The Number of Triangles Formed by Intersecting Diagonals of a Regular Polygon

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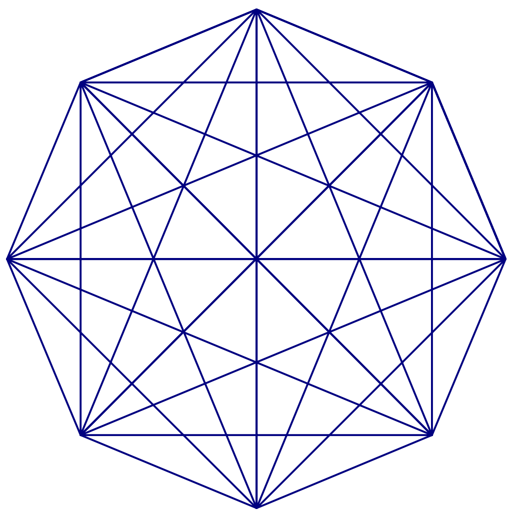
Abstract

We consider the number of triangles formed by the intersecting diagonals of a regular polygon. Basic geometry provides a slight overcount, which is corrected by applying a result of Poonen and Rubinstein [Poonen98]. The number of triangles is 1, 8, 35, 110, 287, 632, 1302, 2400, 4257, 6956 for polygons with 3 through 12 sides.

Introduction

If we connect all corners of a regular polygon with N equal length sides, we have a figure with $\binom{N}{2}$ or $N(N-1)/2$ lines. For N=8, the figure is:

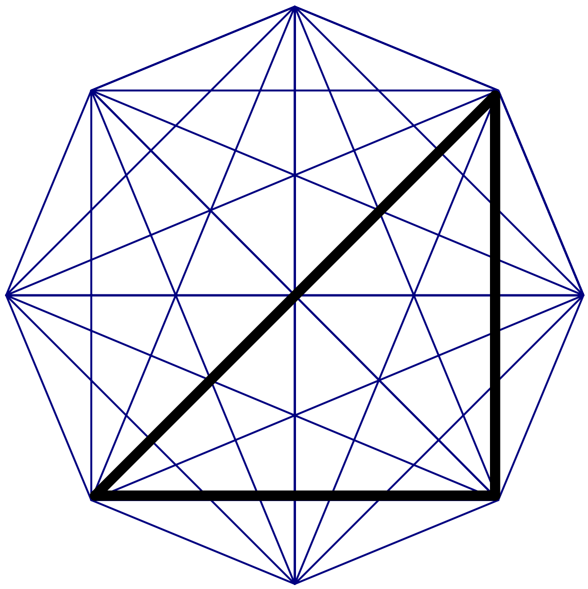


Careful counting shows that there are 632 triangles in this eight sided figure.

Derivation

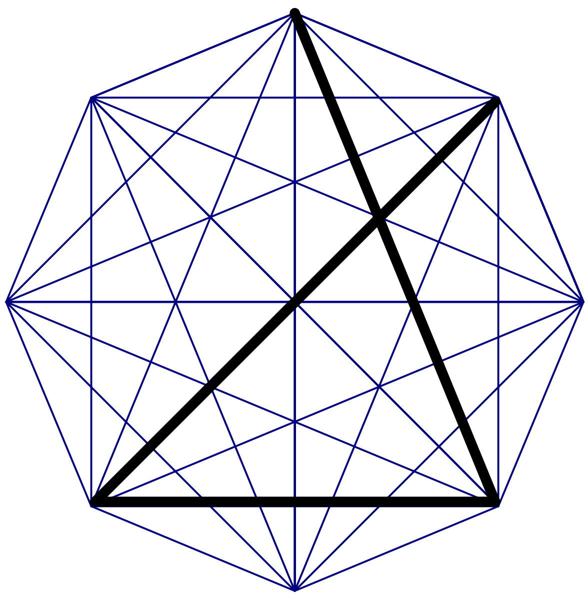
All triangles are formed by the intersection of three diagonals at three different points. There are five arrangements of three diagonals to consider. We classify them based on the number of distinct diagonal endpoints. We will directly count the number of triangles with 3, 4 and 5 endpoints (top three figures). We will count the number of *potential* triangles with 6 endpoints, then correct for the false triangles. In each of the following five figures, a sample triangle is highlighted.

Three, Four and Five Diagonal Endpoints



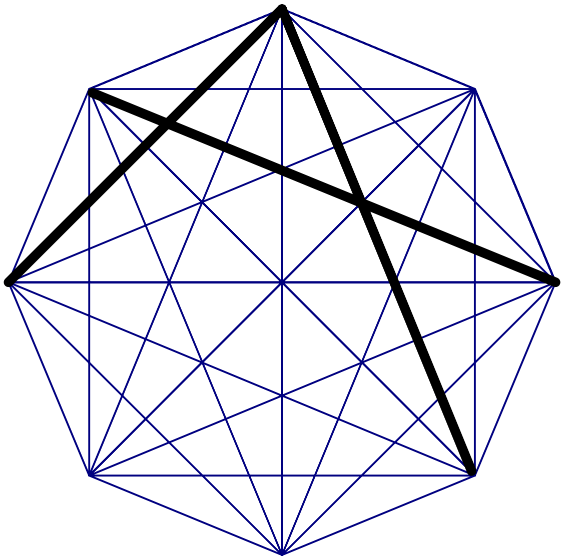
3 diagonal endpoints. There are 56 such triangles in the figure at left.

The number of triangles formed by diagonals with a total of three endpoints is simply $\binom{N}{3}$.



4 diagonal endpoints. There are 280 such triangles in the figure at left.

There are $\binom{N}{4}$ combinations of the four diagonal endpoints. For each set of four endpoints, there are four triangle configurations. Thus there are $4\binom{N}{4}$ triangles formed.

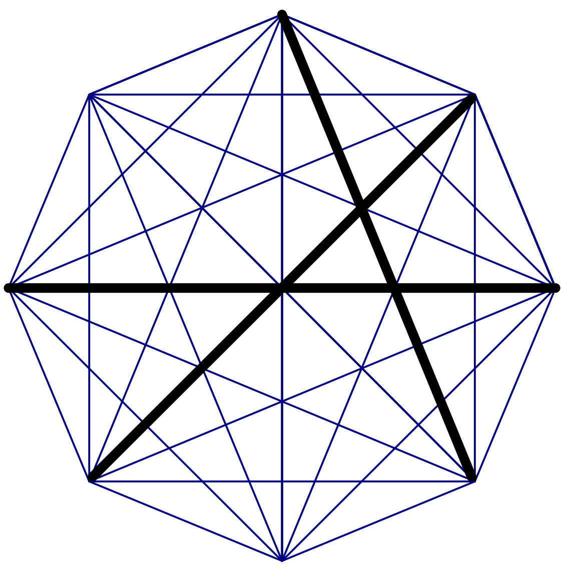


5 diagonal endpoints There are 280 such triangles in the figure at left.

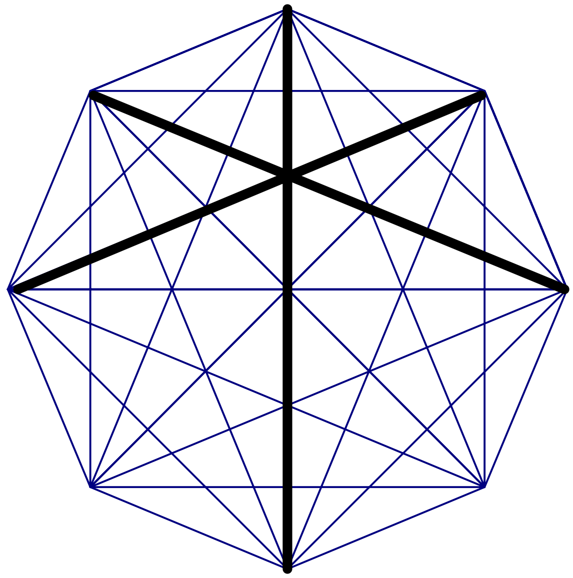
For each of the N corners of the polygon, there are four other diagonal endpoints which can be placed on the N-1 remaining locations. Thus there are $N\binom{N-1}{4}$ triangles formed. This is equal to $5\binom{N}{5}$.

Six diagonal endpoints

The number of *potential* triangles formed by 6 line segments is $\binom{N}{6}$, since there are 6 segment endpoints to be chosen from a pool of N. Often potential triangles are not created by three overlapping line segments because the line segments intersect at a single point. $\binom{N}{6}$ counts both of the following two situations.



6 diagonal endpoints, resulting in triangle. There are 16 such triangles in the figure at left.



6 diagonal endpoints, false triangle. There are 9 interior intersection points in the figure at left where such false triangles can be formed.

We use a result of [Poonen98] to count these false triangles. Using the notation of that paper, let $a_m(N)$ be the number of interior points other than the center for a *N* sided regular polygon's diagonals where there are *m* diagonals intersecting. Surprisingly, there can be only 2, 3, 4, 5, 6, or 7 intersecting diagonals in the polygon interior, other than at the center. The necessary intermediate result from [Poonen98] is reproduced here.

$$a_3(N)/N = (5N^2 - 48N + 76)/48 \bullet \delta_2(N) + 3/4 \bullet \delta_4(N) + (7N - 38)/6 \bullet \delta_6(N)$$
$$- 8 \bullet \delta_{12}(N) - 20 \bullet \delta_{18}(N) - 16 \bullet \delta_{24}(N) - 19 \bullet \delta_{30}(N) + 8 \bullet \delta_{42}(N)$$
$$+ 68 \bullet \delta_{60}(N) + 60 \bullet \delta_{84}(N) + 48 \bullet \delta_{90}(N) + 60 \bullet \delta_{120}(N) + 48 \bullet \delta_{210}(N)$$

$$a_4(N)/N = (7N - 42)/12 \bullet \delta_6(N) - 5/2 \bullet \delta_{12}(N) - 4 \bullet \delta_{18}(N) + 3 \bullet \delta_{24}(N)$$
$$+ 6 \bullet \delta_{42}(N) + 34 \bullet \delta_{60}(N) - 6 \bullet \delta_{84}(N) - 6 \bullet \delta_{120}(N)$$

$$a_5(N)/N = (N - 6)/4 \bullet \delta_6(N) - 3/2 \bullet \delta_{12}(N) - 2 \bullet \delta_{24}(N) + 4 \bullet \delta_{42}(N)$$
$$+ 6 \bullet \delta_{84}(N) + 6 \bullet \delta_{120}(N)$$

$$a_6(N)/N = 4 \bullet \delta_{30}(N) - 4 \bullet \delta_{60}(N)$$

$$a_7(N)/N = \delta_{30}(N) + 4 \bullet \delta_{60}(N)$$

where $\delta_m(N) = 1$ if $N \equiv 0 (\mathrm{mod}\, m)$, 0 otherwise.

If there are *K* line segments that intersect at one common point, where *K*>2, there are $\binom{K}{3}$ false triangles corresponding to that point. Thus the correction term for false triangles is

$$a_3(N)\binom{3}{3}+a_4(N)\binom{4}{3}+a_5(N)\binom{5}{3}+a_6(N)\binom{6}{3}+a_7(N)\binom{7}{3}+\delta_2(N)\binom{N/2}{3}$$

where the last term represents the contribution of the center point for even *N*. The correction is 0 for odd *N*. The number of triangles formed by line segments with six endpoints on the polygon is then:

$$\binom{N}{6}-(a_3(N)\binom{3}{3}+a_4(N)\binom{4}{3}+a_5(N)\binom{5}{3}+a_6(N)\binom{6}{3}+a_7(N)\binom{7}{3}+\delta_2(N)\binom{N/2}{3})$$

Result

The table below summaries the results for $N \leq 20$. These values were checked through use of a computer program performing an exhaustive search.

N	Triangles with 3 diagonal endpoints	Triangles with 4 diagonal endpoints	Triangles with 5 diagonal endpoints	Triangles with 6 diagonal endpoints	Total Number of Triangles
3	1	0	0	0	**1**
4	4	4	0	0	**8**
5	10	20	5	0	**35**
6	20	60	30	0	**110**
7	35	140	105	7	**287**
8	56	280	280	16	**632**
9	84	504	630	84	**1302**
10	120	840	1260	180	**2400**
11	165	1320	2310	462	**4257**
12	220	1980	3960	796	**6956**
13	286	2860	6435	1716	**11297**
14	364	4004	10010	2856	**17234**
15	455	5460	15015	5005	**25935**
16	560	7280	21840	7744	**37424**
17	680	9520	30940	12376	**53516**
18	816	12240	42840	17508	**73404**
19	969	15504	58140	27132	**101745**
20	1140	19380	77520	38160	**136200**

The first few terms, 1, 8, 35, 110, 287, 632, are listed as M4513 (V. Meally) in Sloane's integer sequences. To summarize the final result, the number of triangles generated by intersecting diagonals of an N-regular polygon is:

$$\binom{N}{3}+4\binom{N}{4}+5\binom{N}{5}+\binom{N}{6}$$
$$-(a_3(N)\binom{3}{3}+a_4(N)\binom{4}{3}+a_5(N)\binom{5}{3}+a_6(N)\binom{6}{3}+a_7(N)\binom{7}{3}+\delta_2(N)\binom{N/2}{3})$$

References

[Poonen98] Bjorn Poonen, Michael Rubinstein, "The Number of Intersection Points Made by the Diagonals of a Regular Polygon." SIAM J. on Disc. Math. 11, no 1 (Feb 1998), 133-156. Note that Theorem 1 has a typographical error. In the second line, 232 should be replaced by 262.