Journal of Integer Sequences, Vol. 9 (2006),
Article 06.4.5

General Properties Involving Reciprocals of Binomial Coefficients

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Abstract

Using the properties of the Beta function, we investigate the representation of infinite series involving the reciprocals of binomial coefficients. We confirm and generalize some of the recent results of Sury, Wang and Zhao.

1 Introduction

The binomial coefficients are defined by

$$\binom{n}{m} = \begin{cases} \dfrac{n!}{m!\,(n-m)!}; & n \geq m, \\ 0; & n < m \end{cases}$$

for n and m non-negative integers.

Binomial coefficients play an important role in many areas of mathematics, including number theory, statistics and probability. Reciprocal binomial coefficients are also prolific in the mathematical literature and many results on reciprocals of binomial coefficient identities may be seen in the papers of Mansour [1], Pla [2], Rockett [3], Sury [5], Sury, Wang and Zhao [6], Trif [7], and Zhao and Wang [9].

Sury [5] used the Beta function

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

to observe that

$$\frac{1}{\binom{n}{m}} = \frac{m!\,(n-m)!}{n!} = \frac{\Gamma(m+1)\,\Gamma(n-m+1)}{\Gamma(n+1)} = (n+1)\int_0^1 t^m (1-t)^{n-m}\,dt$$

Utilising the integral identity for the inverse binomial coefficients, Sury and Trif further showed that

$$\sum_{m=0}^{n} \frac{1}{\binom{n}{m}} = \frac{n+1}{2^n}\sum_{m=0}^{n}\frac{2^m}{m+1} = \frac{n+1}{2^n}\sum_{j\ \text{odd}}\frac{1}{j}\binom{n+1}{j}.$$

Sury, Wang and Zhao [6] proved the following theorem.

Theorem 1.1. *In the ring of $Q[T]$ of rational polynomials, the identity*

$$\sum_{r=m}^{n}\frac{T^r(1-T)^{n-r}}{\binom{n}{r}} = (n+1)\sum_{r=m}^{n}\frac{T^{n+1}(1-T)^{n-r}}{r+1} + (n+1)\sum_{r=0}^{n-m}\frac{T^{n-r}(1-T)^{n-m+1}}{(m+r+1)\binom{m+r}{r}} \tag{1}$$

holds for $m \le n$. An equivalent form is that for $\lambda \neq -1$

$$\sum_{r=m}^{n}\frac{\lambda^r}{\binom{n}{r}} = (n+1)\sum_{r=0}^{n-m}\frac{\lambda^{m+r}}{(\lambda+1)^{r+1}}\sum_{i=0}^{n-m-r}\binom{n-m-r}{i}\frac{(-1)^i}{m+1+i} + (n+1)\frac{\lambda^{n+1}}{(\lambda+1)^{n+2}}\sum_{r=m}^{n}\frac{(\lambda+1)^{r+1}}{r+1}. \tag{2}$$

By the use of Theorem 1.1 and noting that for $|x|<1$,

$$\sum_{r=1}^{\infty}\frac{(2x)^{2r}}{r\binom{2r}{r}} = \frac{2x\arcsin(x)}{\sqrt{1-x^2}} = \int_0^1\frac{4x^2t}{1-4x^2t(1-t)}dt.$$

Sury, Wang and Zhao [6] showed, among other results, that

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)\cdots(n+j)} = \frac{1}{(j-1)(j-1)!}, \tag{3}$$

$$\sum_{n=0}^{\infty}\frac{1}{(3n+1)(3n+2)(3n+3)} = \frac{\pi\sqrt{3}-3\ln 3}{12}, \tag{4}$$

$$\sum_{n=0}^{\infty}\frac{1}{(4n+1)(4n+2)(4n+3)(4n+4)} = \frac{6\ln 2-\pi}{24} \tag{5}$$

and

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{n+j}{n}}=j2^{j-1}\left(\ln 2-\sum_{r=1}^{j-1}\frac{1}{r}\right)-j\sum_{r=1}^{j-1}(-1)^r\binom{j-1}{r}\frac{2^{j-1-r}}{r},$$
$$\text{for } j=1,2,\ldots \tag{6}$$

Identities (3) to (6) are reciprocal binomial identities of the form

$$S(a,j)=\frac{1}{j!}\sum_{n=0}^{\infty}\frac{1}{\binom{an+j}{an}} \tag{7}$$

and

$$T(a,j)=\frac{1}{j!}\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{an+j}{an}} \tag{8}$$

for $j=1,2,3\ldots,\ a\in\mathbb{R}^+\setminus\{0\}$.

The identity (3)

$$S(1,j)=\frac{1}{(j-1)(j-1)!}$$

and

$$S(2,j)=\frac{2^{j-2}}{(j-1)!}\left[\ln 2+\sum_{r=1}^{j-2}(-1)^r\binom{j-2}{r}\left(\frac{2^r-1}{r\cdot 2^r}\right)\right]$$
$$=j!\ {}_3F_2\left[\left.\begin{matrix}\frac{1}{2},1,1,\\ \frac{1+j}{2},\frac{2+j}{2}\end{matrix}\right|1\right]$$

and others were also previously given by Sofo [4].

In this paper we shall extend the range of identities for $S(a,j)$, $T(a,j)$, give particular closed form representations to $S\left(\frac{1}{b},j\right)$ and $T\left(\frac{1}{b},j\right)$ for $b=2,3,\ldots$.

Finally, we shall give a generalization to both $S(a,j)$ and $T(a,j)$.

2 Identity Representations

Transform techniques are used extensively in the analysis of series and in their representation in closed form.

In his work, Wheelon [8], and later Sofo [4] essentially showed that

$$\begin{aligned} S(a,j) &= \frac{1}{j!}\sum_{n=0}^{\infty}\frac{1}{\binom{an+j}{an}} \\ &= \sum_{n=0}^{\infty}\frac{1}{\prod_{k=1}^{j}(an+k)} \\ &= \frac{1}{(j-1)!}\int_{x=0}^{1}\frac{(1-x)^{j-1}}{1-x^a}dx \\ &= \frac{1}{j!}\ {}_{j+1}F_j\left[\left.\begin{matrix} 1,\frac{1}{a},\frac{2}{a},\frac{3}{a},\dots,\frac{j}{a} \\ 1+\frac{1}{a},1+\frac{2}{a},1+\frac{3}{a},\dots,1+\frac{j}{a}\end{matrix}\right| 1\right]. \end{aligned}$$

We can use the ideas of Sury, Wang and Zhao [6] by implementing the Beta function to state the following theorem.

Theorem 2.1. *Let* $a \in \mathbb{R}^+ \setminus \{0\}$ *and* $j = 2, 3, 4, \dots$. *Then*

$$S(a,j) = \frac{1}{j!}\sum_{n=0}^{\infty}\frac{1}{\binom{an+j}{an}} \tag{9}$$

$$= \sum_{n=0}^{\infty}\frac{1}{\prod_{k=1}^{j}(an+k)} \tag{10}$$

$$= \frac{1}{(j-1)!}\int_{x=0}^{1}\frac{(1-x)^{j-1}}{1-x^a}dx \tag{11}$$

$$= \frac{1}{j!}\ {}_{j+1}F_j\left[\left.\begin{matrix} 1,\frac{1}{a},\frac{2}{a},\frac{3}{a},\dots,\frac{j}{a} \\ 1+\frac{1}{a},1+\frac{2}{a},1+\frac{3}{a},\dots,1+\frac{j}{a}\end{matrix}\right| 1\right] \tag{12}$$

and similarly

$$T(a,j) = \frac{1}{j!}\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{an+j}{an}} \tag{13}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{k=1}^{j}(an+k)} \tag{14}$$

$$= \frac{1}{(j-1)!}\int_{x=0}^{1}\frac{(1-x)^{j-1}}{1+x^a}dx \tag{15}$$

$$= \frac{1}{j!}\ {}_{j+1}F_j\left[\left.\begin{matrix} 1,\frac{1}{a},\frac{2}{a},\frac{3}{a},\dots,\frac{j}{a} \\ 1+\frac{1}{a},1+\frac{2}{a},1+\frac{3}{a},\dots,1+\frac{j}{a}\end{matrix}\right| -1\right]. \tag{16}$$

Proof. Consider the alternating case

$$\begin{aligned}
T(a,j) &:= \frac{1}{j!}\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{an+j}{an}} \\
&= \frac{1}{(j-1)!}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(j)\,\Gamma(an+1)}{\Gamma(an+j+1)} \\
&= \frac{1}{(j-1)!}\sum_{n=0}^{\infty}(-1)^n B(an+1,j) \\
&= \frac{1}{(j-1)!}\sum_{n=0}^{\infty}\int_{x=0}^{1}(-1)^n x^{an}(1-x)^{j-1}\,dx.
\end{aligned}$$

Interchanging the sum and integral, we have

$$\begin{aligned}
T(a,j) &= \frac{1}{(j-1)!}\int_{x=0}^{1}\sum_{n=0}^{\infty}(-1)^n x^{an}(1-x)^{j-1}\,dx \\
&= \frac{1}{(j-1)!}\int_{x=0}^{1}\frac{(1-x)^{j-1}}{1+x^a}dx
\end{aligned}$$

which is the result (15).

$$B(\alpha,\beta) = \int_{u=0}^{1}u^{\alpha-1}(1-u)^{\beta-1}\,du, \quad \text{for } \alpha>0 \text{ and } \beta>0$$

is the classical Beta function.

From the ratio of successive terms of (13)

$$\frac{V_{n+1}}{V_n} = -\frac{\prod_{j=1}^{k}(an+k)}{\prod_{j=1}^{k}(an+a+k)}$$

we arrive at the result (16).

The proof of $S(a,j)$ follows a similar argument as that used for $T(a,j)$ and will not be done here. □

In the next theorem we consider a particular case for the value of a.

Theorem 2.2. *In the case that* $a=\frac{1}{b}$, *for* b *an even positive integer, then* $T\left(\frac{1}{b},j\right)$ *forms the rational numbers*

$$\begin{aligned}
T\left(\frac{1}{b},j\right) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{k=1}^{j}\left(\frac{n}{b}+k\right)} \qquad (17) \\
&= \frac{1}{(j-1)}\sum_{\mu=0}^{b-1}\frac{(-1)^\mu}{\prod_{\nu=1}^{j-1}\left(\nu+\frac{\mu}{b}\right)}
\end{aligned}$$

and similarly for $b = 2, 3, \dots,$

$$S\left(\frac{1}{b}, j\right) = \sum_{n=0}^{\infty} \frac{1}{\prod_{k=1}^{j}\left(\frac{n}{b}+k\right)} = \frac{1}{(j-1)} \sum_{\mu=0}^{b-1} \frac{1}{\prod_{\nu=1}^{j-1}\left(\nu+\frac{\mu}{b}\right)}. \tag{18}$$

Proof. Consider

$$\begin{aligned}
T\left(\frac{1}{b}, j\right) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\prod_{k=1}^{j}\left(\frac{n}{b}+k\right)} \\
&= \frac{1}{(j-1)!} \int_{x=0}^{1} \frac{(1-x)^{j-1}}{1+x^{\frac{1}{b}}} dx \\
&= \frac{1}{(j-1)!} \int_{x=0}^{1} (1-x)^{j-2} \left(\frac{1-x}{1+x^{\frac{1}{b}}}\right) dx \\
&= \frac{1}{(j-1)!} \int_{x=0}^{1} (1-x)^{j-2} \sum_{\mu=0}^{b-1} (-1)^{\mu} x^{\frac{\mu}{b}} dx \\
&= \frac{1}{(j-1)!} \int_{x=0}^{1} \sum_{r=0}^{j-2} (-1)^r \binom{j-2}{r} x^r \sum_{\mu=0}^{b-1} (-1)^{\mu} x^{\frac{\mu}{b}} dx \\
&= \frac{1}{(j-1)!} \sum_{r=0}^{j-2} (-1)^r \binom{j-2}{r} \sum_{\mu=0}^{b-1} \frac{(-1)^{\mu}}{r+1+\frac{\mu}{b}} \\
&= \frac{1}{(j-1)!} \sum_{\mu=0}^{b-1} \frac{(-1)^{\mu}}{\left(1+\frac{\mu}{b}\right)\binom{j-1+\frac{\mu}{b}}{j-2}} \\
&= \frac{1}{(j-1)} \sum_{\mu=0}^{b-1} \frac{(-1)^{\mu}}{\prod_{\nu=1}^{j-1}\left(\nu+\frac{\mu}{b}\right)},
\end{aligned}$$

which is the result (17).

From (16) it is also interesting to note that for b an even positive integer

$${}_{j+1}F_j\left[\left.\begin{matrix} 1, b, 2\cdot b, 3\cdot b, \dots, j\cdot b \\ 1+b, 1+2\cdot b, 1+3\cdot b, \dots, 1+j\cdot b \end{matrix}\right| -1\right] = j \sum_{\mu=0}^{b-1} \frac{(-1)^{\mu}}{\prod_{\nu=1}^{j-1}\left(\nu+\frac{\mu}{b}\right)}$$

The result (18) can be proved in the same way and will not be detailed here.

Again from (12) it is interesting to note that for $b = 2, 3, \dots$

$${}_{j+1}F_j\left[\left.\begin{matrix} 1, b, 2\cdot b, 3\cdot b, \dots, j\cdot b \\ 1+b, 1+2\cdot b, 1+3\cdot b, \dots, 1+j\cdot b \end{matrix}\right| 1\right] = j \sum_{\mu=0}^{b-1} \frac{1}{\prod_{\nu=1}^{j-1}\left(\nu+\frac{\mu}{b}\right)}.$$

□

3 Examples

In the case when a is a positive integer, by known properties of the hypergeometric function we may state that

$$ {}_{j+1}F_j\left[\begin{matrix} 1, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{j}{a} \\ 1+\frac{1}{a}, 1+\frac{2}{a}, 1+\frac{3}{a}, \ldots, 1+\frac{j}{a} \end{matrix} \middle| -1\right] = {}_{a+1}F_a\left[\begin{matrix} 1, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{a}{a} \\ \frac{1+j}{a}, \frac{2+j}{a}, \frac{3+j}{a}, \ldots, \frac{a+j}{a} \end{matrix} \middle| -1\right]. $$

Consequently, we may write

$$ \begin{aligned} T(a,j) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\prod_{k=1}^{j}(an+k)} \\ &= \frac{1}{j!}\; {}_{a+1}F_a\left[\begin{matrix} 1, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{a}{a} \\ \frac{1+j}{a}, \frac{2+j}{a}, \frac{3+j}{a}, \ldots, \frac{a+j}{a} \end{matrix} \middle| -1\right]. \end{aligned} $$

(i) For $a = 1$ we have,

$$ \begin{aligned} T(1,j) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\prod_{k=1}^{j}(n+k)} \\ &= \frac{1}{j!}\; {}_2F_1\left[\begin{matrix} 1, 1 \\ 1+j \end{matrix} \middle| -1\right] \\ &= \frac{2^{j-1}\ln 2}{(j-1)!} + \frac{2^{j-1}}{(j-1)!}\sum_{r=1}^{j-1}(-1)^r \binom{j-1}{r}\left(\frac{2^r-1}{r\cdot 2^r}\right) \\ &= \begin{cases} 2\ln 2 - 1; & \text{for } \; j=2 \\ \frac{2^{j-2}}{(j-2)!}\left\{ {}_3F_2\left[\begin{matrix} 1,1,2-j \\ 2,2 \end{matrix} \middle| -1\right] - {}_3F_2\left[\begin{matrix} 1,1,2-j \\ 2,2 \end{matrix} \middle| -\frac{1}{2}\right]\right\}; & \text{for } \; j>2. \end{cases} \end{aligned} $$

This confirms the result (1.6), in the paper of Sury, Wang and Zhao [6].

(ii) For $a=2$ and $j=4m+5$, $m=0,1,2,\ldots$, we may extract the following result

$$\begin{aligned} T(2,4m+5) &:= \sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{k=1}^{4m+5}(2n+k)} \\ &= \frac{1}{(4m+5)!}\,{}_3F_2\left[\begin{matrix}\frac{1}{2},1,1\\ 2m+3,\frac{4m+7}{2}\end{matrix}\middle| -1\right] \\ &= \frac{2^{2m+2}}{(4m+4)!}\left\{(-1)^{m+1}\frac{\pi}{4}+\sum_{p=0}^{m}\frac{(-1)^p}{2m-2p+1}\right. \\ &\qquad \left. +\sum_{r=1}^{2m+2}\left(-\frac{1}{2}\right)^r\binom{2m+2}{r}\sum_{s=0}^{r-1}\binom{r-1}{s}\frac{1}{2m+2s+3-r}\right\} \\ &= \frac{-(-1)^m 4^m\pi}{(4m+4)!}+\frac{2^{2m+2}}{(4m+4)!}\left\{\sum_{p=0}^{m}\frac{(-1)^p}{2m-2p+1}\right. \\ &\qquad \left. +\sum_{r=1}^{2m+2}\frac{\left(-\frac{1}{2}\right)^r}{2m+3-r}\binom{2m+2}{r}\,{}_2F_1\left[\begin{matrix}1-r,m+\frac{3}{2}-\frac{r}{2}\\ m+\frac{5}{2}-\frac{r}{2}\end{matrix}\middle| -1\right]\right\}. \end{aligned}$$

A rearrangement of the above result highlights an identity for π;

$$\begin{aligned} (-1)^m\frac{\pi}{4} &= \sum_{p=0}^{m}\frac{(-1)^p}{2m-2p+1}+\sum_{r=1}^{2m+2}\left(-\frac{1}{2}\right)^r\binom{2m+2}{r}\sum_{s=0}^{r-1}\frac{\binom{r-1}{s}}{2m+2s+3-r} \\ &\qquad -\frac{(4m+4)!}{2^{2m+2}}\sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{k=1}^{4m+5}(2n+k)};\quad m=0,1,2,\ldots. \end{aligned}$$

In particular, for $m=0$

$$\frac{\pi}{4}=\frac{5}{6}-6\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)},$$

for $m=1$

$$\frac{\pi}{4}=\frac{109}{140}+2520\sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{k=1}^{9}(2n+k)}.$$

Note: For $n=0$, the first term bounds π as follows:

$$\frac{3958}{1260}<\pi<\frac{3959}{1260}<\frac{22}{7}.$$

Remark. The series (10), $S(a,j)$ and (14), $T(a,j)$ can be expressed in terms of the Lerch transcendent.

In particular, from (14)

$$T(a,j) := \sum_{n=0}^{\infty} \frac{(-1)^n}{\prod_{k=1}^{j}(an+k)}$$
$$= \sum_{n=0}^{\infty} (-1)^n \sum_{k=1}^{j} \frac{A_{j,k}}{(an+k)},$$

where

$$A_{j,k} = \lim_{n\to\left(-\frac{k}{a}\right)} \left\{ \frac{an+k}{\prod_{k=1}^{j}(an+k)} \right\}$$
$$= \frac{(-1)^{k+1}}{(k-1)!\,(j-k)!}.$$

Hence

$$T(a,j) = \sum_{n=0}^{\infty} (-1)^n \sum_{k=1}^{j} \frac{(-1)^{k+1}}{(k-1)!\,(j-k)!\,(an+k)} \tag{19}$$
$$= \sum_{k=1}^{j} \frac{(-1)^{k+1}}{(k-1)!\,(j-k)!} \sum_{n=0}^{\infty} \frac{(-1)^n}{(an+k)}$$
$$= \frac{1}{2a} \sum_{k=1}^{j} \frac{(-1)^{k+1}}{(k-1)!\,(j-k)!} \left\{ \psi\left(\frac{1}{2}+\frac{k}{2a}\right) - \psi\left(\frac{k}{2a}\right) \right\},$$

where $\psi(z)$ is the Psi, or digamma function.

The Lerch transcendent, $\phi(z,s,\alpha)$ is defined as

$$\phi(z,s,\alpha) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s},$$

where the $(n+\alpha)=0$ term is excluded from the sum.

The polygamma functions $\psi^{(k)}(z)$, $k \in \mathbb{N}$ are defined by

$$\psi^{(k)}(z) := \frac{d^{k+1}}{dz^{k+1}} \log\Gamma(z) = \frac{d^k}{dz^k}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right), \quad k \in \mathbb{N}_0 := \mathbb{N}\cup\{0\},$$

where $\psi^{(0)}(z) = \psi(z)$, denotes the Psi, or digamma function, defined by

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\Gamma(z) = \int_1^z \psi(t)\,dt.$$

From (19)

$$T(a,j) = \sum_{k=1}^{j} \frac{(-1)^{k+1}}{(k-1)!\,(j-k)!} \cdot \frac{1}{a} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n+\frac{k}{a}\right)}$$
$$= \sum_{k=1}^{j} \frac{(-1)^{k+1}}{(k-1)!\,(j-k)!a} \phi\left(-1,1,\frac{k}{a}\right).$$

Similar technical details can be written about the series (10), however, they will not be detailed here.

4 Generalization

Both series (9) and (13) for $S(a,j)$ and $T(a,j)$ can be generalized in the following manner.

Theorem 4.1. *For $m \geq 1$ and $a > 0$ and j a positive integer, then*

$$S(a,j,m) = \frac{1}{j!}\sum_{n=0}^{\infty}\frac{\binom{n+m-1}{n}}{\binom{an+j}{an}} = \frac{1}{j!}\sum_{n=0}^{\infty}\frac{\binom{n+m-1}{m-1}}{\binom{an+j}{j}} \tag{20}$$

$$= \frac{1}{(j-1)!}\int_0^1 \frac{(1-x)^{j-1}}{(1-x^a)^m}dx \tag{21}$$

$$= \frac{1}{j!}\ {}_{j+1}F_j\left[\begin{matrix} m, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{j}{a} \\ 1+\frac{1}{a}, 1+\frac{2}{a}, 1+\frac{3}{a}, \ldots, 1+\frac{j}{a}\end{matrix}\middle| 1\right] \tag{22}$$

$$= \sum_{n=0}^{\infty}(m)_n \Bigg/ n!\prod_{k=1}^{j}(an+k) \tag{23}$$

and

$$T(a,j,m) = \frac{1}{j!}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{n+m-1}{n}}{\binom{an+j}{an}} \tag{24}$$

$$= \frac{1}{(j-1)!}\int_0^1 \frac{(1-x)^{j-1}}{(1+x^a)^m}dx \tag{25}$$

$$= \frac{1}{j!}\ {}_{j+1}F_j\left[\begin{matrix} m, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{j}{a} \\ 1+\frac{1}{a}, 1+\frac{2}{a}, 1+\frac{3}{a}, \ldots, 1+\frac{j}{a}\end{matrix}\middle| -1\right] \tag{26}$$

$$= \sum_{n=0}^{\infty}(-1)^n(m)_n \Bigg/ n!\prod_{k=1}^{j}(an+k)\ . \tag{27}$$

Proof. Consider

$$T(a,j,m) = \frac{1}{j!}\sum_{n=0}^{\infty}(-1)^n\frac{\binom{n+m-1}{n}}{\binom{an+j}{an}} = \frac{1}{j!}\sum_{n=0}^{\infty}(-1)^n\frac{\binom{n+m-1}{m-1}}{\binom{an+j}{j}}$$

$$= \frac{1}{(j-1)!}\sum_{n=0}^{\infty}(-1)^n\binom{n+m-1}{n}\frac{\Gamma(an+1)\Gamma(j)}{\Gamma(an+1+j)}$$

$$= \frac{1}{(j-1)!}\sum_{n=0}^{\infty}(-1)^n\binom{n+m-1}{n}B(an+1,j)$$

$$= \frac{1}{(j-1)!}\sum_{n=0}^{\infty}(-1)^n\binom{n+m-1}{n}\int_0^1 x^{an}(1-x)^{j-1}dx,$$

interchanging sum and integral, we have

$$\begin{aligned}T(a,j,m) &= \frac{1}{(j-1)!}\int_0^1 (1-x)^{j-1}\sum_{n=0}^{\infty}(-1)^n\binom{n+m-1}{n}x^{an}dx\\ &= \frac{1}{(j-1)!}\int_0^1 \frac{(1-x)^{j-1}}{(1+x^a)^m}dx,\end{aligned}$$

which is (25).

Next, by considering the ratio of terms in (24), the hypergeometric identity (26) follows. Notice that in the case when a is a positive integer, because of the properties of the hypergeometric function, we may also write

$$T(a,j,m) = \frac{1}{j!}\ {}_{a+1}F_a\left[\begin{array}{c} m, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{a}{a} \\ \frac{1+j}{a}, \frac{2+j}{a}, \frac{3+j}{a}, \ldots, \frac{a+j}{a}\end{array}\middle| -1\right].$$

To deduce (27) we may write

$$\begin{aligned}T(a,j,m) &= \frac{1}{j!}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n+1)}\cdot\frac{j!\Gamma(an+1)}{\Gamma(an+1+j)}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(m)_n}{n!(an+1)_j},\end{aligned}$$

where

$$(p)_\alpha = p(p+1)\cdots(p+\alpha-1) = \frac{\Gamma(p+\alpha)}{\Gamma(p)}$$

is Pochhammer's symbol.

Now we can state

$$T(a,j,m) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\frac{(m)_n}{\prod_{k=1}^{j}(an+k)}.$$

The identities (21) and (22) of $S(a,j,m)$ in (20) follow in a similar fashion as above and will not be detailed here. □

Some examples now follow, with the minimum of detail.

Example 4.1.

$$S(1,j,m) = \frac{1}{j!}\sum_{n=0}^{\infty}\frac{\binom{n+m-1}{n}}{\binom{n+j}{n}} = \frac{1}{(j-m)(j-1)!},$$

hence

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\binom{n+m-1}{n}}{\binom{n+j}{n}} &= \frac{j}{j-m}\quad \textit{for}\ \ j\neq m\\ &= {}_2F_1\left[\begin{array}{c}1,m\\1+j\end{array}\middle|1\right],\end{aligned}$$

this generalizes the result (3).

Example 4.2.

$$S\left(2,9,\frac{9}{2}\right)=\frac{1}{9!}\sum_{n=0}^{\infty}\frac{\binom{n+\frac{7}{2}}{n}}{\binom{2n+9}{2n}},$$

hence

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\binom{n+\frac{7}{2}}{n}}{\binom{2n+9}{2n}} &= \frac{9\pi}{2}-\frac{456}{35}\\ &= \sum_{n=0}^{\infty}\frac{\left(\frac{9}{2}\right)_n}{n!\binom{2n+9}{2n}} = {}_3F_2\left[\begin{array}{c}\frac{1}{2},1,\frac{9}{2}\\ 2,\frac{11}{2}\end{array}\middle|\,1\right].\end{aligned}$$

Also,

$$T\left(2,9,\frac{9}{2}\right)=\frac{1}{9!}\sum_{n=0}^{\infty}(-1)^n\frac{\binom{n+\frac{7}{2}}{n}}{\binom{2n+9}{2n}},$$

hence

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{n+\frac{7}{2}}{n}}{\binom{2n+9}{2n}} &= 9\ln\left(\sqrt{2}+1\right)+\frac{2559\sqrt{2}}{35}-\frac{552}{5}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{9}{2}\right)_n}{n!\binom{2n+9}{2n}}\\ &= {}_3F_2\left[\begin{array}{c}\frac{1}{2},1,\frac{9}{2}\\ 2,\frac{11}{2}\end{array}\middle|\,-1\right].\end{aligned}$$

Example 4.3.

$$S(3,5,4)=\frac{1}{5!}\sum_{n=0}^{\infty}\frac{\binom{n+3}{n}}{\binom{3n+5}{3n}}$$

hence

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\binom{n+3}{n}}{\binom{3n+5}{3n}} &= \frac{100\sqrt{3}\pi}{243}-\frac{10}{9}\\ &= {}_4F_3\left[\begin{array}{c}\frac{1}{3},\frac{2}{3},1,4\\ 2,\frac{7}{3},\frac{8}{3}\end{array}\middle|\,1\right].\end{aligned}$$

Also

$$T(3,5,4)=\frac{1}{5!}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{n+3}{n}}{\binom{3n+5}{3n}}$$

hence

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{n+3}{n}}{\binom{3n+5}{3n}} = \frac{10}{9} + \frac{40\ln 2}{27} - \frac{160\sqrt{3}\pi}{729}$$

$$= {}_4F_3\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, 1, 4 \\ 2, \frac{7}{3}, \frac{8}{3} \end{matrix} \middle| -1\right].$$

Note: The series (20) and (24) can be generalized further, these results will be reported in another forum.

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2000 *Mathematics Subject Classification*: Primary 11B65.
Keywords: Binomial coefficients, combinatorial identities, integral representations.

Received February 20 2006; revised version received July 27 2006. Published in *Journal of Integer Sequences*, September 7 2006.

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