Journal of Integer Sequences, Vol. 9 (2006),
Article 06.2.6

The Number of Topologies on a Finite Set

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Abstract

Let X be a finite set having n elements. How many different labeled topologies one can define on X? Let $T(n,k)$ be the number of topologies having k open sets. We compute $T(n,k)$ for $2 \leq k \leq 12$, as well as other results concerning T_0 topologies on X having $n+4 \leq k \leq n+6$ open sets.

1 Introduction

Recall that a topology τ on the set $X \neq \phi$ is a subset of $P(X)$ that contains ϕ and X, and is closed under union and finite intersection. A topology on X is a sublattice of $(P(X), \subseteq)$ with the maximum element X, denoted by 1, and the minimum element, which is ϕ, denoted by 0. Let X be an $n-$element set. The number $T(n)$ of topologies on X is exactly the number of sublattices of $P(X)$, with 0 and 1.

There is another connection between the number of topologies on X, and the number of some kind of binary relations on it. A relation on X is called a *preorder* if it is reflexive and transitive. Let Q_n denotes the number of such relations. It is known that $T(n) = Q_n$. A preorder on X which is transitive is a *partial order*. The subset $A \subset X$ of the preordered set X is called an *ideal* if $x \in A$ and $y \leq x$ implies $y \in A$. Let P_n denotes the total number of partial orders on X. Then P_n is also the number of T_0 topologies one can define on X. Note that the open sets in the topology correspond to the ideals in the preorder: a topology on X having k open sets, corresponds to a preorder with k ideals and vice versa.

Efficient computation of the total number of labeled topologies $T(n)$ one can define on X is still an open question. There is no known simple formula giving $T(n)$. For small values of n, this may be done by hand; for example, $T(1) = 1$, $T(2) = 4$, and $T(3) = 29$. For $n \geq 4$, the calculations are complicated. The online encyclopedia of N. J. A. Sloane [2] gives the value of $T(n)$ for $1 \leq n \leq 14$.

An approach towards the determination of $T(n)$ is as follows. Let $t(n, k)$ be the set of all labeled topologies on X and having k open sets (or, which is the same, the number of preorders on X having k ideals), $2 \leq k \leq 2^n$, and $T(n, k) = |t(n, k)|$. So, $T(n) = \sum_{k \geq 2} T(n, k)$. Obviously, we have

$$\begin{aligned} T(n, 2) &= T(n, 2^n) = 1 \\ T(n, 3) &= 2^n - 2 \end{aligned}$$

For $k \geq 4$, the determination of $T(n, k)$ is not as straightforward as for $T(n, 2)$ and $T(n, 3)$. The numbers $T(n, k)$ have been determined for some values of k. For instance, R. Stanley [3] computed $T(n, k)$ for large values of k, viz.; $3 \cdot 2^{n-3} < k < 2^n$. Also, he determined labeled T_0 topologies on X having either $n + 1$, $n + 2$, or $n + 3$ open sets.

In this paper, we compute $T(n, k)$ for $2 \leq k \leq 12$, as well as the total number of labeled T_0 topologies on X having $n+4$, $n+5$, $n+6$ open sets. We also give different proofs (shorter or simpler) of some known results in [1, 3, 2].

We need some preliminary definitions and results. Let us recall the definition of Stirling numbers of the second kind:

Definition 1.1. *The number of partitions of a finite set with n elements into k blocks, is the Stirling number of the second kind. It is denoted $S(n, k)$.*

The explicit, and somewhat complicated formula for Stirling numbers of the second kind is

$$S(n, k) = S_{n,k} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n. \tag{1}$$

Lemma 1.2. *Let τ be a topology on a finite set X. Then $\tau^c = \{A^c,\ A \in \tau\}$ is also a topology on X.*

Remark 1.3. *The previous lemma is not necessarily true if X is infinite. Note, also, if τ is a topology, it may happen that $\tau = \tau^c$. Take $X = \{a,\ b,\ c\}$, and $\tau = \{\phi,\ X,\ \{a\},\ \{b,\ c\}\}$.*

Definition 1.4. *A chain topology on X, is a topology whose open sets are totally ordered by inclusion.*

2 Topologies with small number of open sets

In this section we compute $T(n, k)$ for $2 \leq k \leq 12$. We need the following lemma

Lemma 2.1. *(D. Stephen) Let $C(n, k)$ be the number of chain topologies on X having k open sets. Then*

$$C(n, k) = \sum_{l=1}^{n-1} \binom{n}{l} C(l, k - 1) = (k - 1)! S(n, k - 1) \tag{2}$$

Proof. This proof is shorter than that in Stephen [4]. First, there is a bijective correspondence between the $k-$ordered partitions (partitions having k blocks) of the set X and the chains of subsets of X having k (non empty and different from X) members: the chain $\phi \neq A_1 \subsetneqq A_2 \subsetneqq A_3 \ldots \subsetneqq A_k \subsetneqq X$, is associated with the partition $(B_1,\ B_{2,} B_{3,} \ldots B_k)$, where $B_1 = A_1,\ B_i = A_i - A_{i-1},\ A_{k+1} = X - A_k$. In the other hand, if $(B_1,\ B_{2,} B_{3,} \ldots B_k)$ is an ordered partition, the chain $\phi \neq A_1 \subsetneqq A_2 \subsetneqq A_3 \ldots \subsetneqq A_{k-} \subsetneqq X$, where $A_1 = B_1,\ A_i = A_{i-1} \cup B_i,\ 2 \leq i \leq k-1$, is associated with the partition $(B_1,\ B_{2,} B_{3,} \ldots B_k)$. Now the cardinal of the ordered k-partitions is $k!S(n,k)$, which is the desired result.

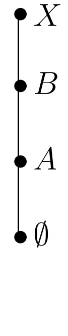
For the recursion, note that a chain topology on a subset $A \subseteq X,\ 1 \leq |A| = l \leq n-1$, having $k-1$ open sets, is a chain topology on X with k open sets. The total number of such topologies on A is $\binom{n}{l} C(l,k-1)$. So, $C(n,k = \sum\limits_{l=1}^{n-l} \binom{n}{l} C(l,k-1)$. ■

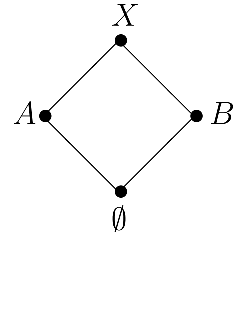
Now we are ready to prove our main result.

Theorem 2.2. *For every $n \geq 1$, we have*

$$\begin{aligned}
T(n,4) &= S_{n,2} + 3!S_{n,3} = 3^n - 5\cdot 2^{n-1} + 2 \\
T(n,5) &= 3!S_{n,3} + 4!S_{n,4} = 4^n - 3^{n+1} + 3\cdot 2^n - 1 \\
T(n,6) &= 3!S_{n,3} + \frac{3}{2}4!S_{n,4} + 5!S_{n,5} \\
T(n,7) &= \frac{9}{4}.4!S_{n,4} + 2.5!S_{n,5} + 6!S_{n,6} \\
T(n,8) &= S_{n,3} + 2.4!S_{n,4} + \frac{15}{4}.5!S_{n,5} + \frac{5}{2}.6!S_{n,6} + 7!S_{n,7} \\
T(n,9) &= \frac{5}{6}4!S_{n,4} + 5.5!S_{n,5} + \frac{11}{2}6!S_{n,6} + 3.7!S_{n,7} + 8!S_{n,8} \\
T(n,10) &= 4!S_{n,4} + \frac{11}{2}5!S_{n,5} + \frac{73}{8}6!S_{n,6} + \frac{15}{2}7!S_{n,7} + \frac{7}{2}8!S_{n,8} + 9!S_{n,9} \\
T(n,11) &= \frac{25}{6}5!S_{n,5} + \frac{79}{6}6!S_{n,6} + \frac{29}{2}7!S_{n,7} + \frac{39}{4}8!S_{n,8} + 4.9!S_{n,9} + 10!S_{n,10}. \\
T(n,\ 12) &= \frac{1}{2}4!S_{n,4} + \frac{9}{2}5!S_{n,5} + 16.6!S_{n,6} + \frac{295}{12}7!S_{n,7} + \frac{85}{4}8!S_{n,8} + \frac{49}{4}.9!S_{n,9} \\
&\qquad + \frac{9}{2}.10!S_{n,10} + 11!S_{n,11}.
\end{aligned}$$

Proof. For every $T(n,k)$, we list all the forms of the topologies in $t(n,k)$, and then compute the topologies of each form. Let $\tau = \{\phi,\ X, A, B\} \in t(n,4)$, then either τ is a chain or has the form ($A \cap B = \phi$ and $A \cup B = X$).





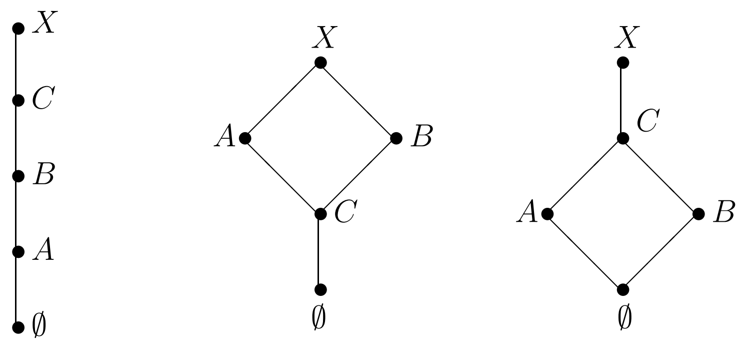
The two cases are disjoint.

Case (1): this is the number of chain topologies having 4 open sets; so the number is $3!S(n,3)$.

Case (2): This is the total number of partitions of X into two blocks, and is done in $S(n,2) = 2^{n-1} - 1$ different ways. Finally, the desired number is

$$T(n,4) = 3!S(n,3) + S(n,2).$$

For $T(n,5)$, there are 3 forms:



1) Chain topologies with 5 open sets.
2)($\phi \subset C \subset B$, $\phi \subset C \subset A$, $A \cup B = X$) or
3)($\phi \subset A \subset C$, $\phi \subset B \subset C$, $A \cup B = C \subsetneq X$).

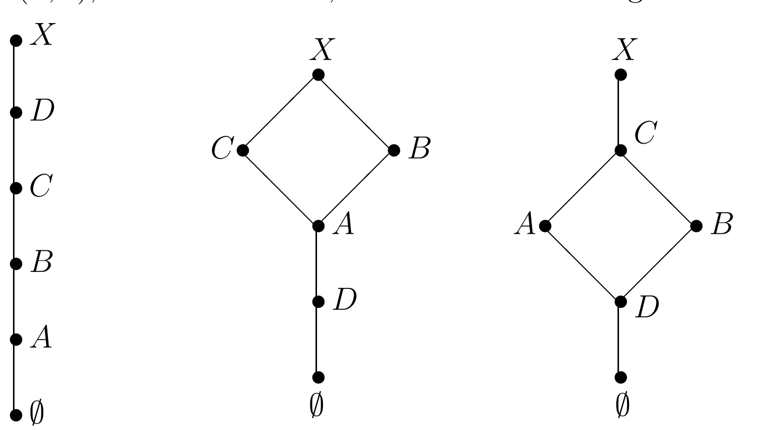
The number of topologies in the first case is $4!S(n,4)$. Cases (2) and (3) are symmetric (and different, i.e., these cases correspond to t and t^c). So, we compute only one, case (3). Let $C \subsetneq X$ be such that $|C| = k$, $2 \leq k \leq n-1$. This is chosen in $\binom{n}{k}$different ways, and then it is partitioned into two disjoint blocks: this is done in $S(k,2) = 2^{k-1} - 1$ different ways. Furthermore, the number in case (3) is :

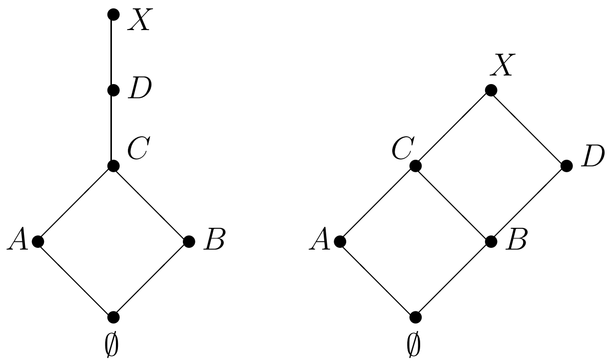
$$\sum_{k=2}^{n-1} \binom{n}{k} \left(2^{k-1} - 1\right) = \frac{3^n - 3 \cdot 2^n + 3}{2} = \frac{3!S(n,3)}{2}.$$

So, the total number for (2) and (3) is $3!S(n,3)$. Consequently, we get

$$T(n,5) = 3!S(n,3) + 4!S(n,4) = 4^n - 3^{n+1} + 3 \cdot 2^n - 1.$$

For $T(n,6)$, we have 5 forms, as indicated in the figure below:





1) Chain topologies with 6 open sets.

2)$(A\cap B=\ \phi,\ A\cup B=C,\ A\subset C,\ C\cap D=B,\ C\cup D=X)$

3)$(\phi\ \subsetneq A_1\subsetneq A_3\subsetneq A_4\subsetneq X$, and $\phi\subsetneq A_2\subsetneq A_3\subsetneq A_4\subsetneq X, A_1\cap A_2=\phi)$, and its symmetric case.

4)$\phi\subsetneq A_1\subsetneq A_2\subsetneq A_4\subsetneq X$,and $\phi\subsetneq A_1\subsetneq A_3\subsetneq A_4\subsetneq X,\ A_2\cup A_3=A_4$.

The number of topologies in the first case is $5!S(n,5)$. The number in the second case is computed as follows: let $C\subsetneq X$, such that $2\le|C|=k\le n-1$. We then partition C in to two blocks A,B. To each partition (A,B) corresponds two topologies:

$$(\phi,\ A,\ B,\ A\cup B=C,\ A\cup(X-B),\ X)$$

and

$$(\phi,\ A,\ B,\ A\cup B=C,\ B\cup(X-A),\ X)\,.$$

So, the total number of topologies in this case is

$$2\sum_{k=2}^{n-1}\binom{n}{k}\left(2^{k-1}-1\right)=3^n-3.2^n+3=3!S(n,3).$$

For the third case, we choose A_3, $2\le|A_3|=k\le n-2$, in $\binom{n}{k}$ different ways and then partition it into 2 blocks, $(A_1,\ A_2)$ with $S(k,2)=2^{k-1}-1$ different ways. There remain $(n-k)$ elements, from which we choose the elements of A_4: note that $|A_4|=k+i\le n-1$. The number of these choices is $\binom{n}{k}\binom{n-k}{i}S(k,2)$. Finally the total number for the third case and its reciprocal is

$$2\sum_{k=2}^{n-2}\sum_{i=1}^{n-k-1}\binom{n}{k}\binom{n-k}{i}\left(2^{k-1}-1\right)=4^n-4\cdot 3^n+3\cdot 2^{n+1}-4=4!S(n,4).$$

The last case is computed similarly and is equal to

$$\frac{4!S(n,4)}{2}.$$

So, we obtain

$$T(n,6)=3!S(n,3)+\frac{3}{2}4!S(n,4)+5!S(n,5).$$

The computation of the remaining cases is similar to $T(n,5)$ and $T(n,6)$, but much longer. We note that for $T(n,7)$, we have 8 forms, for $T(n,8)$, there are 15 forms, for $T(n,9)$ there are 26 cases, and so on. ■

3 Topologies with large number of open sets

The following result is due to H. Sharp [1] and D. Stephen [4].

Theorem 3.1. *For $n \geq 3$, $T(n,k) = 0$ for $3 \cdot 2^{n-2} < k < 2^n$.*

Sharp [1] proved this result using graph theory. Stephen's proof [4]used topological facts. Here is another one, which is direct and allows us to compute $T(n, 3 \cdot 2^{n-2})$.

Proof. Since we are looking for a non-discrete topology τ having the maximum of open sets, it must not contain all the singletons, so, there is an $a \in X$, such that $\{a\} \notin \tau$ .We have to remove all subsets from $P(X)$ such that their intersections give $\{a\}$, those are all $\{a, x\}$, except one, say, $\{a, y\}$. The number of these removed sets is $\binom{n}{k}$. Sets of the form $\{a,\ x_1,\ x_2\}$ are also removed. Their number is $\binom{n-2}{2}$. In general all subsets of the form $\{a, x_1,\ x_2,\ \ldots, x_k\}$, $3 \leq k \leq n-2$ must be removed. Their number is $\binom{n-2}{k}$. Finally, the total number of the removed sets is

$$\sum_{k=0}^{n-2} \binom{n-2}{k} = 2^{n-2}.$$

The remaining elements form a topology having $2^n - 2^{n-2} = 3 \cdot 2^{n-2}$ open sets. ■

The following theorem gives the number of topologies for large k. The notation $(n)_k = n(n-1) \cdots (n-k+1)$ is used.

Theorem 3.2. *(R. Stanley) For $n \geq 5$, we have the following values*

$$\begin{aligned}
T(n, 3 \cdot 2^{n-2}) &= (n)_2 \\
T(n, 5 \cdot 2^{n-3}) &= (n)_3 \\
T(n, 9 \cdot 2^{n-4}) &= \frac{5(n)_5}{6} \\
T(n, 17 \cdot 2^{n-5}) &= \frac{(n)_5}{12} \\
T(n, 15 \cdot 2^{n-5}) &= (n)_5 \\
T(n, 7 \cdot 2^{n-4}) &= \frac{9}{4}(n)_5 + (n)_5 \\
T(n, 2^{n-1}) &= (n)_4 + (n)_3 + \frac{(n)_2}{2}.
\end{aligned}$$

Proof. We give only the proof of the first assertion, which is related to the previous Theorem. The element a is chosen in $\binom{n}{1} = n$ ways. The other one , i.e; $\{a, y\}$, in $\binom{n-1}{1} = (n-1)$ ways. So the total number is $n(n-1)$. ■

Now let $T_0(n,k)$ be the number of labeled T_0 topologies on X having k open sets. This is also the number of labeled posets on X having k ideals. Since a topology is T_0 if and only if it has a minimal base of $n+1$, it follows then that $T_0(n,k) = 0$ for $2 \leq k \leq n$. R. Stanley [3] determined $T_0(n,k)$ for $n+1 \leq k \leq n+3$. We now determine $T_0(n, n+4)$, $T_0(n, n+5)$, $T_0(n, n+6)$:

Theorem 3.3. *We have*

$$T_0(n,n+4) = \frac{(n-3)(n^2+15n+20)}{48}n!, \quad n \geq 3.$$
$$T_0(n,n+5) = \frac{n^4+26n^3+35n^2-478n-248}{384}n!, \ n \geq 4, \ T_0(3,8)=1.$$
$$T_0(n,n+6) = \frac{n^5-15n^4+1885n^3-15265n^2+53954n-97680}{3840}n!, \ n \geq 5$$

Proof. A topology with $n+4$ open sets, on a set of n-element, is T_0 if and only if it contains 3 copies of the graph in the Figure on the right.

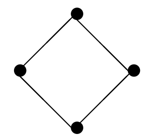
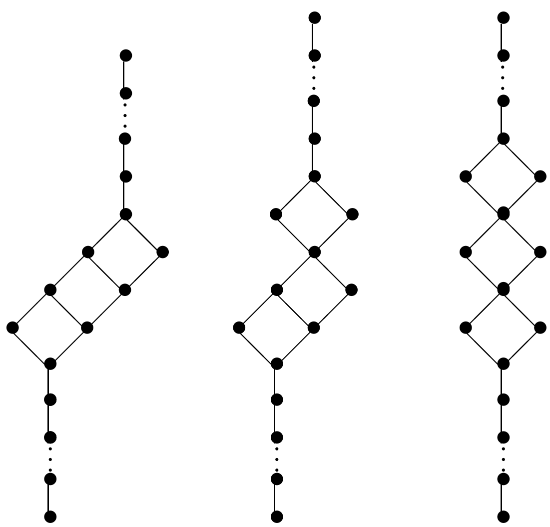


Figure 1

Those are 8 elements, inserted in any place in the chain formed by the remaining elements, as indicated in the following figure:



The total number in the first case is $2(n-3)n!$, in the second case is $(n-3)(n-4)n!/2$, and the total number in the last case is $(n-3)(n-4)(n-5)n!/48$. Summing, we obtain the desired result. Also, for $T_0(n,n+5)$, these topologies are constituted by 4 copies of the graph in Figure 1, or a copy of a boolean algebra having 8 elements as indicated in Figure 2 (note that $T_0(3,8)=1$).

According to the disposition of these copies we have 5 cases: in the first, the number is $\frac{(n+3)(n-4)}{2}n!$, in the second we have $\frac{(2n-9)(n-4)+1}{2}n!$ in the third case the number is $\frac{(n-4)(n-5)(n-6)}{8}n!$. In the fourth case, $\frac{(n-4)(n-5)(n-6)(n-7)}{384}n!$. In the last one, for the topologies having a copy of a boolean algebra of 8 elements, the number is $\frac{(n-2)}{6}n!$. The total number is obtained by summing these numbers in all the previous cases. For $T_0(n,n+6)$, we proceed in the same manner: A topology with $(n+6)$

open sets on an n-element set is T_0 if and only if it contains 5 copies of the graph in Figure 1, or a copy of a boolean algebra with 8 elements and a copy of Figure 2. Note that $T_0(4,10) = 48$. Let $n > 4$. Here too, according to the disposition of the graph in the chain, we have 6 cases: $2(n^2 - 6n + 6)n!$ in the first case . $\dfrac{(n-5)(n-6)(n+1)}{4}n!$ in the second case. The number in the third case is $\dfrac{(n-5)(n^2-12n+38)}{4}n!$. The number in the fourth case is $\dfrac{(n-5)(n-6)(n-7)(n-8)}{192}n!$.The number in the fifth case is $\dfrac{(n-5)(n-6)(n-7)(n-8)(n-9)}{3840}n!$. In the last case , we have a copy of a boolean algebra and a copy of the graph in Figure 1. The total number in this case is $\dfrac{(n^2+5n-12)}{12}n!$. The total number is obtained by computing all topologies in all cases. ■

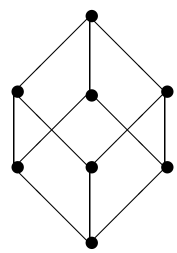


Figure 2

Let $T_0^h(n,k)$ be the number of homeomorphic T_0 topologies with k open sets. From the last theorem, we easily deduce

Theorem 3.4. *We have*

$$\begin{aligned}
T_0^h(n,n+4) &= \frac{(n-3)(n^2-3n+8)}{6},\ n \geq 3.\\
T_0^h(n,n+5) &= \frac{(n-1)(n-3)(n^2-6n+32)}{24},\ n \geq 4,\ T_0^h(3,8) = 1.\\
T_0^h(n,n+6) &= \frac{n^5 - 25n^4 + 345n^3 - 2015n^2 + 5054n - 4320}{120},\ n \geq 5,\ T_0^h(4,10) = 2.
\end{aligned}$$

For small n, we can use the previous results to compute $T(n)$.

$T(3,2) = 1$, $T(3,3) = 6$, $T(3,4) = 9$, $T(3,5) = 6$, $T(3,6) = 6$, $T(3,7) = 0$, $T(3,8) = 1$.

For $n = 4$, we have

$$\begin{aligned}
T(4,2) &= 1,\ T(4,3) = 14,\ T(4,4) = 43,\ T(4,5) = 60,\ T(4,6) = 72,\ T(4,7) = 54\\
T(4,8) &= 54\,,\ T(4,9) = 20,\ T(4,10) = 24,\ T(4,11) = 0,\ T(4,12) = 12,\ T(4,16) = 1\\
T(4,k) &= 0 \ \text{ for } 12 < k < 16.
\end{aligned}$$

So, $T(4) = 355$.

4 Remarks and questions

There are some interesting questions related to the sequence $T(n, k)$: where its maximum is reached? Perhaps it is near $n + k_0$, where k_0 is the integer which maximizes the Stirling numbers of the second kind. Is it true that $T(n, k) \neq 0$, for $2 \leq k \leq 2^{n-2}$. It is easy to prove that $T(n, k) \neq 0$, for $2 \leq k \leq 2n$.

5 Acknowledgments

My sincere thanks to Prof. Y. Attic, for his kind help in drawing the graphs in this paper. I also thank the anonymous referee for his/her helpful comments.

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2000 *Mathematics Subject Classification*: Primary 11B73; Secondary 11B65.
Keywords: Labeled topology, Stirling number .

(Concerned with sequences A000798, A001930, and A008277.)

Received August 29 2005; revised version received May 20 2006. Published in *Journal of Integer Sequences*, May 24 2006.

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