A7}$$

2) The functional τ of the binary function variable F, where:

$\tau(F)(x,y) = F(x*y,x)$

can be denoted by the w.f.e.:

$C[\![C, s_1, \overline{mult}, \overline{s_1}]\!]$.

Now $\tau(f)$, for f in the previous example, can be computed as follows:

$C[\![C, s_1, \overline{mult}, \overline{s_1}]\!][\![C[\![div, mult, sub]\!]]\!]$

$= C[\![C[\![div,mult, sub]\!], mult, s_1]\!]$. $\square$ - A_2,A_4

The last w.f.e. is a reasonably compact denotation for the resulting function $\tau(f)$ (which would otherwise have to be found by cumbersome substitution) i.e.

$\tau(f)(x,y) = ((x*y)*x)/((x*y)-y)$.

3) Consider the well known recursive definition:

F(x) <= <u>if</u> x = 1 <u>then</u> 1 <u>else</u> x * F(x-1).

The second member, considered as a functional of the variable F, can be denoted by the w.f.e.:

$C[\![C,\overline{if},C[\![C,\overline{eq},\overline{s_1},1]\!],1,C[\![C,\overline{mult},\overline{s_1},C[\![C,s_1,C[\![C,\overline{sub},\overline{s_1},1]\!]]\!]]\!]]\!]$.

Let's call this w.f.e. τ.

The w.f.e. $Y[\![\tau]\!]$ then represents the function g denoted by the above recursive definition.

Now consider the Algol-like program:

<u>begin</u> F(x) ⇐ <u>if</u> x = 1 <u>then</u> 1 <u>else</u> x * F(x-1); F(2)

<u>end</u>.

This program can be denoted by the following w.f.e.:

C⟦eval,Y,2⟧⟦τ⟧

Let's "run" the program:

C⟦eval,Y,2⟧⟦τ⟧

= eval⟦Y⟦τ⟧,2⟧ - A2,A9

= Y⟦τ⟧⟦2⟧ - A5

= τ⟦Y⟦τ⟧⟧⟦2⟧ - A3

= C⟦if,C⟦eq,s_1,1⟧,1,C⟦mult,s_1,C⟦Y⟦τ⟧,
C⟦sub,s_1,1⟧⟧⟧⟧⟦2⟧ - A2,A9,A6

= if⟦eq⟦2,1⟧,1,mult⟦2,Y⟦τ⟧⟦sub⟦2,1⟧⟧⟧⟧ - A2

= if ⟦false,1,mult⟦2,Y⟦τ⟧⟦1⟧⟧⟧ - A7,A8

= mult⟦2,τ⟦Y⟦τ⟧⟧⟦1⟧⟧ - A4,A3

= mult⟦2,C⟦if,C⟦eq,s_1,1⟧,1,C⟦mult,s_1,
C⟦Y⟦τ⟧,C⟦sub,s_1,1⟧⟧⟧⟧⟦1⟧⟧ - A2,A9

= mult⟦2,if⟦eq⟦1,1⟧,1,mult⟦1,Y⟦τ⟧⟦sub⟦-1,1⟧⟧⟧⟧⟧ - A2,A9,A6

= mult⟦2,if⟦true,1,mult⟦1,Y⟦τ⟧⟦sub⟦-1,1⟧⟧⟧⟧⟧ - A8

= mult⟦2,1⟧ - A4

= 2 . - A7

□

This is a reasonably short <u>completely</u> formal proof that the running of the program yields 2 as the result.

II. CONSTRUCTION OF A MODEL FOR THE CALCULUS

Introduction:

We construct a complete lattice D_∞ such that D_∞ is isomorphic to the lattice of all continuous functions with value in D_∞ and having an arbitrary number of arguments in D_∞. Then, using a simple interpretation rule, we assign an element (i.e. a "meaning") of D_∞ to each w.f.e. of our calculus. This interpretation is such that every reduction rule of **the calculus preserves the meanings of the w.f.e's on which it operates.**

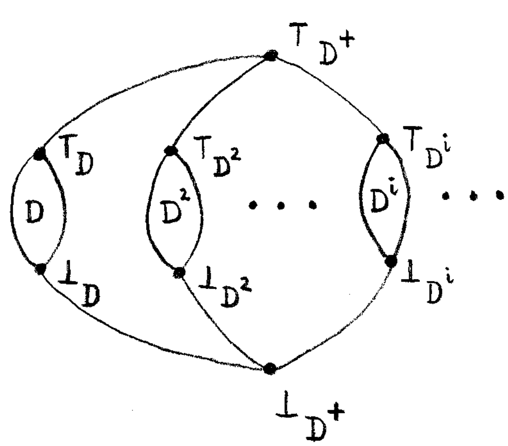
When compared to known constructions of models for the Lambda-calculus (by D. Scott), our construction differs only in the fact that we consider functions of an arbitrary number of arguments, **instead of functions of only one argument.**

Yet the resulting lattice D_∞ is shown to possess a much richer algebraic structure, useful for specifying the semantics of programming languages.

Most of the terminology is borrowed and many of the proofs are adapted from [JR]. (i.e. the proofs marked with $*$).

Definition 1:

Let D be any complete lattice. We define D^+ to be the lattice



where D^i is the usual $i^{\underline{th}}$ cartesian power of D (D^i is a complete lattice: [JR] p.69). All elements (vectors) of D^+, except $\top_{D^+}$ and $\bot_{D^+}$, will be denoted by

$$\langle a_1, \ldots, a_i \rangle .$$

It must be understood that $\langle a_1, \ldots, a_i \rangle$ belongs to the sublattice D^i of D^+.

Definition 2:

Let D and D' be complete lattices. By $D \to D'$ we denote the lattice of all continuous functions from D to D'. The partial order of $D \to D'$ is defined to be the pointwise induced order.

Proposition 1:

Let D be any complete lattice then D^+ is a complete lattice.

Proof: See [JR] p.75. □

Proposition 2:

Let D and D' be complete lattices then $D \to D'$ (i.e. the set of all continuous functions from D to D') is also a complete lattice. Moreover, for all $F \subseteq D \to D'$

$$(\sqcup F)(x) = \sqcup \{f(x) \mid f \in F\} .$$

Proof: See [JR] p.16. □

Definition 3:

Let $\quad f \in R^+ \to Q$

$\quad\quad\quad g \in S \to R$

where Q, R, S are complete lattices, by $f \triangledown g$ we mean a function from S^+ to Q such that

1) $[f \triangledown g](\langle a_1,\ldots,a_p\rangle) = f(\langle g(a_1),\ldots,g(a_p)\rangle)$ for all $\langle a_1,\ldots,a_p\rangle \in S^+$.

2) $[f \triangledown g](\top_{S^+}) = f(\top_{R^+})$ and $[f \triangledown g](\bot_{S^+}) = f(\bot_{R^+})$.

Proposition 3:

1) $f \triangledown g \in S^+ \to Q$.

2) $f \triangledown I_R = f$ (I_R is the identity function on R).

3) If $h \in T \to S$ where T is another complete lattice then: $[f \triangledown g] \triangledown h = f \triangledown [g \circ h] \in T^+ \to Q$.

4) If $k \in Q \to P$ where P is another complete lattice then: $k \circ [f \triangledown g] = [k \circ f] \triangledown g$.

5) $f \triangledown g$ is continuous in f and in g.

Proof:

1) Suppose $X \subseteq S^+$ directed and contain two vectors of different lengths,

$$\langle a_1, \ldots, a_p\rangle \text{ and } \langle b_1, \ldots, b_q\rangle$$

with $p \neq q$, then X must contain $\top_{S^+}$. Thus the set $\{[f \triangledown g](v) \mid v \in X\}$ must contain $[f \triangledown g](\top_{S^+}) = f(\top_{R^+})$. Therefore:

$$f(\top_{R^+}) = \bigsqcup\{[f \triangledown g](v) \mid v \in X\} .$$

On the other hand we have:

$$[f \triangledown g](\bigsqcup X) = [f \triangledown g](\top_{S^+}) = f(\top_{R^+}) .$$

Hence

$$[f \triangledown g](\bigsqcup X) = \bigsqcup\{[f \triangledown g](v) \mid v \in X\} .$$

Suppose all elements of X (except possibly $\top_{S^+}$ and $\perp_{S^+}$) of same length: Obvious.

2), 3), 4), and 5): Obvious. □

Definition 4:

Let D and D' be complete lattices, and

$$\phi \in D \to D'$$
$$\psi \in D' \to D .$$

We write $D \overset{\phi,\psi}{\lesssim} D'$ whenever

$$\psi \circ \phi = I_D$$
$$\phi \circ \psi \sqsubseteq I_{D'} .$$

Definition 5:

An infinite sequence of complete lattices $D_0, D_1, \ldots,$ and continuous functions $\phi_0, \phi_1, \ldots$ and $\psi_0, \psi_1, \ldots$ such that

$$D_0 \overset{\phi_0,\psi_0}{\lesssim} D_1 \overset{\phi_1,\psi_2}{\lesssim} D_2 \overset{\phi_2,\psi_2}{\lesssim} \ldots$$

is called a retraction sequence.

Definition 6:

The completion of a retraction sequence

$$D_0 \overset{\phi_0,\psi_0}{\lesssim} D_1 \overset{\phi_1,\psi_1}{\lesssim} \ldots$$

is the partially ordered set D_∞ of infinite sequences:

$$D_\infty = \{\langle x_0, x_1, \ldots\rangle \mid (\forall n \geq 0) x_n \in D_n \text{ and } x_n = \psi_n(x_{n+1})\}$$

where $x \underset{D_\infty}{\sqsubseteq} y$ iff, for all $n \geq 0$

the n^{th} component of $x \underset{D_n}{\sqsubseteq}$ the n^{th} component of y.

Proposition 4:

The completion of a retraction sequence is a complete lattice.

Proof: See [JR] p.93. □

Definition 7:

Let $D_0 \overset{\phi_0,\psi_0}{\lesssim} D_1 \overset{\phi_1,\psi_1}{\lesssim} \cdots$ be a retraction sequence.

Let $\theta_{mn} \in D_m \to D_n$ be the function such that:

$$\text{If } \begin{bmatrix} m < n \\ m = n \\ m > n \end{bmatrix} \quad \text{then} \quad \theta_{mn} = \begin{bmatrix} \phi_{n-1} \circ \cdots \circ \phi_m \\ I_{D_m} \\ \psi_n \circ \cdots \circ \psi_{m-1} \end{bmatrix} .$$

Obviously θ_{mn} are continuous [JR].

Proposition 5:

1) $\theta_{mn} \circ \theta_{km} = \theta_{kn}$ whenever $m \geq k$ or $m \geq n$.

2) $\theta_{mn} \circ \theta_{km} \sqsubseteq \theta_{kn}$.

3) When $m \leq n$ then $D_m \overset{\theta_{mn},\theta_{nm}}{\lesssim} D_n$.

Proof: See [JR] p.92. □

Definition 8:

Let $\theta_{n\infty} \in D_n \to D$,

$\theta_{\infty n} \in D_\infty \to D_n$ and

$\theta_{\infty\infty} \in D_\infty \to D_\infty$ be the functions

such that

$$\theta_{n\infty}(x) = \langle \theta_{n0}(x), \theta_{n1}(x), \ldots \rangle$$

$\theta_{\infty n}(x)$ = The $n^{\underline{th}}$ component of the vector x

$\theta_{\infty\infty}(x) = x$.

Proposition 6:

$\theta_{n\infty}$, $\theta_{\infty n}$ and $\theta_{\infty\infty}$ are continuous and the proposition 5 remains true when free occurrences of the subscripts are permitted to take on ∞ as well as integer values.

Proof: (For the continuity of $\theta_{n\infty}$, $\theta_{\infty n}$ and $\theta_{\infty\infty}$ see [JR] p.96)

1)

i) Suppose $m \neq \infty$, $k \neq \infty$ and $n \neq \infty$: Obvious. - P5

ii) Suppose $m = k$ or $m = n$: Obvious.

iii) Suppose $m = \infty$:

We have

$$\theta_{\infty n} \circ \theta_{k\infty} = \theta_{kn}$$

because

$$\theta_{k\infty}(x) = \langle\theta_{k0}(x),\ \theta_{k1}(x),\ \ldots\ \rangle \qquad \text{- D8}$$

implies

$$[\theta_{\infty n} \circ \theta_{k\infty}](x) = \theta_{kn}(x)\ .$$

iv) Suppose $n = \infty$, $k \neq \infty$, $m \neq \infty$ and $m > k$:

We have

$$\theta_{m\infty} \circ \theta_{km} = \theta_{k\infty}$$

because

$$[\theta_{m\infty} \circ \theta_{km}](x) = \langle\theta_{m1}(\theta_{km}(x)),\ \theta_{m2}(\theta_{km}(x)),\ \ldots\ \rangle \qquad \text{- D8}$$

$$= \langle\theta_{k1}(x),\ \theta_{k2}(x),\ \ldots\ \rangle \qquad \text{- P5}$$

$$= \theta_{k\infty}(x)\ . \qquad \text{- D8}$$

v) Suppose $k = \infty$, $m \neq \infty$, $n \neq \infty$ and $m > n$:

We have

$$\theta_{\infty n}(x) = \psi_n(\theta_{\infty,n+1}(x)) \qquad \text{- D8}$$

$$= \psi_n(\psi_{n+1}(\theta_{\infty,n+2}(x))$$

$$= [\psi_n \circ \psi_{n+1} \circ \ldots \circ \psi_{m-1}](\theta_{\infty m}(x))$$

$$= [\theta_{mn} \circ \theta_{\infty m}](x) \,. \qquad \text{- D7}$$

Therefore

$$\theta_{mn} \circ \theta_{\infty m} = \theta_{\infty n} \,.$$

2)

i) Suppose $m \geq n$ or $m \geq k$: Obvious by 1).

ii) Suppose $n = \infty$, $m \neq \infty$, $k \neq \infty$ and $m < k$:

We have

$$\theta_{m\infty} \circ \theta_{km} \sqsubseteq \theta_{k\infty}$$

because

$$[\theta_{m\infty} \circ \theta_{km}](x) = \langle \theta_{m1}(\theta_{km}(x), \theta_{m2}(\theta_{km}(x), \ldots \rangle$$

$$\sqsubseteq \langle \theta_{k1}(x), \theta_{k2}(x), \ldots \rangle \qquad \text{- P5}$$

$$\sqsubseteq \theta_{k\infty}(x) \,. \qquad \text{- D8}$$

iii) Suppose $k = \infty$, $m \neq \infty$, $n \neq \infty$ and $m < n$:

We have

$$\theta_{\infty m}(x) = \psi_m(\theta_{\infty,m+1}(x)) \qquad \text{- D8}$$

$$\vdots$$

$$= [\psi_m \circ \psi_{m+1} \circ \ldots \circ \psi_{n-1}](\theta_{\infty n}(x))$$

$$= \theta_{nm}(\theta_{\infty n}(x)) \,. \qquad \text{- D7}$$

Hence

$$\theta_{mn} \circ \theta_{\infty m} = \theta_{mn} \circ \theta_{nm} \circ \theta_{\infty n}$$

$$\sqsubseteq \theta_{\infty n} \; . \qquad \text{- P5}$$

3) Obvious. □

Proposition 7:

1) $\{\theta_{n\infty} \circ \theta_{\infty n} \mid n = 0,1,2...\}$ is a directed sequence

(i.e. $\theta_{n\infty} \circ \theta_{\infty n} \sqsubseteq \theta_{n+1,\infty} \circ \theta_{\infty,n+1}$ for $n = 0,1,2...$) .

2) $\coprod_{n=0}^{\infty} \theta_{n\infty} \circ \theta_{\infty n}$ = the identity function on D_∞ .

Proof:

1) $\theta_{n\infty} \circ \theta_{\infty n} = \theta_{n\infty} \circ \theta_{n+1,n} \circ \theta_{\infty,n+1}$ - P6

$= \theta_{n+1,\infty} \circ \theta_{n,n+1} \circ \theta_{n+1,n} \circ \theta_{\infty,n+1}$ - P6

$\sqsubseteq \theta_{n+1,\infty} \circ \theta_{n+1,n+1} \circ \theta_{\infty,n+1}$ - P6

$\sqsubseteq \theta_{n+1,\infty} \circ \theta_{\infty,n+1}$. - P6

2) $\left(\coprod_{n=0}^{\infty} \theta_{n\infty} \circ \theta_{\infty n}\right)(x) = \coprod_{n=0}^{\infty} \left(\theta_{n\infty} \circ \theta_{\infty n}\right)(x)$ - P2

$= \coprod_{n=0}^{\infty} <[\theta_{n0} \circ \theta_{\infty n}](x), [\theta_{n1} \circ \theta_{\infty n}](x),$

$..., [\theta_{nn} \circ \theta_{\infty n}](x), ... >$ - D8

$= \coprod_{n=0}^{\infty} <\theta_{\infty 0}(x), \theta_{\infty 1}(x), ..., \theta_{\infty n}(x), \theta_{n,n+1} \circ \theta_{\infty n}(x), ...>$ - P6

$= <\theta_{\infty 0}(x), \theta_{\infty 1}(x), ..., \theta_{\infty n}(x), ... >$

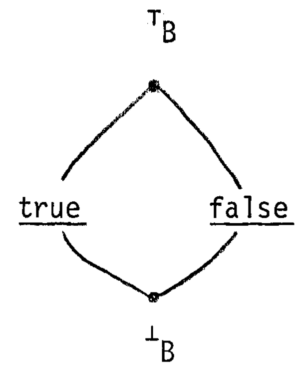
because the ℓ.u.b is the limit of a directed squence (by P6).

Hence: $[\bigsqcup_{n=0}^{\infty} \theta_{n\infty} \circ \theta_{\infty n}](x) = x$. □ - D8

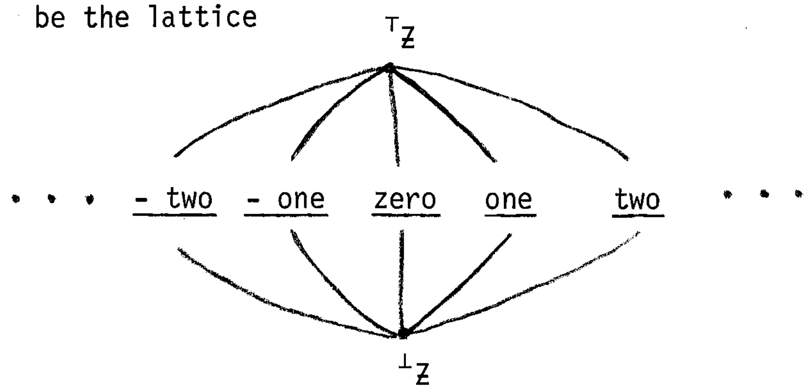
<u>Definition 9</u>:

Let

$\mathbb{B}$ be the lattice

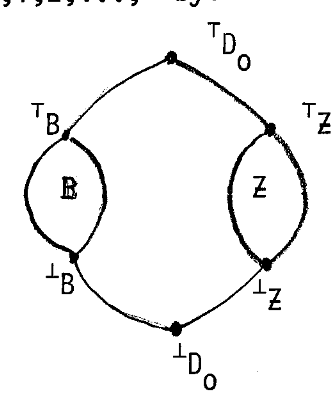


$\mathbb{Z}$ be the lattice



Define D_n , for $n = 0,1,2,...,$ by:

D_0 :



then recursively:

$$D_{n+1} = D_n^+ \to D_n , \qquad (n \geq 0).$$

Obviously D_n are complete lattices for $n = 0,1,2,\ldots$ ([JR] p.80 and P1).

Definition 10:

Define $\phi_n \in D_n \to D_{n+1}$

$\psi_n \in D_{n+1} \to D_n$

for $n = 0,1,2,\ldots,$ by:

$\phi_0(x)$ = the constant function with value x of $D_0^+ \to D_0$

$\psi_0(f) = f(\perp_{D_0^+})$

then recursively:

$$\phi_{n+1}(x) = \phi_n \circ x \nabla \psi_n$$

$$\psi_{n+1}(f) = \psi_n \circ f \nabla \phi_n \ .$$

Proposition 8:

For $n = 0,1,2,\ldots$ we have:

$$D_n \overset{\phi_n,\psi_n}{\lesssim} D_{n+1} \ .$$

Proof: (by induction)

ϕ_0 and ψ_0 are obviously continous.

Moreover:

$$\psi_0(\phi_0(x)) = x \qquad \text{for } x \in D_0 \ ,$$

and

$$\phi_0(\psi_0(f)) = \text{the constant function from } D_0^+ \text{ to } D_0 \text{ with value } f(\bot_{D_0^+})$$

$$\sqsubseteq f$$

for all $f \in D_1$.

The induction step is

$$\psi_{n+1}(\phi_{n+1}(x)) = \psi_n \circ [\phi_{n+1}(x)] \triangledown \phi_n$$

$$= \psi_n \circ [\phi_n \circ x \triangledown \psi_n] \triangledown \phi_n \qquad \text{- D10}$$

$$= \psi_n \circ \phi_n \circ x \triangledown [\psi_n \circ \phi_n] \qquad \text{- P3}$$

$$= I_{D_n} \circ x \triangledown I_{D_n} \qquad \text{- induction hypothesis}$$

$$= x \quad (\text{for all } x \in D_{n+1}) \qquad \text{- P3}$$

and

$$\phi_{n+1}(\psi_{n+1}(f)) = \phi_n \circ [\psi_{n+1}(f)] \triangledown \psi_n \qquad \text{- D10}$$

$$= \phi_n \circ [\psi_n \circ f \triangledown \phi_n] \triangledown \psi_n \qquad \text{- D10}$$

$$= \phi_n \circ \psi_n \circ f \triangledown [\phi_n \circ \psi_n] \qquad \text{- P3}$$

$$\sqsubseteq I_{D_{n+1}} \circ f \triangledown I_{D_{n+1}} \qquad \text{- induction hypothesis}$$

$$\sqsubseteq f \quad (\text{for all } f \in D_{n+2}). \qquad \text{- P3}$$

□

<u>Definition 11</u>:

We have proved that

$$D_0 \overset{\phi_0,\psi_0}{\lesssim} D_1 \overset{\phi_1,\psi_1}{\lesssim} D_2 \overset{\phi_2,\psi_2}{\lesssim} \cdots$$

is a retraction sequence. Let D_∞ be its completion.

Proposition 9:

The θ_{mn} satisfy the following equation:

$$\theta_{m+1,n+1}(f) = \theta_{mn} \circ f \triangledown \theta_{nm} , \quad \text{for all} \quad f \in D_{m+1} .$$

* Proof:

Suppose $m < n$. Then, for all $f \in D_{m+1}$:

$$\begin{aligned}
\theta_{m+1,n+1}(f) &= \phi_n(\phi_{n-1}(\ldots(\phi_{m+1}(f))) && \text{- D7} \\
&= \phi_{n-1} \circ [\phi_{n-1}(\ldots(\phi_{m+1}(f)))] \triangledown \psi_{n-1} && \text{- D10} \\
&= \phi_{n-1} \circ [\phi_{n-2} \circ [\ldots(\phi_{m+1}(f))] \triangledown \psi_{n-2}] \triangledown \psi_{n-1} && \text{- D10} \\
&= \phi_{n-1} \circ \phi_{n-2} \circ [\ldots(\phi_{m+1}(f))] \triangledown [\psi_{n-2} \circ \psi_{n-1}] && \text{- P3} \\
&\;\;\vdots \\
&= \phi_{n-1} \circ \ldots \circ \phi_m \circ f \triangledown [\psi_m \circ \ldots \circ \psi_{n-1}] && \text{- P3} \\
&= \theta_{mn} \circ f \triangledown \theta_{nm} . && \text{- D7}
\end{aligned}$$

Suppose $m = n$. Then, for all $f \in D_{m+1}$:

$$\begin{aligned}
\theta_{m+1,n+1}(f) &= f \\
&= I_{D_m} \circ f \triangledown I_{D_m} && \text{- P3} \\
&= \theta_{mn} \circ f \circ \theta_{nm} . && \text{- D7}
\end{aligned}$$

Suppose $m > n$. Then, for all $f \in D_{m+1}$:

$$\theta_{m+1,n+1}(f) = \psi_{n+1}(\psi_n(\ldots(\psi_m(f)))) \qquad \text{- D7}$$

$$= \psi_n \circ [\psi_n(\ldots(\psi_m(f)))] \triangledown \phi_n \qquad \text{- D10}$$

$$= \psi_n \circ [\psi_{n-1} \circ [\ldots(\psi_m(f))] \triangledown \phi_{n-1}] \triangledown \phi_n \qquad \text{- D10}$$

$$= \psi_n \circ \psi_{n-1} \circ [\ldots(\psi_m(f))] \triangledown [\phi_{n-1} \circ \phi_n] \qquad \text{- P3}$$

$$\vdots$$

$$= \psi_n \circ \psi_{n-1} \circ \ldots \circ \psi_{m-1} \circ f \triangledown [\phi_{m-1} \circ \ldots \circ \phi_{n-1} \circ \phi_n]$$

- P3

$$= \theta_{mn} \circ f \triangledown \theta_{nm} . \qquad \square \qquad \text{- D7}$$

Definition 12:

Let $\phi \in D_\infty \to [D^+ \to D\ ]$

$\psi \in [D^+_\infty \to D_\infty] \to D_\infty$

be the functions such that

(1) $$\phi(x) = \coprod_{n=0}^{\infty} \theta_{n\infty} \circ [\theta_{\infty,n+1}(x)] \triangledown \theta_{\infty n}$$

(2) $$\psi(f) = \coprod_{n=0}^{\infty} \theta_{n+1,\infty}(\theta_{\infty n} \circ f \triangledown \theta_{n\infty}) .$$

Proposition 10:

The ℓ.u.b's in the second member of (1) and (2) above are limits of directed sequences.

* Proof:

For all $x \in D_\infty$ we have:

$$\theta_{n\infty} \circ [\theta_{\infty,n+1}(x)] \;\nabla\; \theta_{\infty n}$$

$$= \theta_{n+1,\infty} \circ \theta_{n,n+1} \circ [\theta_{\infty,n+1}(x)] \;\nabla\; [\theta_{n+1,n} \circ \theta_{\infty,n+1}] \qquad \text{- P6}$$

$$= \theta_{n+1,\infty} \circ \theta_{n+1,n+2}[\theta_{\infty,n+1}(x)] \;\nabla\; \theta_{\infty,n+1} \qquad \text{- P9,P3}$$

$$\sqsubseteq \theta_{n+1,\infty} \circ [\theta_{\infty,n+2}(x)] \;\nabla\; \theta_{\infty,n+1} \qquad \text{- P6}$$

and for all $f \in D_\infty^+ \to D_\infty$ we have:

$$\theta_{n+1,\infty}(\theta_{\infty n} \circ f \;\nabla\; \theta_{n\infty})$$

$$= \theta_{n+1,\infty}(\theta_{n+1,n} \circ \theta_{\infty,n+1} \circ f \;\nabla\; [\theta_{n+1,\infty} \circ \theta_{n,n+1}]) \qquad \text{- P6}$$

$$= \theta_{n+1,\infty}(\theta_{n+1,n} \circ [\theta_{\infty,n+1} \circ f \;\nabla\; \theta_{n+1,}\ ] \;\nabla\; \theta_{n,n+1}) \qquad \text{- P3}$$

$$= \theta_{n+1,\infty}(\theta_{n+2,n+1}(\theta_{\infty,n+1} \circ f \;\nabla\; \theta_{n+1,\infty})) \qquad \text{- P9}$$

$$\sqsubseteq \theta_{n+2,\infty}(\theta_{\infty,n+1} \circ f \;\nabla\; \theta_{n+1,\infty}) \,. \quad \square \qquad \text{- P6}$$

Proposition 11:

ϕ and ψ are continuous.

* Proof:

Suppose $X \subseteq D_\infty$ directed. We have:

$$\phi(\sqcup X) = \coprod_{n=0}^{\infty} \theta_{n\infty} \circ [\theta_{\infty,n+1}(\sqcup X)] \;\nabla\; \theta_{\infty n} \qquad \text{- D12}$$

$$= \coprod_{n=0}^{\infty} \theta_{n\infty} \circ [\sqcup\{\theta_{\infty,n+1}(x) \mid x \in X\}] \;\nabla\; \theta_{\infty n} \qquad \text{- continuity}$$

$$= \coprod_{n=0}^{\infty} [\sqcup\{\theta_{n\infty} \circ [\theta_{\infty,n+1}(x)] \;\nabla\; \theta_{\infty n} \mid x \in X\}] \qquad \text{- continuity}$$

$$= \coprod \{ \coprod_{n=0}^{\infty} \theta_{n\infty} \circ [\theta_{\infty,n+1}(x)] \nabla \theta_{\infty n} \mid x \in X\}$$ - continuity and P10

$$= \coprod \{\phi(x) \mid x \in X\} .$$ - D12

The proof of the continuity of ψ is similar. □

Proposition 12:

ϕ and ψ are inverses to each other.

* Proof:

As an anxiliary argument we show that

$$\phi(\theta_{m+1,\infty}(f)) = \theta_{m\infty} \circ f \nabla \theta_{\infty m}$$

and

$$\theta_{\infty,m+1}(\psi(f)) = \theta_{\infty m} \circ f \nabla \theta_{m\infty} .$$

Infact, for all $f \in D_m^+ \to D_m$ we have:

$$\phi(\theta_{m+1,\infty}(f)) = \coprod_{n=0}^{\infty} \theta_{n\infty} \circ [\theta_{\infty,n+1}(\theta_{m+1,}\ (f))] \nabla \theta_{\infty n} .$$ - D12

But:

$$\theta_{n\infty} \circ [\theta_{\infty,n+1}(\theta_{m+1,\infty}(f))] \nabla \theta_{\infty n} = \theta_{n\infty} \circ [\theta_{m+1,n+1}(f)] \nabla \theta_{\infty n}$$ - P6

$$= \theta_{n\infty} \circ \theta_{mn} \circ f \nabla [\theta_{nm} \circ \theta_{\infty n}] .$$ - P9,P3

Hence:

$$\phi(\theta_{m+1,\infty}(f))$$

$$= \coprod_{n=0}^{\infty} \theta_{n\infty} \circ \theta_{mn} \circ f \nabla [\theta_{nm} \circ \theta_{\infty n}]$$ - P6

$$= \coprod_{n>m}^{\infty} \theta_{n\infty} \circ \theta_{mn} \circ f \nabla (\theta_{nm} \circ \theta_{\infty n})$$ - because we had a directed sequence (by P7)

$$= \coprod_{n>m}^{\infty} \theta_{m\infty} \circ f \nabla \theta_{\infty m}$$ - P6

$$= \theta_{m\infty} \circ f \nabla \theta_{\infty m} .$$ - because the indexing is independent of n

Similarly, for all $f \in D_{\infty}^{+} \to D_{\infty}$ we have:

$\theta_{\infty,m+1}(\psi(f)) = \theta_{\infty m} \circ f \nabla \theta_{m\infty}$ (End of the auxiliary argument).

We have (for all $x \in D_{\infty}$):

$$\psi(\phi(x)) = \coprod_{m=0}^{\infty} \theta_{m+1,\infty}(\theta_{\infty,m+1}(\psi(\phi(\coprod_{n=0}^{\infty} \theta_{n+1,\infty}(\theta_{\infty,n+1}(x))))))$$ - D12,P7

$$= \coprod_{m=0}^{\infty} [\coprod_{n=0}^{\infty} \theta_{m+1,\infty}(\theta_{\infty,m+1}(\psi(\phi(\theta_{n+1,\infty}(\theta_{\infty,n+1}(x))))))]$$ - continuity

$$= \coprod_{m=0}^{\infty} \theta_{m+1,\infty}(\theta_{\infty,m+1}(\psi(\phi(\theta_{m+1,\infty}(\theta_{\infty,m+1}(x))))))$$

- property of double limit of directed sequences

$$= \coprod_{m=0}^{\infty} \theta_{m+1,\infty}(\theta_{\infty,m+1}(\psi(\theta_{m\infty} \circ [\theta_{\infty,m+1}(x)] \nabla \theta_{\infty m})))$$

- anxiliary argument

$$= \coprod_{m=0}^{\infty} \theta_{m+1,\infty}(\theta_{\infty m} \circ \theta_{m\infty} \circ [\theta_{\infty,m+1}(x)] \nabla [\theta_{\infty m} \circ \theta_{m\infty}])$$

- anxiliary argument + P3

$$= \coprod_{m=0}^{\infty} \theta_{m+1,\infty}(I_{Dm} \circ [\theta_{\infty,m+1}(x)] \nabla I_{Dm})$$ - P6

$$= \bigsqcup_{m=0}^{\infty} \theta_{m+1,\infty}(\theta_{\infty,m+1}(x)) \qquad \text{- P3}$$

$$= x . \qquad \text{- P7}$$

We also have (for all $f \in D_{\infty}^{+} \to D_{\infty}$):

$$\phi(\psi(f)) = \phi(\bigsqcup_{m=0}^{\infty} \theta_{m+1,\infty}(\theta_{\infty,m+1}(\psi(f)))) \qquad \text{- P7}$$

$$= \bigsqcup_{m=0}^{\infty} \phi(\theta_{m+1,\infty}(\theta_{\infty,m+1}(\psi(f)))) \qquad \text{- continuity}$$

$$= \bigsqcup_{m=0}^{\infty} \phi(\theta_{m+1,\infty}(\theta_{\infty m} \circ f \nabla \theta_{m\infty})) \qquad \text{- anxiliary argument}$$

$$= \bigsqcup_{m=0}^{\infty} \theta_{m\infty} \circ [\theta_{\infty m} \circ f \nabla \theta_{m\infty}] \nabla \theta_{\infty} \qquad \text{- auxiliary argument}$$

$$= \bigsqcup_{m=0}^{\infty} \theta_{m\infty} \circ \theta_{\infty m} \circ f \nabla [\theta_{m\infty} \circ \theta_{\infty m}] . \qquad \text{- P3}$$

Hence for all $<a_1, \ldots, a_p> \in D_{\infty}^{+}$ we have:

$$\phi(\psi(f))(<a_1,\ldots,a_p>)$$

$$= \bigsqcup_{m=0}^{\infty} \theta_{m\infty}(\theta_{\infty m}(f(<\theta_{m\infty}(\theta_{\infty m}(a_1)), \ldots, \theta_{m\infty}(\theta_{\infty m}(a_p))>))) \qquad \text{- P2}$$

$$= \bigsqcup_{m=0}^{\infty} \theta_{m\infty}(\theta_{\infty m}(f(\bigsqcup_{n=0}^{\infty} <\theta_{n\infty}(\theta_{\infty n}(a_1)), \ldots >))) \qquad \text{- property of double limit of directed sequences}$$

$$= f(\coprod_{n=0}^{\infty} <\theta_{n\infty}(\theta_{\infty n}(a_1)), \ldots, \theta_{n\infty}(\theta_{\infty n}(a_p))>) \qquad \text{- P7}$$

$$= f(<\coprod_{n=0}^{\infty} \theta_{n\infty}(\theta_{\infty n}(a_1)), \ldots, \coprod_{n=0}^{\infty} \theta_{n\infty}(\theta_{\infty n}(a_p))>) \qquad \text{- D1}$$

$$= f(<a_1, \ldots, a_p>). \qquad \text{- P7}$$

Furthermore, we have:

$$\phi(\psi(f))(\top_{D_\infty^+}) = \coprod_{m=0}^{\infty} \theta_{m\infty}(\theta_{\infty m}(f(\top_{D_\infty^+}))) \qquad \text{- D3}$$

$$= f(\top_{D_\infty^+}) \qquad \text{- D7}$$

and

$$\phi(\psi(f))(\perp_{D_\infty^+}) = \coprod_{m=0}^{\infty} \theta_{m\infty}(\theta_{\infty m}(f(\perp_{D_\infty^+}))) \qquad \text{- D3}$$

$$= f(\perp_{D_\infty^+}) . \qquad \text{- D7}$$

Therefore, for all $f \in D_\infty^+ \to D_\infty$, we have:

$$\phi(\psi(f)) = f \quad .$$

□

Theorem 1:

D_∞ and $D_\infty^+ \to D_\infty$ are isomorphic w.r.t. their lattice structure by the pair of continuous functions ϕ , ψ .

Proof:

Obvious (by Propositions 11 and 12). □

Definition 13:

For $i = 1, 2, \ldots$ let $s_i^!$ be the function from D_∞^+ to D_∞ such that:

1-) For all $\langle a_1, \ldots, a_p\rangle \in D^+$:

$$s_i^!(\langle a_1, \ldots, a_p\rangle) = \begin{cases} a_i & \text{if } p \geq i \\ \bot_{D_\infty} & \text{otherwise} \end{cases}$$

2-) $s_i^!(\top_{D_\infty^+}) = \top_{D_\infty}$

$s_i^!(\bot_{D_\infty^+}) = \bot_{D_\infty}$.

Proposition 13:

The functions $s_i^!$ are continuous.

Proof:

. Suppose $X \subseteq D^+$ directed and contain two vectors of different lengths, i.e. $\langle a_1, \ldots, a_p\rangle$ and $\langle b_1, \ldots, b_q\rangle$ with $p \neq q$, then X must contain $\top_{D_\infty^+}$. Hence:

$$\begin{aligned} s_i^!(\sqcup X) &= s_i^!(\top_{D_\infty^+}) \\ &= \top_{D_\infty} && \text{- D13} \\ &= \sqcup\{s_i^!(v) \mid v \in X\} && \text{- because } \top_{D_\infty} \in \{s_i^!(v) \mid v \in X\} \end{aligned}$$

. Suppose $X \subseteq D^+$ directed and its elements (except possibly $\top_{D_\infty^+}$ and $\bot_{D_\infty^+}$) of the same length, then trivially we have:

$$s_i^!(\sqcup X) = \sqcup\{s_i^!(v) \mid v \in X\} \quad . \qquad \square$$

Definition 14:

Let $\mathcal{Y}$ be the function from $D_\infty^+ \to D_\infty$ to D_∞ such that:

$$\mathcal{Y}(f) = \bigsqcup_{n=0}^{\infty} \mathcal{Y}_n(f)$$

where, for $n = 0, 1, 2, \ldots$ $\mathcal{Y}_n$ are functions from $D_\infty^+ \to D_\infty$ to D_∞ such that, for all $f \in D_\infty^+ \to D_\infty$:

$$\mathcal{Y}_0(f) = \bot_{D_\infty}$$

then recursively:

$$\mathcal{Y}_{n+1}(f) = f(<\mathcal{Y}_n(f)>) \quad .$$

Proposition 14:

$\mathcal{Y}$ is continuous.

* Proof:

. We first prove that, for $n = 0, 1, 2, \ldots,$ $\mathcal{Y}_n$ is continuous.

$\mathcal{Y}_0$ is obviously continuous.

The inductive step is:

Let $F \subseteq D_\infty^+ \to D_\infty$ be directed. We have:

$$\begin{aligned}
\mathcal{Y}_{n+1}(\sqcup F) &= (\sqcup F)(<\mathcal{Y}_n(\sqcup F)>) && \text{- D14} \\
&= (\sqcup F)(<\sqcup\{\mathcal{Y}_n(f) \mid f \in F\}>) && \text{- inductive hypothesis} \\
&= \sqcup\{g(<\sqcup\{\mathcal{Y}_n(f) \mid f \in F\}>) \mid g \in F\} && \text{- P2} \\
&= \sqcup\{\sqcup\{g(<\mathcal{Y}_n(f)>) \mid f \in F\} \mid g \in F\} && \text{- continuity of g}
\end{aligned}$$

$= \sqcup\{f(<\mathcal{Y}_n(f)>) \mid f \in F\}$ — limit of directed set and continuity of f

$= \sqcup\{\mathcal{Y}_{n+1}(f) \mid f \in F\}$. — D14

. Secondly, we prove that, for all f : $\{\mathcal{Y}_n(f) \mid n = 0, 1, 2, \ldots\}$ is a directed sequence (i.e. $\mathcal{Y}_n(f) \sqsubseteq \mathcal{Y}_{n+1}(f)$) .

Obviously: $\mathcal{Y}_0(f) \sqsubseteq \mathcal{Y}_1(f)$

The inductive step is:

Assume: $\mathcal{Y}_n(f) \sqsubseteq \mathcal{Y}_{n+1}(f)$

The monotonicity of f gives:

$$f(<\mathcal{Y}_n(f)>) \sqsubseteq f(<\mathcal{Y}_{n+1}(f)>)$$

i.e. $\mathcal{Y}_{n+1}(f) \sqsubseteq \mathcal{Y}_{n+2}(f)$. — D14

. Now suppose $F \subseteq D_\infty^+ \to D_\infty$ directed, then:

$\mathcal{Y}(\sqcup F) = \bigsqcup_{n=0}^{\infty} \mathcal{Y}_n(\sqcup F)$ — D14

$= \bigsqcup_{n=0}^{\infty} \sqcup\{\mathcal{Y}_n(f) \mid f \in F\}$ — continuity of $\mathcal{Y}_n$

$= \sqcup\left\{\bigsqcup_{n=0}^{\infty} \mathcal{Y}_n(f) \mid f \in F\right\}$ — continuity of $\sqcup$

$= \sqcup\{\mathcal{Y}(f) \mid f \in F\}$. — D14

□

<u>Proposition 15</u>:

For all $f \in D_\infty^+ \to D_\infty$: $f(<\mathcal{Y}(f)>) = \mathcal{Y}(f)$.

<u>Proof</u>:

For all $f \in D_\infty^+ \to D_\infty$, we have:

$$f(<\mathcal{Y}(f)>) = f(<\coprod_{n=0}^{\infty} \mathcal{Y}_n(f)>) \qquad \text{- D14}$$

$$= \coprod_{n=0}^{\infty} f(<\mathcal{Y}_n(f)>) \qquad \text{- continuity of } f$$

$$= \coprod_{n=0}^{\infty} \mathcal{Y}_{n+1}(f) \qquad \text{- D14}$$

$$= \coprod_{n=0}^{\infty} \mathcal{Y}_n(f) \qquad \text{- } \{\mathcal{Y}_n(f) \mid n=0, 1, 2, \ldots\} \text{ is a directed sequence (see second part of proof of P14)}$$

$$= \mathcal{Y}(f) \quad . \qquad \text{- D14}$$

□

Definition 15:

Let Y' be the function from D_∞^+ to D_∞ such that

1-) $Y'(<a_1>) = \mathcal{Y}(\phi(a_1))$

2-) $Y'(<a_1, \ldots, a_p>) = Y'(<a_1>)$

3-) $Y'(\top_{D_\infty^+}) = \top_{D_\infty}$

$Y'(\bot_{D_\infty^+}) = \bot_{D_\infty}$.

Proposition 16:

Y' is continuous.

Proof: Trivial in view of P14. □

Proposition 17:

For all $a_1 \in D_\infty$

$$Y'(<a_1>) = \phi(a_1)(<Y'(<a_1>)>)$$

(recall that $<a_1> \in D_\infty^+$ by definition).

Proof:

For all $a_1 \in D_\infty$, we have:

$$Y'(\langle a_1 \rangle) = \mathcal{Y}(\phi(a_1)) \qquad \text{- D15}$$

$$= \phi(a_1)(\langle \mathcal{Y}(\phi(a_1)) \rangle) \qquad \text{- P15}$$

$$= \phi(a_1)(\langle Y'(\langle a_1 \rangle) \rangle) \quad . \qquad \text{- D15}$$

□

Definition 16:

Let	"true"	denote	$\theta_{0\infty}(\langle \underline{true} \rangle)$
	"false"	denote	$\theta_{0\infty}(\langle \underline{false} \rangle)$
	"0"	denote	$\theta_{0\infty}(\langle \underline{zero} \rangle)$
	"1"	denote	$\theta_{0\infty}(\langle \underline{one} \rangle)$
	"2"	denote	$\theta_{0\infty}(\langle \underline{two} \rangle)$
		⋮	
	"-1"	denote	$\theta_{0\infty}(\langle \underline{-one} \rangle)$
	"-2"	denote	$\theta_{0\infty}(\langle -\underline{two} \rangle)$
		⋮	

(see D9 for the meaning of $\langle \underline{true} \rangle$, $\langle \underline{false} \rangle$, $\langle \underline{zero} \rangle$, $\langle \underline{one} \rangle$, ...)

Obviously: true, false, 0, 1, 2, ..., -1, -2, ... are elements of D_∞ .

Proposition 18:

For all $x \in D_\infty$: $[\theta_{0\infty} \circ \theta_{\infty 0}](x) = x$ iff there exists a unique $y \in D_0$ such that $\theta_{0\infty}(y) = x$. (Or more generally: $[\theta_{n\infty} \circ \theta_{\infty n}](x) = x$ iff there exists a unique $y \in D_n$ such that $\theta_{n\infty}(y) = x$.)

Proof:

. Suppose $[\theta_{n\infty} \circ \theta_{\infty n}](x) = x$.

Take $y = \theta_{\infty n}(x)$

Then $y \in D_n$ - D8

and $\theta_{n\infty}(y) = \theta_{n\infty}(\theta_{\infty n}(x))$

$= [\theta_{n\infty} \circ \theta_{\infty n}](x)$

$= x$.

If $y' \in D_n$ is such that $y' \neq y$ then $\theta_{n\infty}(y') \neq \theta_{n\infty}(y)$, i.e. $\theta_{n\infty}(y') \neq x$.

. Suppose $\theta_{n\infty}(y) = x$. Then

$$[\theta_{n\infty} \circ \theta_{\infty n}](x) = [\theta_{n\infty} \circ \theta_{\infty n} \circ \theta_{n\infty}](y)$$

$$= [\theta_{n\infty} \circ \theta_{nn}](y) \qquad \text{- P6}$$

$$= [\theta_{n} \circ I_{D_n}](y) \qquad \text{- D7}$$

$$= \theta_{n\infty}(y)$$

$$= x \quad .$$

□

Definition 17:

Let if' be the function from $D^{+}_{\infty} \to D_{\infty}$ such that:

1-) For all $a_1, a_2, a_3, \ldots, a_p \in D$ (with $p \geq 3$) :

$$\text{if}'(\langle a_1, a_2, a_3, \ldots, a_p \rangle) = \begin{cases} a_2 & \text{if } \theta_{\infty 0}(a_1) = \underline{\langle\text{true}\rangle} \\ a_3 & \text{if } \theta_{\infty 0}(a_1) = \underline{\langle\text{false}\rangle} \\ \top_{D_\infty} & \text{if } \theta_{\infty 0}(a_1) = \top_B \\ \top_{D_\infty} & \text{if } \theta_{\infty 0}(a_1) = \top_{D_0} \\ \bot_{D_\infty} & \text{otherwise} \end{cases}$$

2-) $\text{if}'(\langle a_1, a_2\rangle) = \text{if}'(\langle a_1, a_2, \perp_{D_\infty}\rangle)$

3-) $\text{if}'(\langle a_1\rangle) = \text{if}'(\langle a_1, \perp_{D_\infty}, \perp_{D_\infty}\rangle)$

4-) $\text{if}'(\top_{D_\infty^+}) = \top_{D_\infty}$

$\text{if}'(\perp_{D_\infty^+}) = \top_{D_\infty}$.

Proposition 19:

if' is continuous.

Proof: Similar to that of P13. □

Definition 18:

Let K' be the function from D_∞^+ to D_∞ such that:

1-) $K'(\langle a_1\rangle) = \psi(A)$

where A is the constant function from D_∞^+ to D_∞ with value a_1 .

2-) $K'(\langle a_1, \ldots, a_p\rangle) = K'(\langle a_1\rangle)$ if $p > 1$

3-) $K'(\top_{D_\infty^+}) = \top_{D_\infty}$

$K'(\perp_{D_\infty^+}) = \perp_{D_\infty}$.

Proposition 20:

K' is continuous.

Proof: Similar to that of P13. □

Definition 19:

Let C' be the function from D_∞^+ to D_∞ such that:

1-) For $p \geq 2$:

$C'(<a_1, \ldots, a_p>) = \psi(A)$

where A is the function from D_∞^+ to D_∞ such that

a.) $A(<b_1, \ldots, b_p>) = \phi(a_1)(<\phi(a_2)(<b_1, \ldots, b_p>), \ldots, \phi(a_p)(<b_1, \ldots, b_p>)>)$ for all $<b_1, \ldots, b_p> \in D_\infty^+$.

b.) $A(\top_{D_\infty^+}) = \top_{D_\infty^+}$

$A(\bot_{D_\infty^+}) = \bot_{D_\infty}$

(Obviously A is continuous, the proof is similar to that of P13.)

2-) $C'(<a_1>) = \bot_{D_\infty}$

3-) $C'(\top_{D_\infty^+}) = \top_{D_\infty}$

$C'(\bot_{D_\infty^+}) = \bot_{D_\infty}$.

Proposition 21:

C' is continuous.

Proof: Similar to that of P13. □

Definition 20:

Let eval' be the function from D_∞^+ to D_∞ such that:

1-) For all $<a_1, \ldots, a_p> \in D_\infty^+$, with $p \geq 2$:

$$eval'(<a_1, \ldots, a_p>) = \phi(a_1)(<a_2, \ldots, a_p>)$$

2-) $eval'(<a_1>) = \bot_{D_\infty}$

3-) $eval'(\top_{D^+_\infty}) = \top_{D_\infty}$

$eval'(\bot_{D^+_\infty}) = \bot_{D_\infty}$.

<u>Proposition 22</u>:

eval' is continuous.

<u>Proof</u>: Similar to that of P13. □

<u>Definition 21a</u>:

Let sum be the function from $D_o x D_o$ to D_o such that, for all $a_1, a_2 \in D_o$:

1-) If $a_1, a_2 \in$ {<<u>zero</u>>, <<u>one</u>>, <<u>-one</u>>, <<u>two</u>>, <<u>-two</u>>, ...} then

$sum(a_1, a_2) = a_3$

where a_3 is obtained by adding a_2 to a_1 .

2-) If $(a_1 = \top_Z$ or $a_1 = \top_{D_o})$ and $(a_2 = \top_Z$ or $a_2 = \top_{D_o})$ then

$$sum(a_1, a_2) = \top_{D_o}$$

3-) $sum(a_1, a_2) = \bot_{D_o}$ otherwise.

Obviously sum is continuous.

(Plus similar definitions for substract, multiply, equal, greater , ...)

Definition 21b:

Let add' be the function from D_∞^+ to D_∞ such that:

1-) For all $a_1, \ldots, a_p \in D_\infty$ with $p \geq 2$

$$\text{add}'(<a_1, \ldots, a_p>) = \theta_{0\infty}(\text{sum}(\theta_{\infty 0}(a_1), \theta_{\infty 0}(a_2))$$

2-) $\text{add}'(<a_1>) = \text{add}'(<a_1, \bot_{D_\infty}>)$

3-) $\text{add}'(\top_{D_\infty^+}) = \top_{D_\infty}$

$\text{add}'(\bot_{D_\infty^+}) = \bot_{D_\infty}$.

(Similar definitions for sub', mult', div', eq', greater', less', ...)

Proposition 23:

add', sul', mult', div', eq', greater', less', ... are continuous.

Proof: Similar to that of P13. □

Definition 22: (Assignment of meaning to each of the symbols used in the calculus)

Let

s_i	denote	$\psi(s_i')$,	for $i = 1, 2, \ldots$
Y	denote	$\psi(Y')$	
K	denote	$\psi(K')$	
C	denote	$\psi(C')$	
eval	denote	$\psi(\text{eval}')$	
if	denote	$\psi(\text{if}')$	
add	denote	$\psi(\text{add}')$	

sub	denote	$\psi(\text{sub}')$
mult	denote	$\psi(\text{mult}')$
div	denote	$\psi(\text{div}')$
eq	denote	$\psi(\text{eq}')$
greater	denote	$\psi(\text{greater}')$
less	denote	$\psi(\text{less}')$ ... .

Obviously the just defined elements belong all to D_∞ .

<u>Definition 23</u>: (Assignment of meaning to the w.f.e.'s of the calculus)

For $x, y_1, y_2, \ldots, y_p \in D_\infty$ $(p \geq 1)$, let $x[\![y_1, \ldots, y_p]\!]$ denote

$$\phi(x)(\langle y_1, \ldots, y_p \rangle) \quad .$$

<u>Theorem 2</u>: (Validity of axiom schema #1)

For $i = 1, 2, \ldots;\ a_1, \ldots, a_p \in D_\infty$:

$$s_i[\![a_1, \ldots, a_p]\!] = \begin{cases} a_i & \text{if } p \geq i \\ \perp_{D_\infty^+} & \text{otherwise .} \end{cases}$$ □

<u>Theorem 3</u>: (Validity of axiom schema #2)

For $A_1, \ldots, A_p \in D_\infty$ $(p \geq 2)$

$B_1, \ldots, B_q \in D_\infty$:

$$C[\![A_1, \ldots, A_p]\!][\![B_1, \ldots, B_q]\!]$$

$$= A_1[\![A_2[\![B_1, \ldots, B_q]\!], \ldots, A_p[\![B_1, \ldots, B_q]\!]]\!] \quad .$$

Proof:

$$C[\![A_1, \ldots, A_p]\!][\![B_1, \ldots, B_q]\!]$$

$$= \phi(\phi(C)(<A_1, \ldots, A_p>))(<B_1, \ldots, B_q>) \qquad \text{- D23}$$

$$= \phi(C'(<A_1, \ldots, A_p>))(<B_1, \ldots, B_q>) \qquad \text{- D22, P12}$$

$$= \phi(A_1)(<\phi(A_2)(<B_1, \ldots, B_q>), \ldots, \phi(A_p)(<B_1, \ldots, B_q>)>) \qquad \text{- D19}$$

$$= A_1[\![A_2[\![B_1, \ldots, B_q]\!], \ldots, A_p[\![B_1, \ldots, B_q]\!]]\!] \quad .$$

□

Theorem 4: (Validity of axiom schema #3)

1-) $Y[\![A_1]\!] = A_1[\![Y[\![A_1]\!]]\!]$.

2-) $Y[\![A_1, \ldots, A_p]\!] = Y[\![A_1]\!]$.

Proof:

1-) $Y[\![A_1]\!] = \phi(Y)(<A_1>)$ - D23

$= \phi(\psi(Y'))(<A_1>)$ - D22

$= Y'(<A_1>)$ - P12

$= \phi(A_1)(<Y'(<A_1>)>)$ - P17

$= \phi(A_1)(<\phi(Y)(<A_1>)>)$ - P12

$= A_1[\![Y[\![A_1]\!]]\!]$. - D23

2-) Obvious.

□

Theorem 5: (Validity of axiom schema #4)

For $a_1, a_2 \in D_\infty$:

1-) $\text{if}[\![\text{true}, a_1, a_2]\!] = a_1$

$\text{if}[\![\text{false}, a_1, a_2]\!] = a_2$

2-) $\text{if}[\![a_1, a_2, a_3, \ldots, a_p]\!] = \text{if}[\![a_1, a_2, a_3]\!]$.

Proof:

1-) $\text{if}[\![\text{true}, a_1, a_2]\!] = \phi(\text{if})(<\text{true}, a_1, a_2>)$ - D23

$= \phi(\psi(\text{if}'))(<\text{true}, a_1, a_2>)$ - D22

$= \text{if}'(<\text{true}, a_1, a_2>)$ - P12

But: $\text{true} = \theta_{0\infty}(<\underline{\text{true}}>)$ - D16

Hence:

$$\theta_{\infty 0}(\text{true}) = \theta_{\infty 0}(\theta_{0\infty}(<\underline{\text{true}}>))$$

$$= \theta_{00}(<\text{true}>) \quad \text{- P6}$$

$$= I_{D_0}(<\underline{\text{true}}>) \quad \text{- D7}$$

$$= <\underline{\text{true}}>$$

Therefore:

$$\text{if}[\![\text{true}, a_1, a_2]\!] = a_1 \quad . \quad \text{- D17}$$

Similarly:

$$\text{if}[\![\text{false}, a_1, a_2]\!] = a_2 \; .$$

2-) Obvious. □

Theorem 6a: (Validity of axiom schema #5)

For $B_1, \ldots, B_p \in D_\infty$ $(p \geq 2)$:

$$\text{eval}[\![B_1, \ldots, B_p]\!] = B_1[\![B_2, \ldots, B_p]\!] \quad .$$

Proof:

For $B_1, \ldots, B_p \in D_\infty$ $(p \geq 2)$, we have:

$$\mathrm{eval}\llbracket B_1, \ldots, B_p \rrbracket$$

$= \phi(\mathrm{eval})(<B_1, \ldots, B_p>)$ - D23

$= \phi(\psi(\mathrm{eval}'))(<B_1, \ldots, B_p>)$ - D22

$= \mathrm{eval}'(<B_1, \ldots, B_p>)$ - P12

$= \phi(B_1)(<B_2, \ldots, B_p>)$ - D20

$= B_1\llbracket B_2, \ldots, B_p \rrbracket$. - D23

□

Theorem 6b: (Validity of axiom schema #6)

For $A_1, B_1, A_2, B_2, \ldots \in D_\infty$:

1-) $K\llbracket A_1 \rrbracket\llbracket B_1, \ldots, B_q \rrbracket = A_1$ if $q \geq 1$

2-) $K\llbracket A_1, \ldots, A_p \rrbracket\llbracket B_1, \ldots, B_q \rrbracket = A_1$ if $p \geq 1$, $q \geq 1$

Proof:

1-) $K\llbracket A_1 \rrbracket\llbracket B_1, \ldots, B_q \rrbracket$

$= \phi(\phi(K)(<A_1>))(<B_1, \ldots, B_q>)$ - D23

$= \phi(\phi(\psi(K'))(<A_1>))(<B_1, \ldots, B_q>)$ - D22

$= \phi(K'(<A_1>))(<B_1, \ldots, B_q>)$ - P12

$= A_1$. - D18

2-) Obvious. □

Theorem 7: (Validity of axiom schemas #7 and #8)

For all $a_1, a_2, \ldots, a_p \in D_\infty$:

1-) $add[\![a_1, a_2]\!] = \theta_{0\infty}(sum(\theta_{\infty 0}(a_1), \theta_{\infty 0}(a_2)))$

2-) $add[\![a_1, a_2, \ldots, a_p]\!] = add[\![a_1, a_2]\!]$.

(Plus similar equalities envolving other arithmetic operations or predicates.)

Proof:

1-) $add[\![a_1, a_2]\!] = \phi(add)(<a_1, a_2>)$ - D23

$= \phi(\psi(add')(<a_1, a_2>)$ - D22

$= add'(<a_1, a_2>)$ - P12

$= \theta_{0\infty}(sum(\theta_{\infty 0}(a_1), \theta_{\infty 0}(a_2)))$. - D21b

2-) Obvious. □

Proposition 24:

For $n = 0, 1, 2, \ldots$, if $x = \theta_{n\infty}(y)$ for some $y \in D_n$ then, for all $m \geq n$:

$$\theta_{\infty,m+1}(x) = \theta_{m,m+1}(\theta_{\infty,m}(x))$$

Proof:

We first prove that, for all $m \geq n$, $\theta_{m\infty}(\theta_{\infty m}(x)) = x$.

. For $m = n$:

By hypothesis we have:

$\theta_{\infty n}(x) = \theta_{\infty n}(\theta_{n\infty}(y))$

$= \theta_{nn}(y)$ - P6

$= y$ - D7

Hence:

$\theta_{n\infty}(\theta_{\infty n}(x)) = \theta_{n\infty}(y) = x$ - hypothesis

. Inductive step:

On one hand we have:

$$\coprod_{m=0}^{\infty} \theta_{m\infty}(\theta_{\infty m}(x)) = x \qquad \text{- P7}$$

On the other hand we have:

$$\theta_{m\infty}(\theta_{\infty n}(x)) \sqsubseteq \theta_{m+1,\infty}(\theta_{\infty,n+1}(x)) \qquad \text{- P7}$$

But $\theta_{n\infty}(\theta_{\infty n}(x)) = x$ - above

Hence: for all $m \geq n$:

$$\theta_{m\infty}(\theta_{\infty m}(x)) = x \quad .$$

Therefore,we have, for all $m \geq n$:

$$\begin{aligned}\theta_{\infty,m+1}(x) &= \theta_{\infty,m+1}(\theta_{m\infty}(\theta_{\infty m}(x))) && \text{- above}\\ &= [[\theta_{\infty,m+1}\circ\theta_{m,\infty}]\circ\theta_{\infty m}](x) \\ &= \theta_{m,m+1}(\theta_{\infty m}(x)) \quad . && \text{- P6}\end{aligned}$$

□

<u>Theorem 8</u>: (Validity of axiom schema #9)

If $x = \theta_{0\infty}(y)$ for some $y \in D_0$ then

$$x[\![a_1, \ldots, a_p]\!] = x \text{ (for all } a_1, \ldots, a_p \in D_\infty)$$

and $\phi(x)(\top_{D_\infty^+}) = \phi(x)(\bot_{D_\infty^+}) = x$.

<u>Proof</u>:

. We first show that, for all $n \geq 0$, $a_1, \ldots, a_p \in D_\infty$:

$$[[\theta_{\infty,n+1}(x)] \nabla \theta_{\infty n}](<a_1, \ldots, a_p>) = \theta_{\infty n}(x)$$

For $n = 0$ we have:

$$[[\theta_{\infty,1}(x)] \nabla \theta_{\infty o}](<a_1, \ldots, a_p>)$$

$$= [[\theta_{o1}(\theta_{\infty o}(x))] \nabla \theta_{\infty o}](<a_1, \ldots, a_p>) \qquad \text{- P24}$$

$$= \theta_{\infty,o}(x) \qquad \text{- property of } \theta_{o1}$$

Inductive step:

We have:

$$[[\theta_{\infty,n+2}(x)] \nabla \theta_{\infty,n+1}](<a_1, \ldots, a_p>)$$

$$= [[\theta_{n+1,n+2}(\theta_{\infty,n+1}(x))] \nabla \theta_{\infty,n+1}](<a_1, \ldots, a_p>) \qquad \text{- P24}$$

$$= [[\theta_{n,n+1} \circ [\theta_{\infty,n+1}(x)] \nabla \theta_{n+1,n}] \nabla \theta_{\infty,n+1}](<a_1, \ldots, a_p>) \qquad \text{- P9}$$

$$= [\theta_{n,n+1} \circ [\theta_{\infty,n+1}(x)] \nabla [\theta_{n+1,n} \circ \theta_{\infty,n+1}]](<a_1, \ldots, a_p>) \qquad \text{- P3}$$

$$= [\theta_{n,n+1} \circ [\theta_{\infty,n+1}(x)] \nabla \theta_{\infty,n}](<a_1, \ldots, a_p>) \qquad \text{- P6}$$

$$= \theta_{n,n+1}(\theta_{\infty n}(x)) \qquad \text{-inductive hypothesis}$$

$$= \theta_{\infty,n+1}(x) \quad . \qquad \text{- P24}$$

. Similarly, we have:

$$[[\theta_{\infty,n+1}(x)] \nabla \theta_{\infty n}](\top_{D_\infty^+})$$

$$= [[\theta_{\infty,n+1}(x)]] \nabla \theta_{\infty n}](\perp_{D_\infty^+})$$

$$= \theta_{\infty n}(x) \quad .$$

. Therefore, for all $a_1, \ldots, a_p \in D_\infty$, we have:

$$\mathbf{x}[\![a_1, \ldots, a_p]\!] = \phi(x)(<a_1, \ldots, a_p>) \qquad \text{- D23}$$

$$= \coprod_{n=0}^{\infty} [\theta_{n\infty} \circ [\theta_{\infty,n+1}(x)] \nabla \theta_{\infty n}](\langle a_1, \ldots, a_p \rangle) \qquad \text{- D12}$$

$$= \coprod_{n=0}^{\infty} [\theta_{n\infty} \circ \theta_{\infty n}](x) \qquad \text{- above}$$

$$= x \ . \qquad \text{- P7}$$

. Similarly: $\phi(x)(\top_{D_\infty^+}) = \phi(x)(\bot_{D_\infty^+}) = x$. □

Theorem 9:

If $x = \theta_{0\infty}(y)$ for some $y \in D_0$ then

$$K[\![x]\!] = x \ .$$

Proof:

We have:

$K[\![x]\!] = \phi(K)(\langle x \rangle)$ - D23

$= \phi(\psi(K'))(\langle x \rangle)$ - D22

$= K'(\langle x \rangle)$ - P12

$= \psi$ (the constant function from D_∞^+ to D_∞ with value x) - D18

$= \psi(\phi(x))$ - T8

$= x$. □

Theorem 10:

$C[\![Y, s_1]\!] = Y$

$C[\![K, s_1]\!] = K$

$C[\![\text{if}, s_1, s_2, s_3]\!] = \text{if}$

$C[\![\text{add}, s_1, s_2]\!] = \text{add}$

(Plus similar theorems for sub, mult, div, eq, greater, less ... .) □

<u>Theorem 11</u>: (Consistency of the inference rules)

The inference rules of the calculus are consistent (i.e. from true premisses we can only infer true conclusions). □

<u>Theorem 12</u>: (Extensionality)

Let $A_1, A_2 \in D_\infty$: If

1-) For all $B_1, \ldots, B_p \in D_\infty$ $(p \geq 1)$:

$$A_1[\![B_1, \ldots, B_p]\!] = A_2[\![B_1, \ldots, B_p]\!]$$

2-) $\phi(A_1)(\top_{D_\infty^+}) = \phi(A_2)(\top_{D_\infty^+})$

$\phi(A_1)(\bot_{D_\infty^+}) = \phi(A_2)(\bot_{D_\infty^+})$

then $A_1 = A_2$.

<u>Proof</u>:

By hypothesis we have:

$$\phi(A_1)(<B_1, \ldots, B_p>) = \phi(A_2)(<B_1, \ldots, B_p>)$$

$$\phi(A_1)(\top_{D^+}) = \phi(A_2)(\top_{D_\infty^+})$$

$$\phi(A_1)(\bot_{D^+}) = \phi(A_2)(\bot_{D_\infty^+})$$

This means that $\phi(A_1)$ and $\phi(A_2)$ are a same function from D_∞^+ to D_∞.

Therefore:

$$\psi(\phi(A_1)) = \psi(\phi(A_2))$$

i.e. $A_1 = A_2$. □

Follows are some obvious but useful results:

Proposition 25:

For all $A_1, A_2, \ldots, A_p \in D$ with $p \geq 2$:

$$\phi(C[\![A_1, \ldots, A_p]\!])(\top_{D_\infty^+}) = \phi(A_1)(<\phi(A_2)(\top_{D_\infty^+}), \ldots, \phi(A_p)(\top_{D_\infty^+})>)$$

$$\phi(C[\![A_1, \ldots, A_p]\!])(\bot_{D_\infty^+}) = \phi(A_1)(<\phi(A_2)(\bot_{D_\infty^+}), \ldots, \phi(A_p)(\bot_{D_\infty^+})>)$$

$$\phi(K[\![A_1, \ldots, A_p]\!])(\top_{D_\infty^+}) = \phi(K[\![A_2, \ldots, A_p]\!])(\bot_{D_\infty^+}) = A_2 .$$

□

Theorem 13:

For all $A_1, A_2, \ldots, A_p \in D$ with $p \geq 2$:

$$C[\![A_1, A_2, \ldots, A_p]\!] = C[\![\text{eval}, \overline{A}_1, A_2, \ldots, A_p]\!]$$

$$C[\![C, \overline{A}_1, A_2, \ldots, A_p]\!] = C[\![C, \overline{\text{eval}}, \overline{\overline{A}}_1, A_2, \ldots, A_p]\!]$$

$$C[\![C, s_i, A_2, \ldots, A_p]\!] = C[\![C, \overline{\text{eval}}, C[\![K, s_i]\!], A_2, \ldots, A_p]\!] .$$

□

III. CHURCH-ROSSER PROPERTY AND NORMAL REDUCTION METHOD

In this chapter we use the terminology of [BR]. [illegible] for [illegible].

A. Church-Rosser property:

To make w.f.e.'s look like trees, let's rewrite each w.f.e. as follows:

(Let α be a special symbol different from the basic symbols of the calculus)

1) Each basic symbol is a w.f.e.

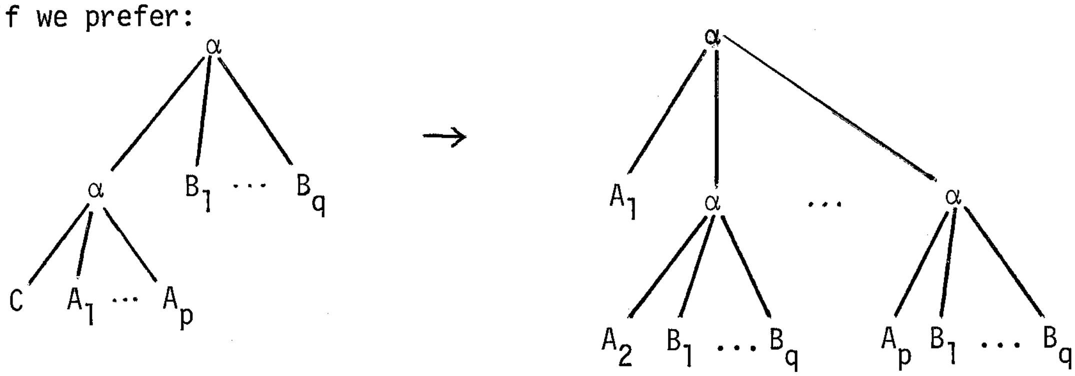
2) If $A_1, A_2, \ldots, A_n$ $(n \geq 2)$ are w.f.e.'s then $\alpha(A_1, A_2, \ldots, A_n)$ is a w.f.e.

3) There are no other kinds of w.f.e.'s.

Thus the second (reduction) rule schema, for example, will appear as follows:

$$\alpha(\alpha(C, A_1, \ldots, A_p), B_1, \ldots, B_q)$$
$$\rightarrow \alpha(A_1,\ (A_2, B_1, \ldots, B_q), \ldots, \alpha(A_p, B_1, \ldots, B_q))$$

or, if we prefer:



The symbols A_i, B_i must now be viewed as tree parameters ranging over the set $\mathcal{E}$ of all w.f.e.'s, each w.f.e. being now viewed as a tree.

Given the set of all rules generated by the rule schemas, we define the binary relation $\Rightarrow$ over $\mathcal{E}$ as the usual extension of the binary relation $\rightarrow$ defined by the set of all rules.

Let $\Rightarrow^*$ be the reflexive transitive closure of $\Rightarrow$. Consider the Post-Brainerd System

$$G = \{V, \mathcal{E}, \Rightarrow, \rightarrow\}$$

where $V = \{\text{basic symbols}\} \cup \{\alpha\}$.

Theorem 1:

The Post-Brainerd System G has the Church-Rosser property, namely:

If R, S, S' are w.f.e.'s and if we have $R \Rightarrow^* S$ and $R \Rightarrow^* S'$, then there exists a w.f.e. T such that $S \Rightarrow^* T$ and $S' \Rightarrow^* T$. (Hence: if S were irreducible then every reduction sequence starting from R can be prolonged to reach S).

Proof:

We use the theorem by B. Rosen ([BR] p.120) to the effect that every unequivocal and closed Post-Brainerd System has the Church-Rosser property.

We have:

1) G is unequivocal: in fact the binary relation $\rightarrow$ is deterministic by construction.

2) G is closed because:

- All tree parameters A_i, B_i appearing in the rule schemas are supposed to range over the set $\mathcal{E}$ of all w.f.e.'s (intuitively this means that we have a "well furnished" repertory of rules).
- The reduction of any w.f.e. always yields a w.f.e.
- Let $X_0 \to Y_0$ and $X' \to Y'$ be a rule and its corresponding rule schema. Now take any other rule $X \to Y$. By the construction of the rule schemas, X can appear nowhere (as a subtree) in X_0, except in the subtrees of X_0 which are delimited by the tree parameters of X'_0, but then either X does'nt reappear at all in Y_0, or X reappears (possibly at several occurrences) unaltered in Y_0.

Conclusion: our calculus has the Church-Rosser property.

□

B. Normal reduction method:

Terminology:

Due to the particular construction of the rule schemas, it is obvious that, in any reduction sequence, each reduction step can be completely identified by the Dewey number (abbreviation: D#) of the node on the current tree where the reduction step was performed.

(For a definition of the Dewey numbering of nodes of a tree please consult [BR], p. 118.)

Definition 1:

By up-shift of a D# $\gamma = n_1 \cdot n_2 \cdot \ldots \cdot n_p$ (with $p \geq 1$) by the D# $\beta = n_1 \cdot n_2 \cdot \ldots \cdot n_j$ (with $j \leq p$) we mean the following D# :

$$\mathcal{U}(\gamma, \beta) = \begin{cases} 0 \cdot n_{j+1} \cdot \ldots \cdot n_p & (\text{if } j < p) \\ 0 & (\text{if } j = p) \end{cases} .$$

By down-shift of a D# $\gamma = n_1 \cdot n_2 \cdot \ldots \cdot n_p$ (with $p \geq 1$) by a D# $\beta = m_1 \cdot m_2 \cdot \ldots \cdot m_j$ we mean the following D# :

$$\mathfrak{D}(\gamma, \beta) = \begin{cases} m_1 \cdot m_2 \cdot \ldots \cdot m_j \cdot n_2 \cdot \ldots \cdot n_p & (\text{if } p > 1) \\ m_1 \cdot m_2 \cdot \ldots \cdot m_j & (\text{if } \gamma = 1) \end{cases} .$$

Definition 2:

By up-shift of a set Γ (resp. down-shift of a set Γ) of D#'s by a D# $\beta = n_1 \cdot \ldots \cdot n_j$ we mean the set $\mathcal{U}(\Gamma, \beta)$ (resp. $\mathfrak{D}(\Gamma, \beta)$) of D#'s each obtained by up-shifting (resp. down-shifting) a different element (among those which can be shifted) of Γ, by β.

Definition 3:

By pivot of a reduction sequence we mean the first reduction step, if any, which is performed at the top of the current tree (i.e., the reduction sequence being a list of D#'s , the first D# which is equal to 0).

When such a pivot exists, we define the head and the tail of a reduction sequence W as the following lists of D#'s :

Let $W = (z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n)$ where $z_i = \text{pivot}(W)$. Then

$$\text{head}(W) = (z_1, \ldots, z_{i-1})$$
$$\text{tail}(W) = (z_{i+1}, \ldots, z_n) \quad .$$

Remark that head(S) or tail(S) , or both, can be nil.

Definition 4:

Let γ and β be two lists of D#'s such that:

$$\gamma = (z_1, \ldots, z_n) \qquad n \geq 0$$
$$\beta = (t_1, \ldots, t_p) \qquad p \geq 0$$

We define the concatenation of γ and β as:

$$\gamma \circ \beta = (z_1, \ldots, z_n, t_1, \ldots, t_p) \quad .$$

Proposition 1:

Let S be a w.f.e. (called the initial source tree). T be an irreducible w.f.e. (called the initial target tree).

If there exists a reduction sequence from S to T (called it the initial source sequence) then we can effectively find another reduction sequence $W = (z_1, \ldots, z_n)$ from S to T such that, for any $1 \leq i \leq n$, if the reduction step z_i were not performed at the top of the current tree (i.e. $z_i \neq 0$) then z_i must have been performed at the left most among the topmost nodes (of the current tree) where reductions were possible.

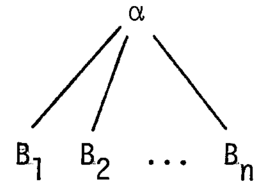
(Remark that, our calculus being deterministic by construction, such a sequence W must be unique; let's call it the initial target sequence).

Proof:

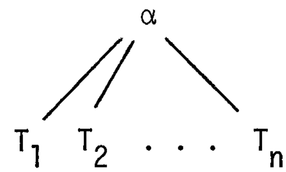
The following procedure (written in an Algol-like terminology) always halts and yields the initial target sequence when called with the three actual parameters: initial source tree, initial target tree and initial source sequence.

List Procedure FIND (SOURCE, TARGET, SEQUENCE);

if length of SEQUENCE ≤ 1
    then FIND: = SEQUENCE
    else
    if SEQUENCE has no pivot
        then SOURCE must have the form



and TARGET must have the form



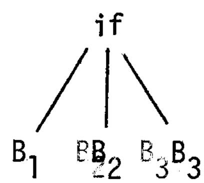
with $T_1, T_2, \ldots, T_n$ irreducible. Therefore we can take:

$$\begin{aligned}\text{FIND:} = \; & \mathfrak{D}(\text{FIND}(B_1, T_1, \mathfrak{U}(\text{SEQUENCE}, 0.0)), 0.0) \\ & \circ\; \mathfrak{D}(\text{FIND}(B_2, T_2, \mathfrak{U}(\text{SEQUENCE}, 0.1)), 0.1) \\ & \vdots \\ & \circ\; \mathfrak{D}(\text{FIND}(B_n, T_n, \mathfrak{U}(\text{SEQUENCE}, 0.n)), 0.n)\end{aligned}$$

else

if pivot(SEQUENCE) corresponds to the reduction of a conditional to the true branch (for example)

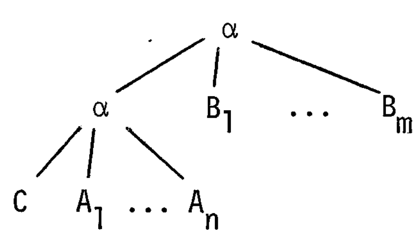
then SOURCE must have the form



Therefore we can take:

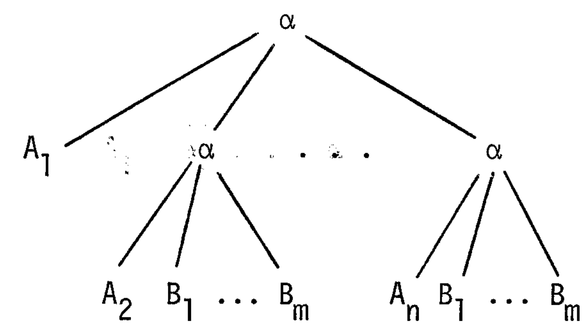
$$\begin{aligned}\text{FIND:} = \; & \mathfrak{D}(\text{FIND}(B_1, \text{'true'}, \mathfrak{U}(\text{head}(\text{SEQUENCE}), 0.0))) \\ & \circ\; (0) \\ & \circ\; \mathfrak{U}(\text{head}(\text{SEQUENCE}), 0.1) \\ & \circ\; \text{tail}(\text{SEQUENCE})\end{aligned}$$

else SOURCE must have the form (for example):



which could be reduced to:

Q =



Therefore we can take

FIND: =

(0)∘FIND(Q, TARGET,

$\mathfrak{D}$ ($\mathfrak{U}$(head(SEQUENCE), 0.0.1), 0.0)

∘ $\mathfrak{D}$ ($\mathfrak{U}$(head(SEQUENCE), 0.0.2), 0.1.0)

∘ $\mathfrak{D}$ ($\mathfrak{U}$(head(SEQUENCE), 0.1), 0.1.1)

⋮

∘ $\mathfrak{D}$ ($\mathfrak{U}$(head(SEQUENCE), 0.1), 0.1.m)

⋮

∘ $\mathfrak{D}$ ($\mathfrak{U}$(head(SEQUENCE), 0.0.n), 0.[n-1].0)

∘ $\mathfrak{D}$ ($\mathfrak{U}$(head(SEQUENCE), 0.1), 0.[n-1].1)

⋮

∘ $\mathfrak{D}$ ($\mathfrak{U}$(head(SEQUENCE), 0.1), 0.[n-1].m)

∘ tail(SEQUENCE))

The convergence of the procedure is ensured by the fact that in any recursive call of FIND from within its body we have as the third actual parameter:

- either a reduction sequence which is shorter than (the reduction sequence which is) the third actual parameter of the containing call of FIND .
- or a reduction sequence whose eventual pivot is situated nearer to the right end (of the sequence).

Therefore we must end up with a reduction sequence having no pivot.

□

The net result of the present section is that:

if, during the process of reduction, we always chose to reduce first at the topmost node where reduction is possible and if such a node doesn't exist, at the leftmost node among the topmost nodes where reductions are possible,

then we can be sure that if this method of reduction, when applied to a given w.f.e., gives a divergent reduction sequence then no other methods of reduction can give a convergent reduction sequence (from the same w.f.e.).

Example of normal reduction:

Consider the well known program:

<u>begin</u>

F(x, y) ⇐ <u>if</u> x=0 <u>then</u> 1 <u>else</u> F(x-1, F(x-y, y));

F(1, 0)

<u>end</u>

This program can be denoted be the w.f.e. $C[\![eval, Y, 1, 0]\!][\![\tau]\!]$

where

$$\tau = C[\![C, \overline{if}, C[\![C, \overline{eq}, \bar{s}_1, 0]\!], 1, C[\![C, s_1, C[\![C, \overline{sub}, \bar{s}_1, 1]\!], C[\![C, s_1, \overline{sub}, s_2]\!]]\!]]\!] .$$

Let's "run" this program by using the normal method. Let

ω_1 ="$C[\![C, \overline{eq}, \bar{s}_1, 0]\!]$"

ω_2 ="$C[\![C, s_1, C[\![C, \overline{sub}, \bar{s}_1, 1]\!], C[\![C, s_1, \overline{sub}, s_2]\!]]\!]$".

$C[\![eval, Y, 1, 0]\!][\![\tau]\!]$

$= eval[\![Y[\![\tau]\!], 1[\![\tau]\!], 0[\![\tau]\!]]\!]$ - A2

$= Y[\![\tau]\!][\![1[\![\tau]\!], 0[\![\tau]\!]]\!]$ - A5

$= \tau[\![Y[\![\tau]\!]]\!][\![1[\![\tau]\!], 0[\![\tau]\!]]\!]$ - A3

(Let ω_3 ="$[\![Y[\![\tau]\!]]\!]$")

$= C[\![\overline{if}\ \omega_3, \omega_1\ \omega_3, 1\ \omega_3, \omega_2\ \omega_3]\!][\![1[\![\tau]\!], 0[\![\tau]\!]]\!]$ - A2

(Let ω_4 ="$[\![1[\![\tau]\!], 0[\![\tau]\!]]\!]$", ω_5 ="$[\![\omega_3]\!][\![\omega_4]\!]$")

$= \overline{if}\ \omega_3[\![\omega_1\ \omega_5, 1\ \omega_5, \omega_2\ \omega_5]\!]$ - A2

$= if[\![\omega_1\ \omega_5, 1\ \omega_5, \omega_2\ \omega_5]\!]$ - A6

$= if[\![C[\![\overline{eq}\ \omega_3, s_1\ \omega_3, 0\ \omega_3]\!][\![\omega_4]\!], 1\ \omega_5, \omega_2\ \omega_5]\!]$ - A2

$= if[\![\overline{eq}\ \omega_3[\![\bar{s}_1\ \omega_5, 0\ \omega_5]\!], 1\ \omega_5, \omega_2\ \omega_5]\!]$ - A2

$= \mathrm{if}\llbracket \mathrm{eq}\llbracket \overline{s}_1 \cdot \omega_5,\ 0 \cdot \omega_5 \rrbracket,\ 1 \cdot \omega_5,\ \omega_2 \cdot \omega_5 \rrbracket$ - A6

$= \mathrm{if}\llbracket \mathrm{eq}\llbracket s_1 \llbracket \omega_4 \rrbracket,\ 0 \cdot \omega_5 \rrbracket,\ 1 \cdot \omega_5,\ \omega_2 \cdot \omega_5 \rrbracket$ - A6

$= \mathrm{if}\llbracket \mathrm{eq}\llbracket 1 \llbracket \tau \rrbracket,\ 0 \cdot \omega_5 \rrbracket,\ 1 \cdot \omega_5,\ \omega_2 \cdot \omega_5 \rrbracket$ - A1

$= \mathrm{if}\llbracket \mathrm{eq}\llbracket 1,\ 0 \cdot \omega_5 \rrbracket,\ 1 \cdot \omega_5,\ \omega_2 \cdot \omega_5 \rrbracket$ - A9

$= \mathrm{if}\llbracket \mathrm{eq}\llbracket 1,\ 0 \llbracket \omega_4 \rrbracket \rrbracket,\ 1 \cdot \omega_5,\ \omega_2 \cdot \omega_5 \rrbracket$ - A9

$= \mathrm{if}\llbracket \mathrm{eq}\llbracket 1,\ 0 \rrbracket,\ 1 \cdot \omega_5,\ \omega_2 \cdot \omega_5 \rrbracket$ - A9

$= \mathrm{if}\llbracket \mathrm{false},\ 1 \cdot \omega_5,\ \omega_2 \cdot \omega_5 \rrbracket$ - A8

$= \omega_2\ \omega_3 \llbracket \omega_4 \rrbracket$ - A4

(Let $\omega_6 =$ "$C\llbracket C,\ \overline{\mathrm{sub}},\ \overline{s}_1,\ 1 \rrbracket$" , $\omega_7 =$ "$C\llbracket C,\ s_1,\ \overline{\mathrm{sub}},\ s_2 \rrbracket$")

$= C\llbracket s_1 \cdot \omega_3,\ \omega_6 \cdot \omega_3,\ \omega_7 \cdot \omega_3 \rrbracket \llbracket \omega_4 \rrbracket$ - A2

$= s_1 \cdot \omega_3 \llbracket \omega_6 \cdot \omega_5,\ \omega_7 \cdot \omega_5 \rrbracket$ - A2

$= Y\llbracket \tau \rrbracket \llbracket \omega_6 \cdot \omega_5,\ \omega_7 \cdot \omega_5 \rrbracket$ - A1

$= \tau \cdot \omega_3 \llbracket \omega_6 \cdot \omega_5,\ \omega_7 \cdot \omega_5 \rrbracket$ - A3

(Let $\omega_8 = \llbracket \omega_6 \cdot \omega_5,\ \omega_7 \cdot \omega_5 \rrbracket$)

$= C\llbracket \overline{\mathrm{if}} \cdot \omega_3,\ \omega_1 \cdot \omega_3,\ 1 \cdot \omega_3,\ \omega_2 \cdot \omega_3 \rrbracket \llbracket \omega_8 \rrbracket$ - A2

(Let $\omega_9 = \omega_3 \llbracket \omega_8 \rrbracket$)

$= \overline{\mathrm{if}} \cdot \omega_3 \llbracket \omega_1 \cdot \omega_9,\ 1 \cdot \omega_9,\ \omega_2 \cdot \omega_9 \rrbracket$ - A2

$= \mathrm{if}\llbracket \omega_1 \cdot \omega_9,\ 1 \cdot \omega_9,\ \omega_2 \cdot \omega_9 \rrbracket$ - A4

$= \mathrm{if}\llbracket \overline{\mathrm{eq}} \cdot \omega_3 \llbracket \overline{s}_1 \cdot \omega_9,\ 0 \cdot \omega_9 \rrbracket,\ 1 \cdot \omega_9,\ \omega_2 \cdot \omega_9 \rrbracket$ - A2

$= \mathrm{if}\llbracket \mathrm{eq}\llbracket s_1 \llbracket \omega_8 \rrbracket,\ 0 \cdot \omega_9 \rrbracket,\ 1 \cdot \omega_9,\ \omega_2 \cdot \omega_9 \rrbracket$ - A4, A1

$= \text{if}[\![\text{eq}[\![\omega_6\,\omega_5,\ 0\cdot\omega_9]\!],\ 1\cdot\omega_9,\ \omega_2\,\omega_9]\!]$ - A1

$= \text{if}[\![\text{eq}[\![C[\![\overline{\text{sub}}\cdot\omega_3,\ \bar{s}_1\,\omega_3,\ 1\cdot\omega_3]\!][\![\omega_4]\!],\ 0\cdot\omega_9]\!],\ 1\cdot\omega_9,\ \omega_2\cdot\omega_9]\!]$ - A2

$= \text{if}[\![\text{eq}[\![\text{sub}[\![s_1[\![\omega_4]\!],\ 1\cdot\omega_5]\!],\ 0\,\omega_9]\!],\ 1\cdot\omega_9,\ \omega_2\cdot\omega_9]\!]$ - A6, A1

$= \text{if}[\![\text{eq}[\![1,\ 1]\!],\ 1\,\omega_9,\ \omega_2\cdot\omega_9]\!]$ - A7, A9

$= \text{if}[\![\text{true},\ 1\,\omega_9,\ \omega_2\,\omega_9]\!]$ - A8

$= 1\cdot\omega_9$ - A4

$= 1$. - A9

The computation halts. □

IV. AN EXTENSION OF THE CALCULUS

Introduction:

We present here a shorthand version of the calculus of chapter I (henceforth called the strict calculus). This shorthand version of the calculus will be referred to as the extended calculus.

Part of the terminology comes from [VW].

Definition 1:

Let's introduce a new set V of symbols (called the set of variables) such that

$$V = \{x_i, \underline{x}_i, \underline{\underline{x}}_i, \underline{\underline{\underline{x}}}_i, \ldots \mid i = 1, 2, 3, \ldots\}$$

We introduce also the symbol $\perp$ (called bottom).

Forms and clauses:

- Every basic symbol is a form.
- Every variable is a form.
- $\perp$, true, false, 0, 1, -1, 2, -2, ... are clauses.
- If A is a form then $\{A\}$ is a clause.
- Any clause is also a form.
- If $A_1, A_2, \ldots, A_n$ $(n \geq 2)$ are forms then $A_1 [\![A_2, \ldots, A_n]\!]$ is a form.
- There are no other kinds of forms or clauses.

Examples 1:

Are forms: $\perp$, 3, $\underline{x}_2$, add$[\![x_1, 5]\!]$, ...

Are clauses: $\{x_5\}$, $\{add[\![x_1, mult]\!]\}$, 3, $\perp$,

$\{div[\![\underline{x}_2, \{sub[\![x_2, x_1]\!]\}]\!]\}$,

$\{x_1[\![\underline{x}_1, \underline{x}_2]\!]\}$, ...

<u>Well formed expressions</u>: (for the extended calculus)

- Every clause is a w.f.e.
- If $A_1, A_2, \ldots, A_n$ $(n \geq 2)$ are w.f.e.'s then $A_1[\![A_2, \ldots, A_n]\!]$ is a w.f.e.
- There are no other kinds of w.f.e.'s.

<u>Remark 1</u>: Every w.f.e. of the strict calculus is a w.f.e. of the extended calculus.

<u>Definition 2</u>:

The <u>reach</u> of a clause $\{A\}$ is defined to be A but excluding all clauses contained in A.

<u>Example 2</u>: The reach of the clause

$$\{A\} = \{div[\![\underline{x}_2, \{sub[\![x_2, \{x_8\}]\!]\}, x_3]\!]\}$$

is indicated by the following figure:

$$\{\underbrace{div[\![\underline{x}_2,}\ \{sub[\![x_2, \{x_8\}]\!]\}\underbrace{, x_3}]\!]\}$$

reach of $\{A\}$.

<u>Definition 3</u>: (Argument Substitution)

Let $\{A\}$ be a clause.

X_A denote the vector $<x_1, x_2, \ldots, x_{m_A}>$ where m_A is the highest among the subscripts of the non-underlined variables occurring within the reach of $\{A\}$.

V denote the vector $<v_1, v_2, \ldots, v_{m_A}>$ where m_A is defined as above and $v_1, v_2, \ldots, v_{m_A}$ are w.f.e.'s (extended).

Then $\{A\} \Big|_{V}^{X_A}$ denotes the clause which is obtained from $\{A\}$ by the following string manipulation:

<u>Step 1</u>: Replace every non-underlined variable x_i $(1 \le i \le m_A)$ occurring within the reach of $\{A\}$ by the w.f.e. v_i .

<u>Step 2</u>: Replace every underlined variable (e.g. $\underline{\underline{x}}_3$) occurring within the reach of $\{A\}$ by a variable having the same subscript but with a degree of underlining one less (e.g. $\underline{\underline{x}}_3$ must be replaced by $\underline{x}_3$).

<u>Examples 3</u>:

$$\{x_1\llbracket \underline{x}_1, \underline{\underline{x}}_2 \rrbracket\} \Big|_{<add>}^{<x_1>} = \{add\llbracket x_1, \underline{x}_2 \rrbracket\}$$

$$\{div\llbracket \underline{x}_2, \{sub\llbracket x_2, \{\underline{\underline{x}}_8\}\rrbracket\}, x_3 \rrbracket\} \Bigg|_{<\bot,\bot,add[3,5]>}^{<x_1,x_2,x_3>}$$

$$= \{div\llbracket x_2, \{sub\llbracket x_2, \{\underline{x}_8\}\rrbracket\}, add\llbracket 3,5 \rrbracket\} \ .$$

Definition 4:

Let $\{A\}$ be a clause containing variables within its reach.

Define degree (A) to be the number of underlines of the most heavily underlined variables among the variables occurring within the reach of the clause $\{A\}$.

Definition 5:

For any w.f.e. A let's denote $C[\![K,A]\!]$ by $\tilde{A}$.

Propositon 1:

If $A, B_1, B_2, \ldots, B_p$ $(p \geq 1)$ are w.f.e.'s then

$$\tilde{A}[B_1,\ldots,B_p] = \overline{A[\![B_1,\ldots,B_p]\!]}$$

Proof: Obvious. □

Remark: We already have

$$\bar{A}[\![B_1,\ldots,B_p]\!] = A \; . \qquad \text{- A6}$$

Proposition 2:

If $B_1, \ldots, B_p$ are w.f.e.'s and $p \geq i$ then:

$$\underbrace{\tilde{\tilde{\cdots\tilde{s}}}}_{i}{}_i[\![B_1,\ldots,B_p]\!] = \underbrace{\overline{\overline{\cdots\overline{B}}}}_{i}{}_i$$

Proof: Obvious. □

<u>Definition 6</u>:

Consider the transformation rules:

(1) $A_1[\![B_1,\ldots,B_p]\!] \rightarrow e[\![A_1,B_1,\ldots,B_p]\!]$

(2) $\text{eval}[\![B_1,\ldots,B_p]\!] \rightarrow e[\![B_1,B_2,\ldots,B_p]\!]$

where

e is a special symbol.

A_1, B_1, ..., B_p are forms with $p \geq 1$ and $A_1 \neq \text{eval}$.

Let A be any form containing no clauses (as subforms). Define $\varepsilon(A)$ as the form obtained by applying recursively and exhaustively the transformation rules (1) and (2) to A.

<u>Example 4</u>:

$\varepsilon(\text{add}[\![x_2,x_3]\!] = e[\![\text{add},x_2,3]\!]$

$\varepsilon(x_2[\![\underline{x}_1,\underline{x}_2]\!]) = e[\![x_2,\underline{x}_1,\underline{x}_2]\!]$

$\varepsilon(\text{eval}[\![\text{add},3,x_1]\!]) = e[\![\text{add},3,x_1]\!]$.

<u>Assignment of meaning to clauses</u>:

To any clause {A} we assign an element of D_∞ by translating {A} into a w.f.e. t({A}) of the strict calculus by the following string manipulation:

<u>Setp 1</u>: Set SOURCE to be the clause {A}.

<u>Step 2</u>: If: SOURCE doesn't contain any clause
then: set t({A}) to be SOURCE and exit.

else: take any innermost clause {B} contained in SOURCE, and assign B to TEMP

if: TEMP contains no variables

then: go to step 2 with a new SOURCE obtained from the old SOURCE by replacing the substring {B} with K⟦TEMP⟧.

else: Perform the following five substeps:

Substep 1:

Reset TEMP to ε(TEMP).

Substep 2:

Reset TEMP to a new value which is obtained from TEMP by replacing every occurrence ω of $\perp$ or basic symbol by

$$\begin{bmatrix} \bar{\omega} \\ \bar{\bar{\omega}} \\ \bar{\bar{\bar{\omega}}} \\ \vdots \end{bmatrix} \quad \text{if degree (TEMP)} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ \vdots \end{bmatrix} .$$

Substep 3:

Reset TEMP to a new value which is obtained from TEMP by replacing every occurrence of "e⟦" by

$$\begin{bmatrix} \text{"C⟦eval"} \\ \text{"C⟦C,}\overline{\text{eval}}\text{"} \\ \text{"C⟦C,}\overline{\text{C}}\text{,}\overline{\overline{\text{eval}}}\text{"} \\ \vdots \end{bmatrix} \quad \text{if degree (TEMP)} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ \vdots \end{bmatrix} .$$

Substep 4:

Reset TEMP to a new value which is obtained from TEMP by replacing every occurrence of

(- if degree (TEMP) = 0:)

$$\text{"}x_i\text{"} \quad \text{by} \quad \text{"}s_i\text{"}$$

$(i = 1,2,\ldots)$

(- if degree (TEMP) = 1:)

$$\begin{bmatrix} \text{"}x_i\text{"} \\ \text{"}\underline{x}_i\text{"} \end{bmatrix} \quad \text{by} \quad \begin{bmatrix} \text{"}\tilde{s}_i\text{"} \\ \text{"}\bar{s}_i\text{"} \end{bmatrix}$$

$(i = 1,2,\ldots)$

(- if degree (TEMP) = 2:)

$$\begin{bmatrix} \text{"}x_i\text{"} \\ \text{"}\underline{x}_i\text{"} \\ \text{"}\underline{\underline{x}}_i\text{"} \end{bmatrix} \quad \text{by} \quad \begin{bmatrix} \text{"}\overset{\approx}{s}_i\text{"} \\ \text{"}\overset{\sim}{\bar{s}}_i\text{"} \\ \text{"}\overset{=}{s}_i\text{"} \end{bmatrix}$$

$(i = 1,2,\ldots)$

(- if degree (TEMP) = 3:)

$$\begin{bmatrix} \text{"}x_i\text{"} \\ \text{"}\underline{x}_i\text{"} \\ \text{"}\underline{\underline{x}}_i\text{"} \\ \text{"}\underline{\underline{\underline{x}}}_i\text{"} \end{bmatrix} \quad \text{by} \quad \begin{bmatrix} \text{"}\overset{\approx}{\tilde{s}}_i\text{"} \\ \text{"}\overset{\approx}{\bar{s}}_i\text{"} \\ \text{"}\overset{\sim}{\overset{=}{s}}_i\text{"} \\ \text{"}\overset{\equiv}{s}_i\text{"} \end{bmatrix}$$

$(i = 1,2,\ldots)$

(- Etc. ...)

Substep 5:

Reset SOURCE to a new value which is obtained from SOURCE by replacing the substring {B} with TEMP, and go to step 2. □

$t(\perp) = \perp$; $t(0) = 0$; $t(1) = 1$; $t(-1) = -1$ ...

Example 5:

$$t(\{if[\![eq[\![\underline{x}_1,1]\!],1,mult[\![\underline{x}_1,x_1[\![sub[\![\underline{x}_1,1]\!]]\!]]\!]]\!]\})$$

$$= C[\![C,\overline{eval},\overline{\overline{if}},C[\![C,\overline{eval},\overline{\overline{eq}},\bar{s}_1,\bar{1}]\!],\bar{1},$$

$$C[\![C,\overline{eval},\overline{\overline{mult}},\bar{s}_1,C[\![C,\overline{eval},\tilde{s}_1,C[\![C,\overline{eval},\overline{\overline{sub}},\bar{s}_1,\bar{1}]\!]]\!]]\!]]\!]$$

(then by T9 and T13 of chapter II:)

$$= C[\![C,\overline{if},C[\![C,\overline{eq},\bar{s}_1,1]\!],1,C[\![C,\overline{mult},\bar{s}_1,C[\![C,s_1,$$

$$C[\![C,\overline{sub},\bar{s}_1,1]\!]]\!]]\!]]\!] \quad .$$ □

Theorem 1:

The above assignment of meaning to clauses is consistent with argument substitution, namely:

Let {A} be any clause, then:

$$t(\{A\})[\![B_1,\ldots,B_p]\!] = t(\{A\} \mid^{X_A}_{V})$$

where V is the vector:

$$\begin{bmatrix} \langle B_1,\ldots,B_{m_A}\rangle \\ \\ \langle B_1,\ldots,B_p,\underbrace{\bot,\ldots,\bot}_{(p-m_A)\text{ times}}\rangle \end{bmatrix} \text{ if } \begin{bmatrix} p \geq m_A \\ \\ p < m_A \end{bmatrix}$$

(for all w.f.e.'s (extended) $B_1,\ldots,B_p$ with $p \geq 1$).

Proof:

The proof is straightforward if we remark that the argument substitution affects only the variables situated within the reach of the clause $\{A\}$. □

Axioms schemas: (for the extended calculus)

We have the same nine axiom schemas as for the strict calculus (naturally A_1, A_2, ..., B_1, B_2, ... now denote arbitrary w.f.e.'s of the extended calculus).

We add one more axiom schema: the one validated by Theorem 1:

10) $$\{A\}[\![B_1,B_2,\ldots,B_p]\!] = \{A\} \Big|_V^{X_A}$$

for all clause $\{A\}$ and all w.f.e.'s B_1, B_2, ..., B_p $(p \geq 1)$ where V is defined as in Theorem 1.

Inference rules: Same as for the strict calculus.

Reduction rule schemas: Same definition as for the strict calculus. Naturally the extended calculus needs not have the Church-Rosser property, but this is an unimportant question.

Example 6:

1) The function f of two variables where

$$f(x,y) = (x*y)/(x-y)$$

can be denoted by the clause:

$$\{div\llbracket mult\llbracket x_1,x_2\rrbracket,sub\llbracket x_1,x_2\rrbracket\rrbracket\}$$

Let's compute $f(5,3)$:

$$\{div\llbracket mult\llbracket x_1,x_2\rrbracket,sub\llbracket x_1,x_2\rrbracket\rrbracket\}\llbracket 5,3\rrbracket$$

$$= \{div\llbracket mult\llbracket 5,3\rrbracket,sub\llbracket 5,3\rrbracket\rrbracket\} \qquad \text{- T1}$$

$$= \{div\llbracket 15,2\rrbracket\} \qquad \text{- A7}$$

$$= \{7\} \qquad \text{- A7}$$

$$= K\llbracket 7\rrbracket \qquad \text{- } t(\{7\})$$

$$= 7.$$

2) The functional τ of the binary function variable F, where:

$$\tau(F)(x,y) = F(x*y,x)$$

can be denoted by the clause:

$$\{x_1\llbracket mult\llbracket \underline{x}_1,\underline{x}_2\rrbracket,\underline{x}_1\rrbracket\}$$

Now $\tau(f)$, for f of the previous example, can be computed as follows:

$$\{x_1\llbracket mult\llbracket \underline{x}_1,\underline{x}_2\rrbracket,\underline{x}_1\rrbracket\}\llbracket\{div\llbracket mult\llbracket x_1,x_2\rrbracket,sub\llbracket x_1,x_2\rrbracket\rrbracket\}\rrbracket$$

$$= \{\{div\llbracket mult\llbracket x_1,x_2\rrbracket,sub\llbracket x_1,x_2\rrbracket\rrbracket\}\llbracket mult\llbracket x_1,x_2\rrbracket,x_1\rrbracket\} \cdot$$

- T1

□

3) Let τ denote the clause:

$$\{if\llbracket eq\llbracket \underline{x}_1,1\rrbracket,1,mult\llbracket \underline{x}_1,x_1\llbracket sub\llbracket \underline{x}_1,1\rrbracket\rrbracket\rrbracket\rrbracket\}$$

Then the Algol-like program of example 3 (chapter I) can be denoted by the w.f.e.

$\{Y[\![x_1]\!][\![2]\!]\}[\![\tau]\!]$

Let's "run" this program:

$\{Y[\![x_1]\!][\![2]\!]\}[\![\tau]\!]$

$= \{Y[\![\tau]\!][\![2]\!]\}$ - A10

$= \{\tau[\![Y[\![\tau]\!]]\!][\![2]\!]\}$ - A3

$= \{\{if[\![eq[\![x_1,1]\!],1,mult[\![x_1,Y[\![\tau]\!][\![sub[\![x_1,1]\!]]\!]]\!]]\!]\}[\![2]\!]\}$ - A10

$= \{\{mult[\![2,Y[\![\tau]\!][\![1]\!]]\!]\}\}$ - A10,8,4

$= \{\{mult[\![2,\tau[\![Y[\![\tau]\!]]\!][\![1]\!]]\!]\}\}$ - A3

$= \{\{mult[\![2,\{\{1\}\}]\!]\}\}$ - A10,8,4

$= \{\{mult[\![2,1]\!]\}\}$ - T9(II)

$= \{\{2\}\}$ - A7

$= 2$. - T9(II)

□

This proof is more readable than that of example 3 (chapter I), yet still is completely formal.

V. AN EXPERIMENT WITH MACRO DEFINITIONS

Consider the system of equation

$$\begin{cases} x = f(x,y) \\ y = g(x,y) \end{cases} .$$

We want to find the least solution vector $\langle x_0, y_0 \rangle$ of this system.

Inspired by [BS], let's consider the elements G_1, $Y_{1.2}$ and $Y_{2.2}$ of D_∞ defined by:

$$G_1 = C\llbracket Y, C\llbracket C, \bar{C}, \tilde{s}_2, \bar{\bar{s}}_1, \bar{s}_1 \rrbracket\rrbracket$$

$$Y_{1.2} = C\llbracket Y, C\llbracket C, s_1, \bar{s}_1, G_1 \rrbracket\rrbracket$$

$$Y_{2.2} = C\llbracket eval, G_1, Y_{1.2} \rrbracket .$$

Let $\tau_1, \tau_2 \in D_\infty$, we want to prove:

(1) $$Y_{1.2}\llbracket \tau_1, \tau_2 \rrbracket = \tau_1 \llbracket Y_{1.2}\llbracket \tau_1, \tau_2 \rrbracket, Y_{2.2}\llbracket \tau_1, \tau_2 \rrbracket\rrbracket$$

(2) $$Y_{2.2}\llbracket \tau_1, \tau_2 \rrbracket = \tau_2 \llbracket Y_{1.2}\llbracket \tau_1, \tau_2 \rrbracket, Y_{2.2}\llbracket \tau_1, \tau_2 \rrbracket\rrbracket .$$

Proof:

We have: $G_1\llbracket \tau_1, \tau_2 \rrbracket = Y\llbracket C\llbracket C, \bar{\tau}_2, \bar{s}_1, s_1 \rrbracket\rrbracket$.

and: $Y_{1.2}\llbracket \tau_1, \tau_2 \rrbracket = Y\llbracket C\llbracket \tau_1, s_1, G_1\llbracket \tau_1, \tau_2 \rrbracket\rrbracket\rrbracket$.

Let:

$$TEMP = C\llbracket \tau_1, s_1, G\llbracket \tau_1, \tau_2 \rrbracket\rrbracket .$$

We have:

$$Y_{1.2}[\![\tau_1,\tau_2]\!] = TEMP[\![Y[\![TEMP]\!]]\!]$$

$$= \tau_1[\![Y[\![TEMP]\!],G_1[\![\tau_1,\tau_2]\!][\![Y[\![TEMP]\!]]\!]]\!]$$

$$= \tau_1[\![Y_{1.2}[\![\tau_1,\tau_2]\!],G_1[\![\tau_1,\tau_2]\!][\![Y_{1.2}[\![\tau_1,\tau_2]\!]]\!]]\!] .$$

On the other hand we have:

$$Y_{2.2}[\![\tau_1,\tau_2]\!] = eval[\![G_1[\![\tau_1,\tau_2]\!],Y_{1.2}[\![\tau_1,\tau_2]\!]]\!]$$

$$= G_1[\![\tau_1,\tau_2]\!][\![Y_{1.2}[\![\tau_1,\tau_2]\!]]\!].$$

Hence:

$$Y_{1.2}[\![\tau_1,\tau_2]\!] = \tau_1[\![Y_{1.2}[\![\tau_1,\tau_2]\!],Y_{2.2}[\![\tau_1,\tau_2]\!]]\!] .$$

This proves (1).

Let: $M = C[\![C,\bar{\tau}_2,\bar{s}_1,s_1]\!]$.

Let's compute farther $Y_{2.2}[\![\tau_1,\tau_2]\!]$:

$$Y_{2.2}[\![\tau_1,\tau_2]\!] = M[\![Y[\![M]\!]]\!][\![Y_{1.2}[\![\tau_1,\tau_2]\!]]\!]$$

$$= C[\![\tau_2,s_1,Y[\![M]\!]]\!][\![Y_{1.2}[\![\tau_1,\tau_2]\!]]\!]$$

$$= \tau_2[\![Y_{1.2}[\![\tau_1,\tau_2]\!],Y[\![M]\!][\![Y_{1.2}[\![\tau_1,\tau_2]\!]]\!]]\!]$$

$$= \tau_2[\![Y_{1.2}[\![\tau_1,\tau_2]\!],Y_{2.2}[\![\tau_1,\tau_2]\!]]\!] .$$

This proves (2). □

Consider the Algol-like program:

```
        begin
             F(x) <= if x = 1 then 1 else x * G(x - 1);
             G(x) <= if x = 1 then 1 else x * F(x - 1);
(**)         F(2)
        end.
```

We can now denote this program by the w.f.e. (extended):

$$\{Y_{1.2}[\![x_1,x_2]\!][\![2]\!]\}[\![\tau_1,\tau_2]\!]$$

where

$$\tau_1 = \{\mathrm{if}[\![\mathrm{eq}[\![\underline{x}_1,1]\!],1,\mathrm{mult}[\![\underline{x}_1,x_2[\![\mathrm{sub}[\![\underline{x}_1,1]\!]]\!]]\!]]\!]\}$$

$$\tau_2 = \{\mathrm{if}[\![\mathrm{eq}[\![\underline{x}_1,1]\!],1,\mathrm{mult}[\![\underline{x}_1,x_1[\![\mathrm{sub}[\![\underline{x}_1,1]\!]]\!]]\!]]\!]\} \quad .$$

Let's "run" this program:

$$\{Y_{1.2}[\![x_1,x_2]\!][\![2]\!]\}[\![\tau_1,\tau_2]\!]$$

$$= \{Y_{1.2}[\![\tau_1,\tau_2]\!][\![2]\!]\} \qquad \text{- A10}$$

$$= \{\tau_1[\![Y_{1.2}[\![\tau_1,\tau_2]\!],Y_{2.2}[\![\tau_1,\tau_2]\!]]\!][\![2]\!]\} \qquad \text{- (1)}$$

$$= \{\{\mathrm{if}[\![\mathrm{eq}[\![x_1,1]\!],1,\mathrm{mult}[\![x_1,Y_{2.2}[\![\tau_1,\tau_2]\!][\![\mathrm{sub}[\![x_1,1]\!]]\!]]\!]]\!]\}[\![2]\!]\} \qquad \text{- A10}$$

$$= \{\{\mathrm{mult}[\![2,Y_{2.2}[\![\tau_1,\tau_2]\!][\![1]\!]]\!]\}\} \qquad \text{- A7,A8}$$

$$= \{\{\mathrm{mult}[\![2,\tau_2[\![Y_{1.2}[\![\tau_1,\tau_2]\!],Y_{2.2}[\![\tau_1,\tau_2]\!]]\!][\![1]\!]]\!]\}\} \qquad \text{- (2)}$$

$$= \{\{\mathrm{mult}[\![2,1]\!]\}\} \qquad \text{- Similar}$$

$$= 2 \ . \qquad \text{- Similar}$$

□

The reasoning is very compact throughout. Naturally we can extend this idea to a system of n recursive definitions of functions for an arbitrary n.

If we had $F(G(3) + 5)$, for example, instead of $F(2)$ at (**) then the corresponding program would be denoted by the w.f.e.

$$\{Y_{1.2}[\![x_1, x_2]\!][\![add[\![Y_{2.2}[\![x_1, x_2]\!][\![3]\!], 5]\!]]\!]\}[\![\tau_1, \tau_2]\!] \quad .$$

This w.f.e. is shorter than corresponding expressions in [BS].

VI. CONCLUSION

The construction of the calculus is elaborate but the end result is simple, as hopefully shown by the various examples. Our calculus consists in fact of two parts.

Part one is a formal deductive system whose w.f.e.'s can be viewed as programs. This deductive system can be neatly merged with the classical formal first order logic, elementary number theory, second order logic or other theories of functions. For instance, the Scott induction principle can be neatly integrated into the formalism. Problems like proving equivalence of $\int_0^\infty x^{n-1}e^{-x}dx$ and the programmer's recursive definition of factorial function can thus be conveniently dealt with. Naturally all alledged proofs are machine checkable (this is a necessity to win the programmer's belief).

Part two (reduction rules) gives some insights to the process of digital manipulation as happens inside the hardware of real computers. For example, overflows may be mathematically explained by the fact that real computers usually don't have a "well furnished" set of reduction rules (e.g. the size of integer number is limited).

Naturally, there is no reason to include only $\mathbb{B}$ and $\mathbb{Z}$ into the set D_0 (from which we construct D_∞). We could take additional primitive sets like: real numbers, complex numbers, matrix, strings of symbols, lists, trees, ..., and take additional primitive operations like: matrix operations, symbolic integration, cdr, car, tree substitution, ... The axioms and rules can be enriched accordingly.

Another possible use of our calculus is to serve as a basis for a development of program optimization techniques. In fact this development could not go far without preestablishing a theory of meanings for programs.

Every result of this report is believed by the author to be original, except when otherwise mentioned. A more complete report is in preparation.

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