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A Recursive Relation for Weighted Motzkin Sequences

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Abstract

We consider those lattice paths that use the steps *Up*, *Level*, and *Down* with assigned weights w, u, and v. In probability theory, the total weight is 1. In combinatorics, we regard weight as the number of colors and normalize by setting $w = 1$. The lattice paths generate Motzkin sequences. Here we give a combinatorial proof of a three-term recursion for a weighted Motzkin sequence and we find the radius of convergence.

1 Introduction

We consider those lattice paths in the Cartesian plane starting from $(0, 0)$ that use the steps U, L, and D, where $U = (1, 1)$, an up-step; $L = (1, 0)$, a level-step; and $D = (1, -1)$, a down-step. Let c and d be positive integers, and color the L steps with d colors and the D steps with c colors. Let $A(n, k)$ be the set of all colored paths ending at the point (n, k), and let $M(n, k)$ be the set of lattice paths in $A(n, k)$ that never go below the x-axis. Let $a_{n,k} = |A(n, k)|$, $m_{n,k} = |M(n, k)|$, and $m_n = |M(n, 0)|$. The number m_n is called the $(1, d, c)$-Motzkin number. Let $B(n, k)$ denote the set of lattice paths in $A(n, k)$ that never return to the x-axis and let $b_{n,k} = |B(n, k)|$. Note that $a_{n,k} = a_{n-1,k-1} + da_{n-1,k} + ca_{n-1,k+1}$. Here we give a combinatorial proof of the following three-term recursion for the $(1, d, c)$-Motzkin sequence:

$$(n + 2)m_n = d(2n + 1)m_{n-1} + (4c - d^2)(n - 1)m_{n-2}.$$

If $\frac{\sqrt{c}}{2} \leq d$, then $\lim_{n\to\infty} \frac{m_n}{m_{n-1}} = k = d + 2\sqrt{c}$.

Example 1. The first few terms of the $(1,3,2)$-Motzkin numbers are $m_0 = 1, 3, 11, 45, 197, 903, \ldots$, $k = 3+2\sqrt{2}$. This sequence is the little Schroeder number sequence and is Sloane's sequence A001003. Some entries of the matrices $(a_{n,k})$, $(m_{n,k})$ and $(b_{n,k})$ are as follows;

$$(a_{n,k}) = \begin{bmatrix} n/k & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 4 & 12 & 13 & 6 & 1 & 0 & 0 \\ 3 & 0 & 8 & 36 & 66 & 63 & 33 & 9 & 1 & 0 \\ 4 & 16 & 96 & 248 & 360 & 321 & 180 & 62 & 12 & 1 \end{bmatrix},$$

$$(m_{n,k}) = \begin{bmatrix} n/k & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 2 & 11 & 6 & 1 & 0 & 0 \\ 3 & 45 & 31 & 9 & 1 & 0 \\ 4 & 197 & 156 & 60 & 12 & 1 \end{bmatrix},$$

$$(b_{n,k}) = \begin{bmatrix} n/k & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 3 & 1 & 0 & 0 & 0 \\ 3 & 0 & 11 & 6 & 1 & 0 & 0 \\ 4 & 0 & 45 & 31 & 9 & 1 & 0 \\ 5 & 0 & 197 & 156 & 60 & 12 & 1 \end{bmatrix}.$$

Example 2. The first few terms of the $(1,2,1)$-Motzkin numbers are $m_0 = 1, 2, 5, 14, 42, \ldots$, $k = 2 + 2\sqrt{1} = 4$. This sequence is the Catalan sequence, Sloane's sequence A000108.

Example 3. The first few terms of the $(1,1,1)$-Motzkin numbers are $m_0 = 1, 1, 2, 4, 9, 21, \ldots$, $k = 1 + 2\sqrt{1} = 3$. The sequence is the Motzkin sequence, discussed by Woan [5], and is Sloane's sequence A001006.

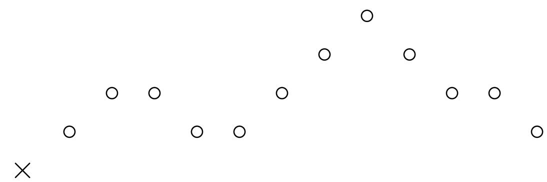
2 Main Results

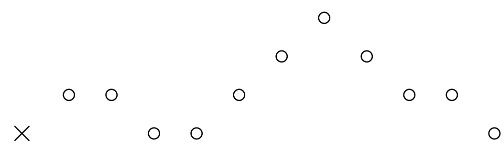
We apply the cut and paste technique to prove the following lemma. Please refer to Dershowitz and Zaks [1] and Pergola and Pinzani [2] for information about the technique.

Lemma 4. *There is a combinatorial proof for the equation* $m_n = b_{n+1,1} = db_{n,1} + cb_{n,2} = dm_{n-1} + cb_{n,2}$.

Proof. Let $P \in B(n+1,1)$. Remove the first step (U) and note that the remaining is in $M(n,0)$. □

For example, the path $P = UULDLUUUDDLD \in B(12,1)$ becomes $Q = ULDLUUUDDLD \in M(11)$ where $\times$ marks the origin.

$P =$  ,

$Q =$  .

Theorem 5. *There is a combinatorial proof for the equation* $(n+1)b_{n+1,1} = a_{n+1,1}$.

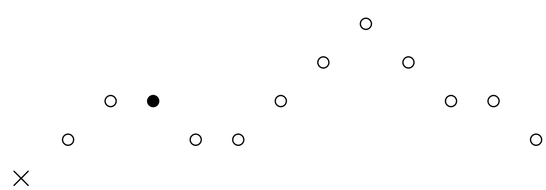
Proof. Wendel [4] proved a similar result. Let

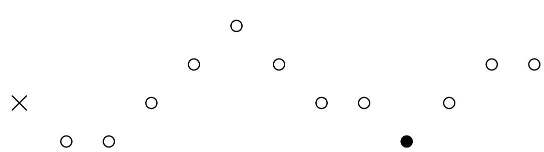
$$S(n+1) = \{P^* : P \in B(n+1,1)$$

where P^* is P with one vertex marked$\}$. Then $|S(n+1)| = (n+1)b_{n+1,1}$. Let $P^* \in S(n+1)$. The marked vertex partitions the path into $P = FB$, where F is the front section and B is the back section. Then $Q = BF \in A(n+1,1)$. Note that, graphically, the point of attachment is the rightmost lowest point of Q.

Conversely, starting with any path we may find the rightmost lowest point of Q and reverse the procedure to create a marked path P^* in $B(n+1,1)$. □

For example,

$P^* =$  $\in S(12)$,

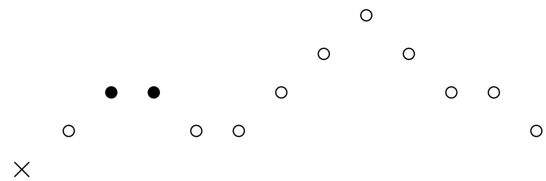
$\rightarrow Q =$  $\in A(12,1)$.

Proposition 6. *The total number of* L *steps in* $M(n)$ *is the same as that in* $B(n+1,1)$ *and is* $da_{n,1}$.

Proof. From the proof of Lemma 4, there is a bijection between $M(n)$ and $B(n+1,1)$. Let $P = FLB \in B(n+1,1)$ with an L step. Then $Q = BF \in A(n,1)$. Note that the attachment point is the rightmost lowest point in Q since $P \in B(n+1,1)$. This identification suggests the inverse mapping. Note that there are d colors for an L step. □

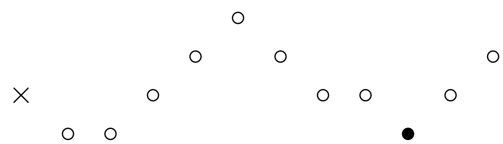
For example,

$P =$



$\in B(12, 1)$,

$\rightarrow Q =$



$\in A(11, 1)$.

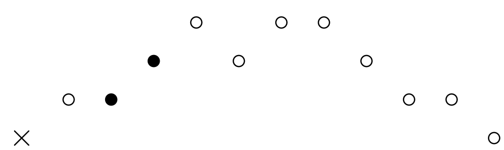
Proposition 7. *There is a combinatorial proof for the equation*

$$a_{n,0} = m_n + \frac{1}{2}(nm_n - da_{n,1}) = b_{n+1,1} + \frac{1}{2}(nb_{n+1,1} - dnb_{n,1}).$$

Proof. Let $T(n) = \{P^e : P \in M(n)$ where P^e is P with a U step marked$\}$. By Theorem 5 and Proposition 6 the number of L steps among all paths in $M(n)$ is $da_{n,1} = dnb_{n,1}$. The total number of steps among all paths in $M(n)$ is $nm_n = nb_{n+1,1}$, hence the total number of U steps among all paths in $M(n)$ is $\frac{1}{2}(nb_{n+1,1} - dnb_{n,1}) = |T(n)|$. Let $P^e = FUB \in T(n)$ with a U step marked. Then $Q = BUF \in A(n, 0) - M(n, 0)$ and the initial point of U in Q is the rightmost lowest point in Q. The inverse mapping starts with the rightmost lowest point. Note that $|M(n, 0)| = m_n = b_{n+1,1}$ □

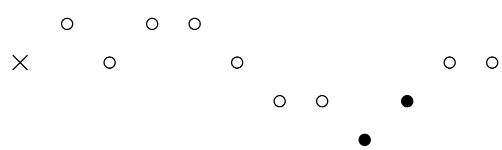
For example,

$P^e =$



$\in T(11)$,

$Q =$



$\in A(11, 0)$.

Proposition 8. *There is a combinatorial proof for the equation*

$$\begin{aligned} a_{n,0} &= a_{n-1,-1} + da_{n-1,0} + ca_{n-1,1} = 2ca_{n-1,1} + da_{n-1,0} \\ &= 2c(n-1)b_{n-1,1} + d(b_{n,1} + \frac{1}{2}((n-1)b_{n,1} - d(n-1)b_{n-1,1}))\,. \end{aligned}$$

Proof. The first equality represents the partition of $A(n, 0)$ by the last step (U, L or D). The second equality represents the fact that $a_{n-1,-1} = ca_{n-1,1}$, since elements in $A(n-1, -1)$ have one more D step than those in $A(n-1, 1)$. And the last equality holds by Theorem 5 and Proposition 7. □

Sulanke [3] proved the following result for the Motzkin sequence.

Theorem 9. $(n+2)m_n = (2dn+d)m_{n-1} + (4c-d^2)(n-1)m_{n-2}$.

Proof. By Propositions 7 and 8

$$b_{n+1,1} + \frac{1}{2}(nb_{n+1,1} - dnb_{n,1}) \quad = \quad 2c(n-1)b_{n-1,1} + db_{n,1} + (\frac{1}{2}((n-1)b_{n,1} - d(n-1)b_{n-1,1}))\,.$$

By Lemma 4

$$m_n + \frac{1}{2}(nm_n - dnm_{n-1}). \quad = \quad 2c(n-1)m_{n-2} + dm_{n-1} + d(\frac{1}{2}((n-1)m_{n-1} - d(n-1)m_{n-2}))\,.$$

Equivalently

$$(n+2)m_n = (2dn+d)m_{n-1} + (4c-d^2)(n-1)m_{n-2}\,.$$

□

Theorem 10. $\lim\limits_{n\to\infty} \frac{m_n}{m_{n-1}} = k = d + 2\sqrt{c}$.

Proof. By Theorem 9, let

$$s_n := \frac{m_n}{m_{n-1}} = \frac{d(2n+1)}{n+2} + (\frac{(4c-d^2)(n-1)}{n+2})\frac{m_{n-2}}{m_{n-1}}\,,$$

and let

$$\begin{aligned} a_n \quad &:= \quad \frac{d(2n+1)}{n+2} = 2d - \frac{3d}{n+2} \\ b_n \quad &:= \quad \frac{(4c-d^2)(n-1)}{n+2} = (4c-d^2)(1 - \frac{3}{n+2}). \end{aligned}$$

Then $s_n = a_{n+\frac{b_n}{s_{n-1}}}$.

If the sequence $s_n = a_{n+\frac{b_n}{s_{n-1}}}$ has limit k, then $k^2 = 2dk + (4c-d^2)$ and $k = \frac{2d+\sqrt{4d^2+4(4c-d^2)}}{2} = d + 2\sqrt{c}$.

Case 1. $4c - d^2 \geq 0$.

If $s_{n-1} \leq k$, then

$$\begin{aligned} s_n \quad &= \quad a_{n+}\frac{b_n}{s_{n-1}} \geq 2d - \frac{3d}{n+2} + \frac{(4c-d^2)\frac{n-1}{n+2}}{k} = k - \frac{3dk + 3(4c-d^2)}{k(n+2)} \\ &= \quad k - \frac{3k^2 - 3dk}{k(n+2)} = k - \frac{3(k-d)}{n+2} \end{aligned}$$

and

$$
\begin{aligned}
s_{n+1} &= a_{n+1} + \frac{b_{n+1}}{s_n} \leq 2d - \frac{3d}{n+3} + \frac{(4c-d^2)(\frac{n}{n+3})}{k - \frac{3(k-d)}{n+2}} \\
&= 2d - \frac{3d}{n+3} + \frac{(4c-d^2)}{k}(1 - \frac{3\ dn + 9d - 3k}{(n+3)(kn-k+3d)}) \\
&= 2d + \frac{(4c-d^2)}{k} - \frac{3d}{n+3} - \frac{(4c-d^2)}{k}\frac{3\ dn + 9d - 3k}{(n+3)(kn-k+3d)} \\
&= k - \frac{3(2dn(k-d) - (k-d)(k-3d))}{(n+3)(kn-k+3d)} \\
&= k - \frac{6(k-d)(d(n+1) - \sqrt{c})}{(n+3)(kn-k+3d)}.
\end{aligned}
$$

Note that $s_1 = \frac{d}{1}$ and $s_2 = \frac{d^2+c}{d} = d + \frac{c}{d}$. If $\frac{\sqrt{c}}{2} \leq d$, then $s_1, s_2 \leq k$. By induction on both odd and even n we have

$$
k - \frac{3(k-d)}{n+3} \leq s_{n+1} \leq k - \frac{6(k-d)(d(n+1) - \sqrt{c})}{(n+3)(kn-k+3d)} \leq k\,.
$$

and $\lim\limits_{n\to\infty} \dfrac{m_n}{m_{n-1}} = k$.

Case 2. $4c - d^2 < 0$.

Inductively, assuming that $s_{n-1} \leq k$ and $\{s_i\}$ is nondecreasing up to $n-1$. Then

$$
\begin{aligned}
s_n &= a_{n+}\frac{b_n}{s_{n-1}} \leq 2d - \frac{3d}{n+2} + \frac{(4c-d^2)\frac{n-1}{n+2}}{k} = k - \frac{3dk + 3(4c-d^2)}{k(n+2)} \\
&= k - \frac{3k^2 - 3dk}{k(n+2)} = k - \frac{3(k-d)}{n+2} < k
\end{aligned}
$$

and

$$\begin{aligned}
s_n - s_{n-1} &= \left(\frac{d(2n+1)}{n+2} + \frac{(4c-d^2)(n-1)}{s_{n-1}(n+2)}\right) - \\
&\quad \left(\frac{d(2n-1)}{n+1} + \frac{(4c-d^2)(n-2)}{s_{n-2}(n+1)}\right) \\
&= d\left(2 - \frac{3}{n+2}\right) + \frac{(4c-d^2)}{s_{n-1}}\left(1 - \frac{3}{n+2}\right) - \\
&\quad \left(d\left(2 - \frac{3}{n+1}\right) + \frac{(4c-d^2)}{s_{n-2}}\left(1 - \frac{3}{n+1}\right)\right) \\
&\geq -\frac{3d}{n+2} + \frac{3d}{n+1} + \frac{(4c-d^2)}{s_{n-1}}\left(-\frac{3}{n+2} + \frac{3}{n+1}\right) \\
&= \frac{3d}{(n+2)(n+1)} + \frac{3(4c-d^2)}{s_{n-1}(n+2)(n+1)} \\
&= \frac{3ds_{n-1} + 3(4c-d^2)}{s_{n-1}(n+2)(n+1)} \geq \frac{12c}{s_{n-1}(n+2)(n+1)} > 0\,.
\end{aligned}$$

By induction $\{s_i\}$ is a bounded nondecreasing sequence and

$$\lim_{n\to\infty} \frac{m_n}{m_{n-1}} = k\,.$$

□

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(Concerned with sequences A000108, A001003, and A001006.)

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