Journal of Integer Sequences, Vol. 8 (2005),
Article 05.5.5

# Sequences That Satisfy $a(n-a(n)) = 0$

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**Abstract**

We explore the properties of some sequences for which $a(n - a(n)) = 0$. Under the natural restriction that $a(n) < n$ the number of such sequences is a Bell number. Adding other natural restrictions yields sequences counted by the Catalan numbers, the Narayana numbers, the triangle of triangular binomial coefficients, and the Schröder numbers.

## 1 Introduction, set partitions

We consider here sequences $a(1), a(2), \ldots, a(n)$ with the property that $a(j - a(j)) = 0$ for all $j = 1, 2, \ldots, n$. Naturally, we must have $1 \leq j - a(j) \leq n$ for $j = 1, 2, \ldots, n$. Let $\mathbf{F}(n)$ be the set of all such sequences, and let $\mathbf{F}(n, m)$ be the subset of those for which $m$ of the $a(j)$ are zero.[3]

**Theorem 1.1.** *For all $1 \leq m \leq n$,*

$$|\mathbf{F}(n, m)| = \binom{n}{m} m^{n-m}.$$

[1]Research supported in part by a NSERC PGS-M scholarship.
[2]Research supported in part by NSERC.
[3]Throughout the paper we will use a bold-case letter, like $\mathbf{X}$ to denote a set of sequences, $\mathbf{X}(n)$ to denote the sequences in $\mathbf{X}$ of length $n$, and $\mathbf{X}(n, m)$ to denote the sequences in $\mathbf{X}(n)$ that contain exactly $m$ zeroes.

*Proof.* There are $\binom{n}{m}$ ways to choose the $m$ indices $J = \{j_1, j_2, \ldots, j_m\}$ for which $a(j) = 0$. Each of the $n - m$ other elements $t$ can take on any value from the $m$-set $\{t - j : j \in J\}$. For such a value of $t$, we have $a(t - a(t)) = a(t - (t - j)) = a(j) = 0$. ☐

The numbers occurring in Theorem 1.1 are not in the OEIS [6] but summing on $m$ yields A000248, the number of labelled forests in which every tree is a star (isomorphic to $K_{1s}$ for some $s$). To get a correspondence with our sequences, let the parent of node $j$ be node $j - a(j)$, with roots regarded as self-parents.

Comtet [1] calls these numbers the "idempotent numbers" (see pp. 91,135). The number of idempotent functions on an $n$-set that have $m$ fixed-points is $\binom{n}{m} m^{n-m}$.

If we take the subset of $\mathbf{F}(n)$ that is closed under the taking of prefixes (or, equivalently, that the $a(j)$ only take on non-negative values), then we get the additional constraint that $a(j) < j$ for $j = 1, 2, \ldots, n$. (Of course, the set of sequences that only satisfy the condition $a(j) < j$ has cardinality $n!$, but our condition is stronger.) Let $\mathbf{A}(n)$ denote the set of all such sequences and let $\mathbf{A}(n, m)$ denote the subset of $\mathbf{A}(n)$ consisting of sequences with exactly $m$ zeroes. Below we list the elements of $\mathbf{A}(n)$ for $n = 1, 2, 3, 4$.

$$\begin{aligned}
\mathbf{A}(1) &= \{0\} \\
\mathbf{A}(2) &= \{00, 01\} \\
\mathbf{A}(3) &= \{000, 001, 002, 010, 012\} \\
\mathbf{A}(4) &= \{0000, 0001, 0002, 0003, 0010, 0012, 0013, 0020, \\
&\quad\; 0022, 0023, 0100, 0101, 0103, 0120, 0123\}
\end{aligned}$$

Note that 011 is missing from $\mathbf{A}(3)$ since then $a(3 - a(3)) = a(3 - 1) = a(2) = 1 \neq 0$. Using the notation of Comtet [1] and Knuth [3], we denote the $n$-th Bell number, A000110, by $\varpi_n$ and Stirling numbers of the second kind, A008277, by $\left\{ {n \atop m} \right\}$.

**Theorem 1.2.** *For all* $0 < m \leq n$,

$$|\mathbf{A}(n)| = \varpi_n \quad \textit{and} \quad |\mathbf{A}(n, m)| = \left\{ {n \atop m} \right\}.$$

*Proof.* Let $j_1, j_2, \cdots, j_m$ be the positions for which $a(j) = 0$. Now define the $i$-th block of a partition to be the set

$$B_i = \{k : k - a(k) = j_i\}.$$

Note that $j_i$ is the smallest element of $B_i$. It should be clear that this specifies a one-to-one correspondence. ☐

**Example:** The sequence that corresponds to the partition $\{1, 3, 4\}, \{2, 5, 8\}, \{6, 7\}$ is $(0, 0, 2, 3, 3, 0, 1, 6)$.

There is a natural pictorial representation of the sequences in $\mathbf{A}(n)$ as what we call a *linear difference diagram*, as shown in Figure 1(i) for the sequence $(0, 0, 0, 3, 2, 4, 0, 1, 2, 7)$. For each value $x \in \{1, 2, \ldots, n\}$, we draw an arc from $x$ to $x - a(x)$, except if $a(x) = 0$. Our condition $a(n - a(n)) = 0$ then translates into the property that each connected component of the underlying graph is a star.

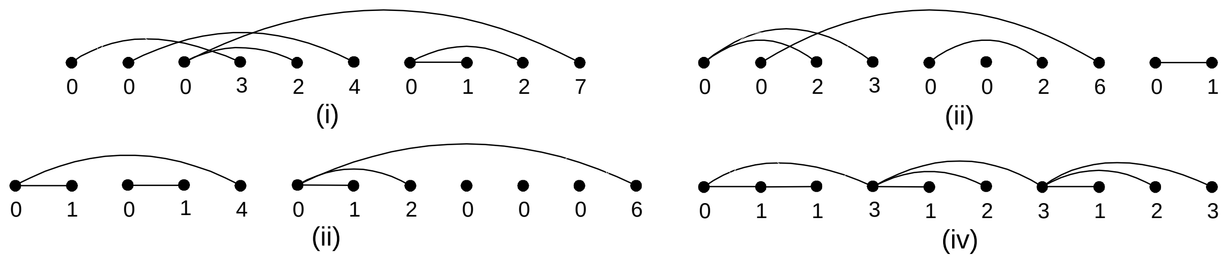


Figure 1: Linear difference diagram representation of an element from: (i) $\mathbf{A}(n)$, (ii) $\mathbf{B}(n)$, (iii) $\mathbf{C}(n)$, and (iv) $\mathbf{D}(n)$.

Define the set $\mathbf{B}(n)$ to be the subset of $\mathbf{A}(n)$ that satisfy the constraint that if $a(j) \neq 0$, then $a(j-1) < a(j)$. We have that $\mathbf{B}(n) = \mathbf{A}(n)$ for $n = 1, 2, 3$ and $\mathbf{B}(4) = \mathbf{A}(4) \setminus \{0022\}$. Let $\mathbf{B}(n, m)$ denote the subset of $\mathbf{B}(n)$ consisting of sequences with exactly $m$ zeroes. The numbers $|\mathbf{B}(n, m)|$ appear in OEIS [6] as sequence A098568 but no combinatorial interpretation is assigned to them. Summing on $m$ gives the sequence A098569.

In terms of set partitions, the set $\mathbf{B}$ corresponds to those in which every element $j$ such that $j$ is not smallest in its block is in a block whose smallest element is no greater than the smallest element of the block containing $j-1$.

**Example** The sequence $(0, 0, 2, 3, 0, 0, 2, 6, 0, 1)$, depicted in Figure 1(ii), is in $\mathbf{B}(n)$.

**Theorem 1.3.** *For all* $1 \leq m \leq n$,

$$|\mathbf{B}(n, m)| = \binom{n - 1 + \binom{m}{2}}{n - m}. \tag{1}$$

*Proof.* Denote $B(n, m) = |\mathbf{B}(n, m)|$. Classify the sequences in $\mathbf{B}(n, m)$ according to the index $k$ of the rightmost zero. The sequences that occur in the first $k-1$ positions are exactly those in $\mathbf{B}(k-1, m-1)$. The values that can go into positions $k+1$ to $n$ must be increasing and can be thought of as a selection with repetition of size $n-k-1$ from the set of positions of the 0's, call them $1 = j_1 < j_2 < \cdots < j_m = k$. Arrange the selection as a nonincreasing sequence $l_{k+1} \geq l_{k+2} \geq \cdots \geq l_n$. Now, if $l_s = j_t$, then set $a(s) = s - j_t$. Note that $a(s - a(s)) = a(j_t) = 0$. Furthermore, $a(s) < a(s+1)$ since $a(s) = s - j_t < s + 1 - j_{t'} = a(s+1)$ where $t' \geq t$. This classification implies that the following recurrence relation holds, with the initial condition that $B(n, 1) = 1$.

$$B(n, m) = \sum_{k=m}^{n} B(k-1, m-1) \binom{n - k + m - 1}{n - k}$$

We will now show that the expression in (1) satisfies this recurrence relation. The following identity is well-known (see Gould [2], equation (3.2)).

$$\binom{x + y + t + 1}{t} = \sum_{j=0}^{t} \binom{j + x}{j} \binom{t - j + y}{t - j} \tag{2}$$

We wish to show that

$$\binom{n-m-1+\binom{m+1}{2}}{n-m} = \sum_{k=m}^{n} \binom{k-m-1+\binom{m}{2}}{k-m}\binom{n-k+m-1}{n-k}.$$

But this is the same as (2) with $j = k-m$, $t = n-m$, $x = \binom{m}{2} - 1$, and $y = m-1$. ☐

# 2 Catalan and Schröder correspondences

Note that the linear difference diagram of Figures 1 (i) and 1 (ii) have crossing arcs. How many such sequences have no crossing arcs?

Define the set $\mathbf{C}(n)$ to be the set of sequences $a(1), a(2), \ldots, a(n)$ for which (a) $0 \leq a(j) < j$ and (b) there is no subsequence such that $i - a(i) < j - a(j) \leq i < j$. Note that $\mathbf{C}(n) = \mathbf{B}(n)$ for $n = 1, 2, 3, 4$ and $\mathbf{C}(5) = \mathbf{B}(5) \setminus \{00203\}$, since $3 - a(3) < 5 - a(5) \leq 3 < 5$. Let $\mathbf{C}(n, m)$ denote the subset of $\mathbf{C}(n)$ consisting of sequences with exactly $m$ zeroes. The numbers $|\mathbf{C}(n, m)|$ appear in OEIS [6] as sequence A001263, the Catalan triangle. Summing on $m$ gives the Catalan numbers A000108.

**Lemma 2.1.** *For all $n \geq 1$, $\mathbf{C}(n) \subseteq \mathbf{A}(n)$.*

*Proof.* Suppose that there is some value $j$ for which $a(j - a(j)) > 0$, and let $i = j - a(j)$. We will show that

$$i - a(i) < j - a(j) = i < j,$$

which will prove the lemma. First note that $a(j) > 0$ since otherwise $a(j - a(j)) = a(j) = 0$. Thus $i < j$. Finally, $i - a(i) = j - a(j) - a(j - a(j)) < j - a(j)$ by our assumption that $a(j - a(j)) > 0$. ☐

**Lemma 2.2.** *For all $n \geq 1$, $\mathbf{C}(n) \subseteq \mathbf{B}(n)$*

*Proof.* If there is some sequence $a(1), a(2), \ldots, a(n)$ that is not in $\mathbf{B}(n)$, then there is some value of $j$ such that $a(j-1) \geq a(j)$ and $a(j) > 0$. Setting $i = j - 1$ we would then have $i - a(i) < j - a(j) \leq i < j$ and so the sequence is not in $\mathbf{C}(n)$ either. ☐

A Dyck path on $2n$ steps is a lattice path in the coordinate plane $(x, y)$ from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$ (*Up*) and $(1, -1)$ (*Down*), never falling below the $x$-axis. Figure 2 shows a typical Dyck path of length 24.

The numbers shown below for $|\mathbf{C}(n, m)|$ are called the Narayana numbers [4]. They count the number of Dyck paths on $2n$ steps with $m$ peaks.

Let $C_n$ denote the $n$-th Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$. The correspondence used in the proof below is mentioned in Stanley [5], problem 6.19($\mathrm{f}^4$).

**Theorem 2.1.**

$$|\mathbf{C}(n)| = C_n \text{ and } |\mathbf{C}(n, m)| = \frac{1}{m}\binom{n-1}{m-1}\binom{n}{m-1}.$$

*Proof.* Considering *Up* steps as left parentheses and *Down* steps as right parentheses, a Dyck path of length $2n$ corresponds to a well-formed parentheses string of equal length. Furthermore, a peak corresponds to a `()` pair.

Let $\mathbf{S}_{2n}$ denote the set of well-formed parenthesis strings of length $2n$. Define the function $f$ from $\mathbf{S}_{2n}$ to $\mathbf{C}(n)$ as $f(s) = (a(1), a(2), \ldots, a(n))$ where $a(j)$ is the number of right parentheses that are properly enclosed by the $j$-th parentheses pair where the parentheses pairs are numbered by the order in which their right parentheses occur. The left parenthesis that matches the $j$-th right parenthesis is referred to as *j's match*. For example, consider $s =$ `((())(()))(((()))()()())`. Subscripting the right parentheses and overlining matching parentheses pairs we obtain

$$\overline{(\overline{(\overline{()}_1)}_2\overline{(\overline{()}_3)}_4)}_5\,\overline{(\overline{(\overline{(\overline{()}_6)}_7)}_8\overline{()}_9\overline{()}_{10}\overline{()}_{11})}_{12}$$

and thus $f(s) = (0, 1, 0, 1, 4, 0, 1, 2, 0, 0, 0, 6)$ (represented as a linear difference diagram in Figure 1(iii)).

We now need to explain why $f(\mathbf{S}_{2n}) \subseteq \mathbf{C}(n)$. Let $s \in \mathbf{S}_{2n}$ and consider $f(s) = (a(1), a(2), \ldots, a(n))$. By definition $a(i) < i$. Furthermore the sequence satisfies the non-crossing property. If it did not, for some $i < j$ we would have that both $j - a(j) \leq i$ and $i - a(i) < j - a(j)$. Notice that $j - a(j)$ is the position of the leftmost right parenthesis to the right of $j$'s match. Now, $j - a(j) \leq i < j$ implies that the $i$-th right parenthesis must lie between $j$ and its match. As well, $i - a(i) < j - a(j)$ implies that the leftmost right parenthesis to the right of $i$'s match is left of the leftmost right parenthesis to the right of $j$'s match. This implies that $i$'s match is left of $j$'s match which contradicts the fact that $s$ is well-formed. Hence $f(s) \in \mathbf{C}(n)$ thus $f(\mathbf{S}_{2n}) \subseteq \mathbf{C}(n)$.

We now show that $f$ is indeed a bijection. Let $b = (b(1), b(2), \ldots, b(n)) \in \mathbf{C}(n)$. We show by induction that it is possible to construct exactly one $s \in \mathbf{S}_{2n}$ such that $f(s) = (a(1), a(2), \ldots, a(n)) = b$. Let $k = 1$ and $s$ be the well-formed parenthesis string `()`. Then $f(s) = (a(1)) = (0) = b(1)$ and $s$ is the only such string. Assume $f(s) = (a(1), a(2), \ldots, a(n)) = (b(1), b(2), \ldots, b(n))$ for some $n \geq 1$ where $s$ is the only such string. Consider $b(n+1)$. If $b(n+1) = 0$ then appending `()` to $s$, in which there is only one way, results in $f(s) = (a(1), a(2), \ldots, a(n), a(n+1)) = (b(1), b(2), \ldots, b(n), b(n+1))$ and $s$ is the only such string. Similarly, if $b(n+1) = n$, then enclosing $s$ within a right and a left parenthesis produces the desired result.

Suppose that $0 < b(n+1) < n$. Consider the substring $s'$ consisting of all elements of $s$ to the right of the $\{n+1-b(n+1)-1\}$-th right parenthesis. If $s'$ is not well-formed then there exists a right parenthesis $i$ with $n+1-b(n+1) \leq i < n+1$ whose match is to the left of the $\{n+1-b(n+1)-1\}$-th right parenthesis. This implies that $i - b(i) \leq n+1-b(n+1)-1$. However then, $i - b(i) < n+1-b(n+1) \leq i < n+1$ which contradicts the fact that $b$ is in $\mathbf{C}(n)$. Therefore $s'$ is well-formed and simply enclosing it in a left and right parentheses pair within $s$ produces the desired result.

Furthermore, note that a zero in an element of $\mathbf{C}(n)$ corresponds to a `()` in an element of $\mathbf{S}_{2n}$ which corresponds to a peak in a Dyck path of length $2n$. Thus $|\mathbf{C}(n, m)|$ is the number of Dyck paths of length $2n$ with $m$ peaks. $\square$

The reader may wonder what happens if we were allowed to have $i - a(i) < j - a(j) = i < j$

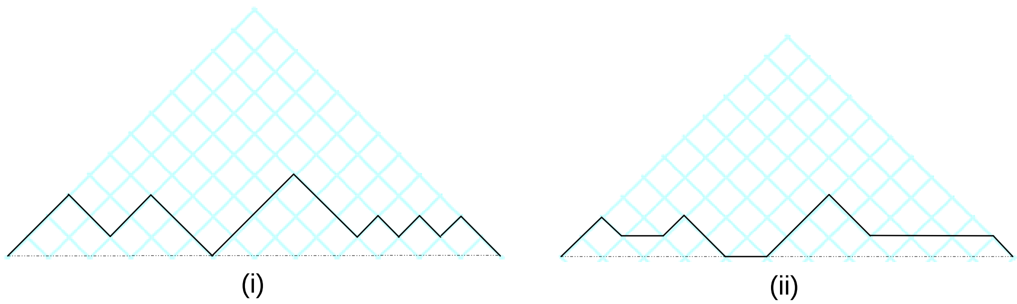


Figure 2: The sequence $(0, 1, 0, 1, 4, 0, 1, 2, 0, 0, 0, 6)$ represented as: (i) a Dyck path of length 24, (ii) a Schröder path of length 11.

but not $i - a(i) < j - a(j) < i < j$. Call the resulting set $\mathbf{D}(n)$. Such sequences need no longer satisfy $a(j - a(j)) = 0$ so strictly speaking are outside the scope of this paper, but the question is interesting nonetheless. They are counted by Schröder numbers.

Let $D_n$ denote, A006318, the $n$-th Schröder number, $D_n = \langle z^n \rangle (1 - z - \sqrt{1 - 6z + z^2})/(2z)$ (see Stanley [4], pg. 178).

**Example** The sequence $(0, 1, 1, 3, 1, 2, 3, 1, 2, 3)$, depicted as a linear difference diagram in Figure 1(iv), is in $\mathbf{D}(n)$ but not $\mathbf{C}(n)$.

A Schröder path is a lattice path in the coordinate plane $(x, y)$ from $(0, 0)$ to $(n, 0)$ with steps $(1, 1)$ (*Up*), $(1, -1)$ (*Down*) and $(1, 0)$ (*Straight*) never falling below the x-axis. The length of a Schröder path is the number of *Up* and *Straight* steps in the path. Figure 2shows a typical Schröder path of length 11.

The numbers $\binom{2n-m-1}{m-1} C_{n-m}$ count the number of Schröder paths from $(0, 0)$ to $(n-1, n-1)$ containing $m - 1$ *Straight* steps (A060693).

**Theorem 2.2.**

$$|\mathbf{D}(n)| = D_{n-1} \textit{ and } |\mathbf{D}(n, m)| = \binom{2n - m - 1}{m - 1} C_{n-m}$$

*Proof.* Let $\mathbf{P}_n$ denote the set of Schröder paths of length $n$. Define the function $g$ from $\mathbf{P}_{n-1}$ to $\mathbf{D}(n)$ as $g(p) = (0, a(1), a(2), \ldots, a(n - 1))$ where $a(j)$ is 0 if the $j$-th counted step is *Straight* or is the number of counted steps (starting with itself) between it and its corresponding *Down* step. For example, let $p$ be the Schröder path shown in Figure 2. Then $g(p) = (0, a(1), a(2), \ldots, a(11)) = (0, 1, 0, 1, 4, 0, 1, 2, 0, 0, 0, 6)$. Notice that $a(i) = 0$ exactly when the $i$-th counted step in $p$ is *Straight*.

We now need to explain why $g(\mathbf{P}_{n-1}) \subseteq \mathbf{D}(n)$. Let $p \in \mathbf{P}_{n-1}$. Consider $g(p) = (0, a(1), a(2), \ldots, a(n - 1)) = (b(1), b(2), \ldots, b(n))$. Since $p$ is a Schröder path $a(i) \leq i$. Since $b(i + i) = a(i)$ we have that $b(i + 1) < i + 1$. Now, suppose that there exists an $1 < i < j \leq n$ such that $i - b(i) < j - b(j) < i < j$. Then there exists some $1 \leq x < y < n$ such that $x - a(x) < y - a(y) < x < y$. Since both $a(y)$ and $a(x)$ must be non-zero to satisfy this inequality, we have that they both count the number of countable steps (beginning with themselves) between them and their respective matches. Now, $x$ lies between $y$ and its match. Furthermore, $y - a(y)$ is the position of the first countable step to the left of $y$'s match. Since $x - a(x) < y - a(y)$, the first countable step to the left of $x$'s match is to the left of the first countable step to the left of $y$'s match which implies that $x$'s match is to the left of $y$'s match. This means that between $y$ and its match there is one more *Up* step then *Down* step thus $y$ and its match are not on the same level contradicting the fact that this is indeed $y$'s match. Hence $g(p) \in \mathbf{D}(n)$ thus $g(\mathbf{P}_{n-1}) \subseteq \mathbf{D}(n)$.

We now show that $g$ is a bijection. Let $b = (b(1), b(2), \ldots, b(n)) \in \mathbf{D}(n)$. We show by induction that it is possible to construct exactly one $p \in \mathbf{P}_{n-1}$ such that $g(p) = (0, a(1), a(2), \ldots, a(n-1)) = b$. Let $k = 2$. If $b(2) = a(1) = 0$ let $p'$ be the Schröder path of length 1 consisting of 1 *Straight* step. Then $g(p') = (0, 0) = (b(1), b(2))$ and there was only one such $p'$. Otherwise let $p'$ be the Schröder path of length 1 consisting of one *Up* step and its match. Then $g(p') = (0, 1) = (b(1), b(2))$ and there was only one such $p'$.

Assume $g(p') = (0, a(1), a(2), \ldots, a(j-1)) = (b(1), b(2), \ldots, b(j))$ for some $j \geq 1$ and $p'$ is the only such path. Consider $b(j+1)$. If $b(j+1) = 0$ then appending a *Straight* step to $p'$, in which there is only one way, results in $g(p') = (0, a(1), a(2), \ldots, a(j)) = (b(1), b(2), \ldots, b(j), b(j+1))$ and $p'$ is the only such string. Similarly, if $b(j+1) = j$ appending an *Up* step to the end $p'$ and placing its match at the front produces the desired result.

Suppose that $0 < b(j+1) < j$. Consider the path $p''$ consisting of all elements of $p'$ to the right of the $j - a(j)$-th countable step. If $p''$ is not a Schröder path then there exists some *Up* step at position $j - a(j) < i < j$ whose match is to the left of the $j - a(j)$-th countable step. However, this implies that the first countable step to the left of $i$'s match is to the left of the first countable step to the left of $j$'s match. This implies that $i - a(i) < j - a(j) < i < j$ and hence $i + 1 - b(i+1) < j + 1 - b(j+1) < i + 1 < j + 1$ contradicting the fact that $b$ is in $\mathbf{D}(n)$. Therefore $p''$ is a Schröder path. Now, within $p'$, simply appending an *Up* step to the end of $p''$ and placing its match at the front of $p''$ produces the desired result.

Furthermore, since a zero in an element of $\mathbf{D}(n)$ in any position other than the first corresponds to a *Straight* step in a Schröder path of length $n-1$, $|\mathbf{D}(n, m)| =$ the number of Schröder paths of length $n-1$ with $m-1$ zeros. □

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2000 *Mathematics Subject Classification*: Primary 05A15; Secondary 68R15.
*Keywords:* Bell number, Catalan number, Schröder number, bijection.

(Concerned with sequences A000108, A000110, A000248, A001263, A006318, A008277, A060693, A098568, and A098569.)

Received March 28 2005; revised version received October 24 2005. Published in *Journal of Integer Sequences*, October 24 2005.

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