On the Cuthill-McKee Algorithm for Ordering Sparse Symmetric Matrices*

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Research Report CS-73-26
October 1973

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* Work supported in part by a University of Waterloo Graduate Bursary

ABSTRACT

In this paper we examine the popular Cuthill-McKee algorithm for ordering sparse positive definite matrices. For a given matrix A, this algorithm is designed to produce a permutation matrix P such that $A_c = PAP^T$ has a small bandwidth. If we exploit zeros to the left of the first nonzero in each row of A, it has been observed that reversing the ordering produced by the Cuthill-McKee algorithm is often very much better than the original ordering. We prove that in general this reversed ordering is always at least as good as the original one in terms of storage and operation count. Some numerical experiments are included.

1. Introduction

Consider the N by N sparse linear algebraic system

$$Ax = b \tag{1.1}$$

where A is symmetric and positive definite. To solve (1.1), we may factor A into the product $\tilde{L}\tilde{D}\tilde{L}^T$, where $\tilde{L}$ is unit lower triangular and $\tilde{D}$ is a positive diagonal matrix, and we then solve $\tilde{L}y = b$, $\tilde{D}z = y$, and $\tilde{L}^T x = z$.

Let P be any permutation matrix. The permuted linear system

$$PAP^T(Px) = Pb \tag{1.2}$$

remains sparse, symmetric and positive definite. It is well known that PAP^T possesses a triangular factorization LDL^T and the factorization is numerically stable [8]. Thus, we might solve (1.2) rather than (1.1). In this case, the relevant factors are L and D. It is well known that an appropriate choice of P can result in a significant reduction on the amount of storage and arithmetic operations required for its direct solution. This remark assumes that some form of compact and efficient storage scheme is used so that some of the zeros in A and L can be exploited.

A popular ordering strategy is the Cuthill-McKee algorithm [2]. Its primary aim is to produce a permutation matrix P for which

$$A_C = PAP^T$$

has a small bandwidth. Following [2], we define the bandwidth of a symmetric matrix M by

$$m = \max\{|i-j| : M_{ij} \neq 0\} \tag{1.3}$$

Although orderings which minimize the bandwidth are often far from optimal in the least-computation and/or least-fill sense, they lend themselves well to convenient and efficient data management.

In this paper, we present a detailed analysis of this widely used Cuthill-McKee algorithm (CM) and its reversed scheme (RCM). We give, in Section 3, the exact storage requirement and arithmetic operations required to factor A_C in terms of its profile structure. If we exploit zeros to the left of the first nonzero in each row of the coefficient matrix, it has been observed that reversing the ordering produced by the CM algorithm is, in certain applications, very much better than the original ordering. See George [3] and Cuthill [1]. In Section 4, we prove that in general this reversed ordering is always at least as good as the original one. For certain classes of problems arising in the application of the finite element method, we present bounds for the improvement. Some numerical experiments are included. We discuss aspects of implementation in Section 5. Section 6 contains our concluding remarks.

2. Some definitions

In this section, we introduce some definitions, notations and simple results which are useful in subsequent sections.

Let A be a given matrix. Let

$$f_i(A) = \min\{j | A_{ij} \neq 0\} , \quad i = 1,2,\ldots,N, \tag{2.1}$$

that is, the column subscript of the first nonzero component in the i-th row of A. We then define the envelope[†] of A, where A is symmetric or lower triangular, by

$$\mathrm{Env}(A) = \{(i,j) | f_i(A) \leq j \leq i\} \tag{2.2}$$

and also the quantities

$$\beta_i(A) = i - f_i(A), \quad i = 1,\ldots,N. \tag{2.3}$$

It is useful to introduce the quantities

$$\mu_i(A) = |\{k | k > i \text{ and } A_{k\ell} \neq 0 \text{ for some } \ell \leq i\}, \quad i = 1,\ldots,N. \tag{2.4}$$

The number $\mu_i(A)$ is simply the number of "active" rows at the i-th step of factorization; that is, the number of rows in the envelope of A, which intersect column i. The following observation is due to George [4].

[†] It is slightly different from the definition given in [5], where the envelope for a general unsymmetric matrix is defined to be $\mathrm{Env}(A) \cup \mathrm{Env}(A^T)$.

Lemma 2.1

If the symmetric factorization of matrix A requires $\theta(A)$ multiplicative operations, then

$$\theta(A) \leq \frac{1}{2} \sum_{i=1}^{N-1} \mu_i(A)[\mu_i(A)+3].$$

Furthermore, it is an equality if A is assumed to have a full envelope.

It is easy to express the size of the envelope in terms of β_i and μ_i.† In fact, we have [1]:

$$|\mathrm{Env}(A)| = \sum_{i=1}^{N} \beta_i + N = \sum_{i=1}^{N} \mu_i + N . \qquad (2.5)$$

We remark here that the bandwidth defined in (1.3) is exactly the quantity $\max_i \beta_i$, while the number $\max_i \mu_i$ is usually referred to as the wavefront or frontwidth of matrix A [4].

Assuming that A has the symmetric factorization LDL^T, it is natural to define

$$\mathrm{Fild}(A) = \{(i,j) | A_{ij} = 0, \ L_{ij} \neq 0\}. \qquad (2.6)$$

Since $\mathrm{Fild}(A) \subset \mathrm{Env}(A)$ (see [5]), the main storage requirement to keep all the nonzero entries of L is always less than or equal to $|\mathrm{Env}(A)|$. In general, zeros within the envelope may also be exploited.

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† When the matrix A under consideration is clear from the context, we denote by β_i and μ_i the quantities $\beta_i(A)$ and $\mu_i(A)$ respectively.

To facilitate the discussion of ordering algorithms, we define the symmetric graph $G(A) = (X(A), E(A))$ associated with the symmetric matrix A. Here $X(A) = \{x_1,\ldots,x_N\}$ is the set of vertices, labelled as implied by A; and $E(A)$ is the set of edges where $\{x_i,x_j\} \in E(A)$ if and only if $A_{ij} \neq 0$, $i > j$. For any permutation matrix P, the graphs $G(A)$ and $G(PAP^T)$ are structurally identical, but the node labels in $G(PAP^T)$ have been permuted according to P.

3. The Cuthill-McKee algorithm

The algorithm proposed by Cuthill and McKee [2] involves a direct and fast method most easily described in terms of labelling the graph structure associated with the matrix A, (that is, the unlabelled graph of G(A)).

Step 0 Choose a starting node and relabel it 1.

Step 1 Let Q = {nodes which have been numbered but are connected to unnumbered nodes}

$$= \{x_{q_1}, x_{q_2}, \ldots, x_{q_{|Q|}}\}$$

where node x_{q_i} is numbered q_i in the CM ordering and $q_1 < q_2 < \ldots < q_{|Q|}$. Number the neighbours of x_{q_1} in order of increasing degree, counting only those connections with unnumbered vertices, then followed by the neighbours of x_{q_2} and so on.

Stop whenever all nodes have been numbered.

Step 2 Go to Step 1.

As remarked in [2], the Cuthill-McKee numbering scheme corresponds to the generation of a spanning tree† of the graph G(A) in a level-by-level fashion. In the case when G(A) has more than one connected component, the process can be continued by selecting a new and unlabelled node to start a new component. Henceforth, we assume that G(A) is connected, or equivalently

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† A tree is a connected graph with N nodes and N-1 edges. A spanning tree of the graph G is a subgraph of G, which is a tree and contains all N nodes.

that the matrix A is irreducible. However, the results hold in the general case.

We begin by showing that A_c satisfies the following monotone profile property which is essential in subsequent discussions:

Lemma 3.1

Let A_c be the matrix ordered by the Cuthill-McKee algorithm. If $j \leq k$, then $f_j \leq f_k$.

Proof It follows from the property of the Cuthill-McKee algorithm that if $f_k < f_j$, then the node with label k should be numbered before the one with j.

Consider the storage requirement in factoring A_C. The following theorem is a restatement, in our notation, of results in [5].

Theorem 3.2

Let $A = LDL^T$ with $f_i(A) < i$. Then Env(L) is full.

It is straightforward to see that A_C satisfies the hypothesis of theorem 3.2, for if there is a row in A_C with $f_i = i$, the monotone profile property of A_C would imply that A_C is reducible, a contradiction to our assumption. The following corollaries are immediate.

Corollary 3.3

Let A_C be factored into LDL^T. The number of nonzero entries in L is exactly $|Env(A_C)|$.

Corollary 3.4

Let A_C be factored into LDL^T. The forward and backward solving can be done in $2\cdot|Env(A_C)| - N$ arithmetic operations.

We now turn to the problem of estimating the amount of arithmetic required for the symmetric factorization of the profile-oriented matrix A_C. The process can be defined by the following equations, for i = 1,...,N,

$$d_{ii} = a_{ii} - \sum_{k=f_i}^{i-1} a'_{ik}\ell_{ik} \tag{3.1a}$$

$$a'_{ij} = a_{ij} - \sum_{k=\max\{f_i,f_j\}}^{j-1} a'_{ik}\ell_{jk}, \quad j = f_i,\ldots,i-1 \tag{3.1b}$$

$$\ell_{ij} = a'_{ij}/d_{jj}, \qquad j = f_i,\ldots,i-1 \tag{3.1c}$$

It should be clear that (3.1) is simply a variant form of the conventional defining equations for the Cholesky decomposition of symmetric matrices (Martin, Peters, and Wilkinson [7]). Here in (3.1), zeros to the left of the first nonzero in each row are exploited. In view of the monotone profile property of A_C, (3.1b) can be rewritten in a more simplified form as:

$$a'_{ij} = a_{ij} - \sum_{k=f_i}^{j-1} a'_{ik}\ell_{jk}, \quad j = f_i,\ldots,i-1. \tag{3.1b}'$$

Recalling that $\beta_i = i-f_i$, we have the following theorem.

Theorem 3.5

The symmetric factorization of the matrix A_C requires $\theta(A_C) = \frac{1}{2}\sum_{i=2}^{N} \beta_i(\beta_i+3)$ arithmetic operations.

Proof From (3.1a) and theorem 3.2, we observe that a'_{ij} are all nonzero at positions within the envelope of A_C. Thus the products $a'_{ik}\ell_{ik}$ and $a'_{ik}\ell_{jk}$ appeared in (3.1a) and (3.1b)' respectively involve nonzero operands. In performing the i-th step defined by (3.1a), (3.1b)' and (3.1a), it is clear that:

a) to compute d_{ii}, we need $i-f_i$ multiplicative operations;

b) for $j = f_i,\ldots,i-1$, a'_{ij} requires $j-1-f_i$ operations, making a total of
$$\sum_{j=f_i}^{i-1} j-1-f_i = \frac{1}{2}(i-f_i)(i-f_i-3);$$

c) to compute ℓ_{ij} $(j = f_i,\ldots,i-1)$, we use $i-f_i$ operations.

Hence, for the i-th step, the number of arithmetic operations required is
$$\frac{1}{2}(i-f_i)(i-f_i+3) = \frac{1}{2}\beta_i(\beta_i+3).$$
Summing up, we can perform the decomposition process in $\frac{1}{2}\sum_{i=2}^{N}\beta_i(\beta_i+3)$ operations, since $\beta_1 = 0$.

It should be noted that for a general profile-oriented matrix, $\frac{1}{2}\sum_{i=2}^{N}\beta_i(\beta_i+3)$ is an upperbound on the total arithmetic required for its triangular factorization. Often, this turns out to be an over-estimate. Only for monotone profile matrices whose triangular factors have full envelopes is the operation count in theorem 3.5 achieved. As an immediate corollary of this observation, we have the well-known result for band matrices (Wilkinson [8]).

Corollary 3.6

The number of operations required for the symmetric factorization of a band matrix with bandwidth m is bounded by $\frac{1}{2}m(m+3)N$.

Proof Set each $\beta_i = m$.

4. The Reversed Cuthill-McKee Ordering

In his study of profile methods, George [3] discovered the reversed Cuthill-McKee algorithm which renumbers the CM ordering in the reversed way. Surprisingly, this simple modification often turns out to be superior to the original one in several aspects, although the bandwidth remains unchanged. The general superior performance of the reversed algorithm has been reported in the thesis of George [3] and in the survey paper by Cuthill [1]. In this section, we shall show that in general the reversed scheme is always at least as good, as far as storage and operation counts are concerned.

As before, we denote by A_C the matrix ordered by the Cuthill-McKee algorithm. Let A_R be the one obtained by reversing the ordering. It is easy to see that

$$A_R = \tilde{P} A_C \tilde{P}$$

where

$$\tilde{P} = \begin{pmatrix} & & & 1 \\ & & \iddots & \\ & 1 & & \\ 1 & & & \end{pmatrix} .$$

It is helpful to consider the labelled symmetric graphs $G(A_C) = (X(A_C), E(A_C))$ and $G(A_R) = (X(A_R), E(A_R))$ associated with matrices A_C and A_R respectively. Let

$$X(A_C) = \{x_1, x_2, \ldots, x_N\}$$

and

$$X(A_R) = \{y_1, y_2, \ldots, y_N\}$$

where x_i is numbered i in the Cuthill-McKee ordering and y_j is the j-th node in the reversed algorithm. $G(A_C)$ and $G(A_R)$ are identical structurally, and x_i and y_{N-i+1} represent the same node in the underlying unlabelled graph.

Lemma 4.1

If $i < j \leq k$ and $(A_C)_{ki} \neq 0$, then there exists an $r \leq i$ such that $(A_C)_{jr} \neq 0$.

Proof Choose $r = f_j(A_C)$. Then nodes x_j and x_r are connected, and it follows from the monotone envelope property of A_C that

$$r = f_j(A_C) \leq f_k(A_C) \leq i$$

The following result is central to much of the subsequent analysis in this section.

Lemma 4.2

Let A_C and A_R be defined as above. For $j = 1,\ldots,N$,

$$\mu_{N-j+1}(A_R) \leq \beta_j(A_C).$$

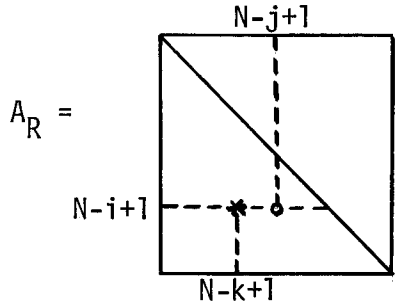
Proof From the definitions in (2.3) and (2.4), it is sufficient to show that if $(A_R)_{N-i+1,N-j+1}$ lies within the envelope of row N-i+1 in A_R where $N-i+1 > N-j+1$ (that is, N-i+1 is an active row during the (N-j+1)-st step of factorization), then the entry $(A_C)_{j,i}$ satisfies the relation $f_j(A_C) \leq i < j$.

For this $(A_R)_{N-i+1,N-j+1}$, we can always find a variable y_{N-k+1} such that

$$N-k+1 \leq N-j+1 < N-i+1$$

and

$$(A_R)_{N-i+1,N-k+1} \neq 0.$$



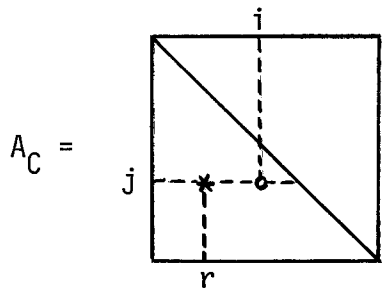
This means that we have a k such that

$$i < j \leq k$$

and

$$(A_C)_{k,i} = (A_R)_{N-i+1,N-k+1} \neq 0$$

By lemma 4.2, there exists a node x_r in $G(A_C)$ with the properties: $r \leq i < j$ and $(A_C)_{j,r} \neq 0$.



Hence the corresponding entry $(A_C)_{j,i}$ lies within the envelope of A_C. The result then follows.

We are now at a position to compare the general performance of the Cuthill-McKee and its reversed algorithms in theoretical terms.

Theorem 4.3

$$|\mathrm{Env}(A_R)| \le |\mathrm{Env}(A_C)|.$$

Proof Use $|\mathrm{Env}(A_R)| = \sum_{i=1}^{N} \mu_i(A_R) + N$

and $|\mathrm{Env}(A_C)| = \sum_{i=1}^{N} \beta_i(A_C) + N$

and apply lemma 4.2.

Theorem 4.4

$$|\mathrm{Fill}(A_R)| \le |\mathrm{Fill}(A_C)|.$$

Proof Let $n(A)$ be the number of nonzeros in the lower triangle of the original matrix A. The symmetric factorization of A_C yields a full envelope, so that $|\mathrm{Fill}(A_C)| = |\mathrm{Env}(A_C)| - n(A)$. On the other hand, we have $|\mathrm{Fill}(A_R)| \le |\mathrm{Env}(A_R)| - n(A)$. Together with theorem 4.3, we obtain the desired result.

Theorem 4.5

$$\theta(A_R) \le \theta(A_C).$$

Proof On combining lemma 2.1, lemma 4.2, and theorem 3.5, we have

$$\theta(A_R) \le \frac{1}{2}\sum_{k=1}^{N-1} \mu_k(A_R)[\mu_k(A_R)+3]$$

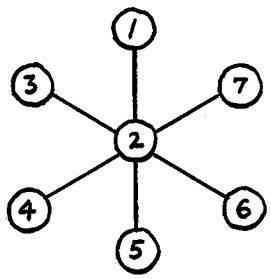
$$\le \frac{1}{2}\sum_{i=2}^{N} \beta_i(A_C)[\beta_i(A_C)+3]$$

$$= \theta(A_C).$$

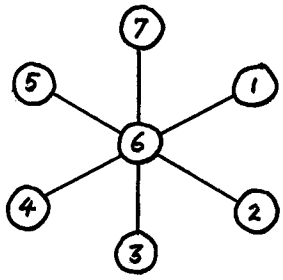
Here the transformation $k = N-i+1$ is used.

Example 1

To illustrate the results in this section, we choose the unlabelled star graph with N nodes. The best possible CM ordering is to start with one of the "planet" vertices. Both orderings for N = 7 are shown in Figure 5.1 below.



CM ordering



RCM ordering

$$A_C = \begin{pmatrix} x & x & & & & & \\ x & x & x & x & x & x & x \\ & x & x & & & & \\ & x & & x & & & \\ & x & & & x & & \\ & x & & & & x & \\ & x & & & & & x \end{pmatrix} \qquad A_R = \begin{pmatrix} x & & & & & & x \\ & x & & & & & x \\ & & x & & & & x \\ & & & x & & & x \\ & & & & x & x & \\ x & x & x & x & x & x & x \\ & & & & & x & x \end{pmatrix}$$

Figure 4.1 Orderings for star graph with 7 nodes

It is clear that for the star graph with N nodes,

$$|\mathrm{Env}(A_C)| - |\mathrm{Env}(A_R)| = |\mathrm{Fill}(A_C)| - |\mathrm{Fill}(A_R)| = \sum_{i=1}^{N-3} i = \frac{1}{2}(N-3)(N-2).$$

and $$\theta(A_C) - \theta(A_R) = \frac{1}{6} N^3 + O(N^2).$$

This example shows that the RCM algorithm can be significantly better than the CM strategy.

Example 2

We now analyse some examples that arise in the application of finite element method. Consider the mesh obtained by subdividing a unit square into n^2 small square elements of side $\frac{1}{n}$, and the finite element system† associated with it. We number the nodes starting at the lower left hand corner using the CM algorithm. The case with n = 4 is given below.

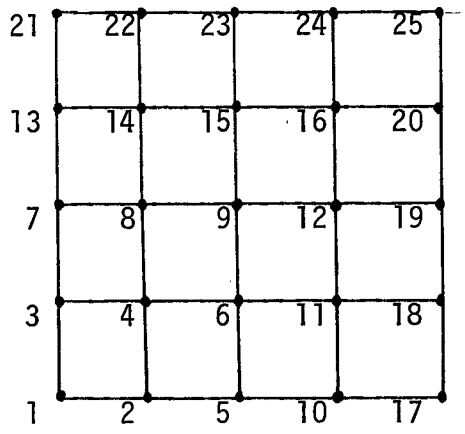


Figure 4.2 4 by 4 regular mesh ordered by CM algorithm

It is not difficult to see that for $n \geq 2$,

(i) $$|Env(A_C)| = 1 + \sum_{k=1}^{n} (4k^2+5k)$$
$$= \frac{2}{3}n(n+1)(2n+1) + \frac{5}{2}n(n+1)+1$$

(ii) $$|Env(A_C)| - |Env(A_R)| = 2n(n-1)$$

† A finite element system or mesh system of equations associated with a mesh M is any N by N symmetric positive definite system Ax = b, with the property that entry A_{ij} is nonzero only if unknowns x_i and x_j are associated with nodes of the same mesh element of M.

(iii) $$\theta(A_C) = \sum_{k=1}^{n} (4k^3+10k^2+3k+1)$$
$$= n^2(n+1)^2 + \frac{5}{3}n(n+1)(2n+1) + \frac{1}{2}n(3n+1)$$

(iv) $$\theta(A_C)-\theta(A_R) \geq \frac{8}{3}n^3 + 3n^2 - \frac{17}{3}n.$$

In table 4.1 we have tabulated the amount saved for different values of n.

Problem		Storage				Operation Count			
n	$N=(n+1)^2$	CM	RCM	Saving	% saved	CM	RCM	Saving	% saved
2	9	36	32	4	11.11	93	71	22	23.66
4	25	171	147	24	14.04	726	530	196	27.00
8	81	997	885	112	11.23	7324	5812	1512	20.65
16	269	6665	6185	480	7.20	89336	77736	11600	12.99
32	1089	48401	46417	1984	4.10	1231088	1140816	90272	7.33

Table 4.1 Theoretical amount saved for the regular square mesh

Example 3

The savings in example 2 are not particularly impressive. We now consider the same model problem with triangular elements (that is, each small square in figure 4.2 is subdivided into two right triangles), and its corresponding mesh system.

Quadratic interpolation is first used, where each triangular element has three vertex unknowns and three edge unknowns. The corresponding CM ordering for n = 2 is given in figure 4.3.

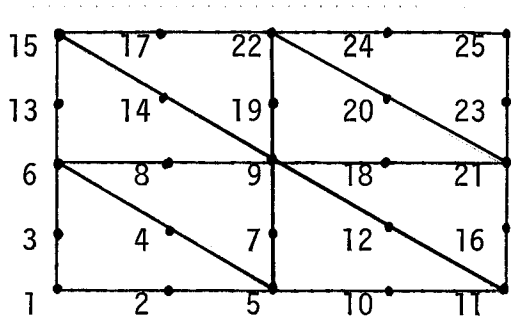


Figure 4.3 2 by 2 regular right triangular mesh with element 2-6† ordered by CM algorithm

Experients using different values of n are tried. The result tabulated in table 4.2 shows that substantial savings in computation and storage can be achieved if RCM rather than CM algorithm is used.

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† We adopt the notation in [3], where the two-part hypenated name refers respectively to the degree of the polynomial and the number of nodes associated with the element.

Problem			Storage				Operation Count			
n	$N=(2n+1)^2$	Nonzeros	CM	RCM	Saving	% saved	CM	RCM	Saving	% saved
2	25	96	176	153	23	13.07	820	589	231	28.17
3	49	207	498	380	118	23.70	3168	1782	1386	43.75
4	81	360	1024	735	289	28.22	7824	3940	3884	49.64
5	121	555	1918	1295	623	32.48	18316	8126	10190	55.63
6	169	792	3102	2019	1083	34.91	34096	14042	20054	58.82
7	225	1071	4822	3025	1797	37.27	60764	23497	37267	61.33
8	289	1392	6922	4241	2681	38.73	97426	35808	61618	63.25
9	361	1755	9744	5837	3907	40.10	153332	54124	99208	64.70

Table 4.2 Experimental savings on the regular right triangular mesh with element 2-6.

In case of cubic interpolation with each element having ten nodal unknowns, the savings are even more dramatic. The mesh with n = 2 would be numbered by the CM ordering as shown below.

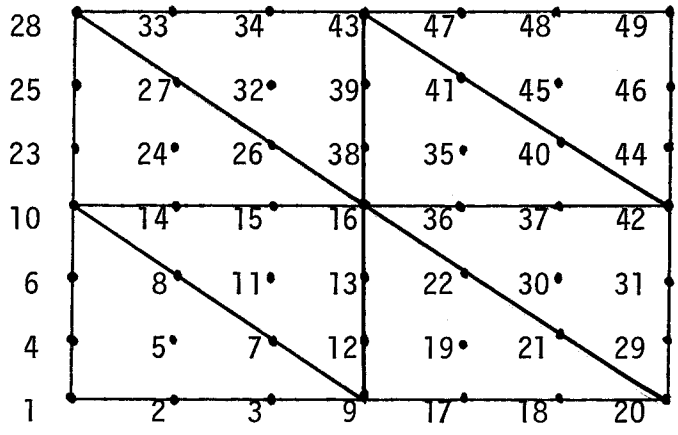

Figure 4.4 2 by 2 regular right triangular mesh with element 3-10 ordered by CM algorithm

As before, we tabulate the storage and operation requirements for the two algorithms using different n.

Problem			Storage				Operation Count			
n	$N=(3n+1)^2$	Nonzeros	CM	RCM	Saving	% saved	CM	RCM	Saving	% saved
2	49	312	628	490	138	21.98	5491	3136	2355	42.89
3	100	684	1978	1252	726	36.70	24564	9429	15135	61.62
4	169	1200	4516	2518	1998	44.24	73595	22046	51549	70.04
5	256	1860	8566	4396	4107	47.95	170809	43624	127185	74.46
6	361	2664	14452	6994	7458	51.61	340101	77574	262527	77.19

Table 4.3 Experimental savings on the regular right triangular mesh with element 3-10.

5. Implementation

In [3], George has advocated the use of profile methods instead of band methods. In fact, it is possible to save about one-third of the storage and to half the computation required to perform the factorization in many cases of practical interest.

Corollary 3.3 strongly suggests the use of Jennings' profile storage scheme [6] in connection with the CM ordering (see [5]). The scheme stores the rows of the envelope of the lower triangle of coefficient matrix A in a linear array. An additional N auxiliary address pointers are needed to locate the positions of the diagonal elements in the main storage array. Thus, the data structure of the scheme requires the computation of these N extra pointers. We point out, however, they can be obtained as an immediate byproduct of the CM ordering process.

This can be best explained in terms of the spanning tree associated with the CM algorithm. To set up the address pointers, it is sufficient to know the number of locations required for each row in the envelope. Note that the storage for row i is exactly $i-f_i+1$. It is easy to see that

$$f_1(A_C) = 1$$

and for $i \neq 1$

$f_i(A_C) = j$, where x_j is the direct ancestor of node x_i in the spanning tree associated with the CM order. We use example 2 with n = 2 to illustrate this observation.

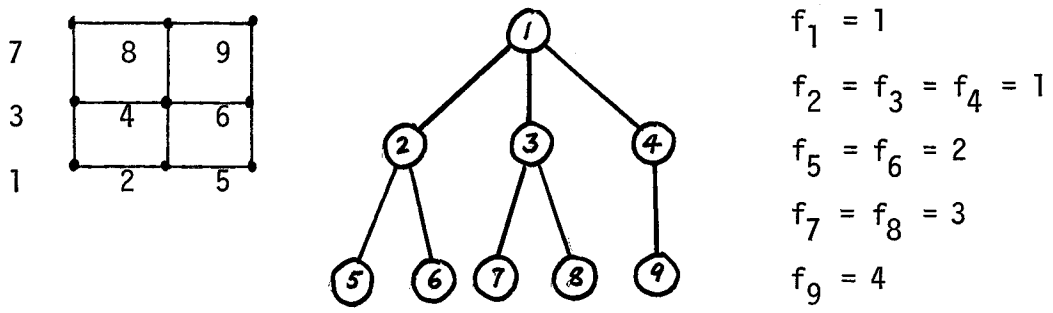

Figure 5.1 Spanning tree associated with the CM ordering for the 2 by 2 regular square mesh.

Hence, the address pointers can be readily obtained during the CM ordering procedure.

To implement the RCM algorithm, it is also attractive to use Jennings' scheme in view of theorem 4.3. Extra work is needed to set up our data structure for the coefficient matrix. If we define

$$\ell_i(A) = \max\{j | A_{ij} \neq 0\} , \qquad i = 1,\ldots,N$$

that is, the column subscript of the last nonzero component in the i-th row of A, then

$$\mathbf{f}_j(A_R) = N - \ell_{N-j+1}(A_C)+1, \quad j = 1,\ldots,N.$$

We can establish the address pointers for the main storage of A_R without much difficulty by keeping track of the quantities $\ell_i(A_C)$.

One final remark is that we should use equations (3.1a), (3.1b) (instead of (3.1b)') and (3.1c) for the symmetric factorization of the matrix A_R. In that case, the number of operations for the decomposition is exactly given by $\frac{1}{2}\sum_{k=1}^{N-1} \mu_k(A_R)[\mu_k(A_R)+3]$.

6. Conclusion

Evidently, in the context of profile or envelope methods, the RCM algorithm compares favourably with the original CM ordering. Thus far we have not brought into the discussion of the execution time in performing the two ordering algorithms. But, there is little doubt that the two strategies under consideration require approximately the same amount of work. In exchange for the extra but comparatively insignificant effort to reverse the ordering and to set up the address pointers for the data management, there is the possibility of substantial reduction in cost of storage and factorization. Moreover, we are guaranteed that the RCM ordering is at least as good, provided we exploit all the zeros to the left of the first nonzero in each row of the matrix.

The savings obtained in example 3 are dramatic. This demonstrates that in practical applications it is possible to reduce over half of the storage requirement and to save over three-quarters of the operations necessary for the symmetric decomposition. Our study provides strong arguments for using the RCM algorithm. Since the CM algorithm is a popular ordering scheme (particularly in structural analysis applications), our investigation in this paper is of practical interest.

Acknowledgement

The author would like to thank Professor Alan George for his many helpful discussions and criticisms.

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