Journal of Integer Sequences, Vol. 8 (2005),
Article 05.3.5

A Self-Indexed Sequence

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Abstract

We investigate the integer sequence $(t_n)_{n\in\mathbb{Z}}$ defined by $t_n = 0$ if $n \le 0$, $t_1 = 1$, and $t_n = \sum_{i=1}^{n-1} t_{n-t_i}$ for $n \ge 2$. This sequence has the following properties: if we consider $f_n(X) := -1 + \sum_{i=1}^{n} X^{t_i}$ and take x_n to be the real positive number such that $f_n(x_n) = 0$, then

$$\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = \lim_{n\to\infty} x_n = 0.410098516\cdots$$

Moreover, if u is the real positive number such that $1 = \sum_{i=1}^{\infty} u^{-t_i}$, then there is a positive constant M such that $t_n \sim M u^n$.

1 Definitions and main results

If we look in the Online Encyclopedia of Integer Sequences of N. J. A. Sloane [1], we find a remarkable sequence by $(t_n)_{n\in\mathbb{Z}}$ defined by $t_n = 0$ if $n \le 0$, $t_1 = 1$ and

$$t_n \quad = \quad \sum_{i=1}^{n-1} t_{n-t_i} \tag{1.1}$$

for $n \ge 2$. This sequence is due to Robert Lozyniak (A052109 in Sloane) and is a cousin of the Hofstadter-Conway \$10,000 challenge sequence (A004001 in Sloane).

We find $t_2 = 1, t_3 = 2, t_4 = 5, t_5 = 12, t_6 = 30, t_7 = 73, \ldots$ Observe that if $n \ge 2$, $t_{n+1} = 2t_n + \cdots \ge 2t_n$. Since $t_4 = 4 + 1$, we have

$$t_n \ge n + 1 \quad (n \ge 4) \tag{1.2}$$

The serie $\sum_{i=1}^{\infty} X^{t_i} = 2X + X^2 + \cdots$ converges for $|X| < 1$. Let u be the real positive number such that

$$\sum_{i=1}^{\infty} u^{-t_i} = 1 \tag{1.3}$$

We easily see that $2 < u < 3$. Indeed we have $\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^{t_i} = \frac{1}{2}+\frac{1}{2}+\cdots > 1$ and $\sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^{t_i} < \frac{2}{3}+\frac{1}{9}\sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^{i} = \frac{5}{6}$.

Theorem 1.1. *There exists a positive constant M such that*

$$\lim_{n\to\infty}\frac{t_n}{u^n} = M.$$

Corollary 1.1. *Let $n \geq 2$ be an integer. Let $f_n(X) = -1+\sum_{i=1}^{n} X^{t_i}$ and take x_n the real positive number such that $f_n(x_n) = 0$. Then*

$$\lim_{n\to\infty}\frac{t_n}{t_{n+1}} = \frac{1}{u} = \lim_{n\to\infty} x_n = 0.410098516\cdots.$$

Proof. By the theorem, $t_n \sim Mu^n$, therefore $\lim_{n\to\infty}\frac{t_n}{t_{n+1}} = \frac{1}{u}$.

In addition, the sequence $(x_n)_{n=1}^{\infty}$ is strictly decreasing. Indeed,

$$\sum_{i=1}^{n} x_n^{t_i} = 1 = \sum_{i=1}^{n+1} x_{n+1}^{t_i}$$

so that

$$\sum_{i=1}^{n}\left(x_{n+1}^{t_i} - x_n^{t_i}\right) = -x_{n+1}^{t_{n+1}} < 0$$

This proves that $x_{n+1} < x_n$ for every n. Moreover, this sequence is bounded, so it converges to an element, say v. The function $f_\infty(x) = -1+\sum_{i=1}^{\infty} x^{t_i}$ is continuous. Let $\epsilon > 0$. We have

$$0 < f_\infty(x_n) = \sum_{i=n+1}^{\infty} x_n^{t_i} < \epsilon$$

if n is big enough. Finally,

$$f_\infty(v) = \lim_{n\to\infty} f_\infty(x_n) = 0.$$

That is, $v = \frac{1}{u}$, because $\frac{1}{u}$ is the only positive zero of f_∞. □

2 Proof of the Theorem

We begin with a lemma:

Lemma 2.1. *If $0 \leq v_n < 1$ for every integer $n > 0$, then the following inequalities are equivalent*

$$\sum_{n=1}^{\infty} v_n < \infty \tag{2.1}$$

$$\prod_{n=1}^{\infty}(1-v_n) > 0. \tag{2.2}$$

Proof. Without loss of generality, we can suppose that $v_n \le \frac{1}{2}$. If an infinite number of $v_n > \frac{1}{2}$, the claims 2.1 and 2.2 are both wrong, otherwise we can take out the finite number of $v_n > \frac{1}{2}$. By Taylor expansion, we have for $0 \le x \le \frac{1}{2}$

$$-\log(1-x) = x + \frac{1}{2}\frac{x^2}{(1-\theta x)^2}, \quad 0 \le \theta \le 1;$$

hence $x \le -\log(1-x) \le 2x$ for $0 \le x \le \frac{1}{2}$. It follows that for every integer $N > 0$ we have

$$\sum_{n=1}^{N} v_n \le -\log \prod_{n=1}^{N}(1-v_n) \le 2\sum_{n=1}^{N} v_n.$$

So the lemma is proved. □

Lemma 2.2. *There exists a constant d such that*

$$0 < d \le \frac{t_n}{u^n} \quad \textit{for every integer } n > 0.$$

Proof. Suppose that $n \ge 4$ and d_n is such that $0 < d_n \le \frac{t_k}{u^k}$ for $1 \le k \le n$. Then we have

$$t_{n+1} = \sum_{i=1}^{n} t_{n+1-t_i} \ge \sum_{\substack{i=1\\ n+1-t_i>0}}^{n} d_n u^{n+1-t_i}.$$

By Eq. (1.2), $n+1-t_n \le 0$. Hence

$$t_{n+1} \ge d_n u^{n+1} \sum_{\substack{i=1\\ n+1-t_i>0}}^{n} u^{-t_i} = d_n u^{n+1} \sum_{\substack{i=1\\ n+1-t_i>0}}^{\infty} u^{-t_i} = d_n u^{n+1}(1-v_n)$$

where $v_n = \sum_{i=1,\, t_i \ge n+1}^{\infty} u^{-t_i}$ by definition of u. So we have

$$d_n(1-v_n) \le \frac{t_k}{u^k} \quad \text{for } 1 \le k \le n+1. \tag{2.3}$$

The series

$$\sum_{n=4}^{\infty} v_n = \sum_{n=4}^{\infty} \sum_{\substack{i=1\\ t_i \ge n+1}}^{\infty} u^{-t_i} = \sum_{i=1}^{\infty} u^{-t_i} \sum_{4 \le n < t_i}^{\infty} 1 = \sum_{i=4}^{\infty} u^{-t_i}(t_i - 4)$$

is convergent; just compare this sum with $\frac{1}{(X-1)^2} = \sum_{i=1}^{\infty} iX^{i-1}$ for $|X| < 1$. Set $d_4 = \min_{1 \le k \le 4} \frac{t_k}{u^k}$. By Lemma 2.1 and (2.3), $d = d_4 \prod_{n=4}^{\infty}(1-v_n)$ is such that $0 < d \le \frac{t_n}{u^n}$ for every positive integer n. □

Lemma 2.3. *For every integer n, one has $t_n \le u^{n-1}$.*

Proof. If $n \leq 1$, this is evident. Suppose that $n \geq 1$ and by induction that $t_i \leq u^{i-1}$ when $i \leq n$. Then by (1.1) and (1.3), we have

$$t_{n+1} = \sum_{i=1}^{n} t_{n+1-t_i} \leq \sum_{i=1}^{n} u^{n-t_i} \leq \sum_{i=1}^{\infty} u^{n-t_i} = u^n.$$

$\square$

Remark 2.1. *Let $N \geq 1$. Set $C_N = \sup_{n \geq N} \left(\frac{t_n}{u^n}\right)$ and $D_N = \inf_{n \geq N} \left(\frac{t_n}{u^n}\right)$. Lemmas 2.2 and 2.3 prove that $C = \lim_{N \to \infty} C_N$ and $D = \lim_{N \to \infty} D_N$ with $0 < D \leq C$ are meaningful. Define $(t'_n)_{n \in \mathbb{Z}}$ by $t'_i = 0$ when $i \leq 1$, $t'_2 = 1$ and for $n \geq 3$ define*

$$t'_n = \sum_{i=1}^{\infty} t'_{n-t_i}. \tag{2.4}$$

Define also $(a_n)_{n \in \mathbb{Z}}$ by $a_n = 0$ when $n \leq 0$, $a_1 = 1$ and for $n \geq 2$:

$$a_n = 1 + \sum_{i=1}^{\infty} a_{n-t_i}. \tag{2.5}$$

Lemma 2.4. *Let n, N, k be positive integers such that $n > N \geq 1$. Then we have*

$$t_{n+k} \leq t'_{k+2} t_n + u^{n+k} C_N \left(1 - \frac{t'_{k+2}}{u^k}\right) + a_k u^N.$$

Proof. By induction on k. For $k = 0$, the claim is true. Suppose that $l \geq 1$ and that the lemma is true for $k = 0, 1, \ldots, (l-1)$. By Eq. (1.1),

$$t_{n+l} = \sum_{i=1}^{n+l-1} t_{n+l-t_i} \leq \sum_{i=1}^{\infty} t_{n+l-t_i} = \Sigma_1 + \Sigma_2 + \Sigma_3$$

where

$$\begin{aligned}
\Sigma_1 &= \sum_{\substack{i=1 \\ n+l-t_i \geq n}}^{\infty} t_{n+l-t_i} \\
\Sigma_2 &= \sum_{\substack{i=1 \\ N \leq n+l-t_i < n}}^{\infty} t_{n+l-t_i} \\
\Sigma_3 &= \sum_{\substack{i=1 \\ n+l-t_i < N}}^{\infty} t_{n+l-t_i}.
\end{aligned}$$

By induction we have

$$\Sigma_1 \leq \sum_{\substack{i=1 \\ l-t_i \geq 0}}^{\infty} \left(t'_{l-t_i+2} t_n + u^{n+l-t_i} C_N \left(1 - \frac{t'_{l-t_i+2}}{u^{l-t_i}}\right) + a_{l-t_i} u^N \right).$$

By the definition of C_N

$$\Sigma_2 \leq \sum_{\substack{i=1 \\ N \leq n+l-t_i < n}}^{\infty} C_N u^{n+l-t_i}.$$

By Lemma 2.3 and the fact that $u > 2$,

$$\Sigma_3 \leq \sum_{i=1}^{N} t_i \leq \sum_{i=0}^{N-1} u^i = \frac{u^N - 1}{u-1} < u^N.$$

Finally

$$t_{n+l} \leq t_n \sum_{\substack{i\geq 1 \\ l-t_i \geq 0}} t'_{l-t_i+2} + C_N \sum_{\substack{i\geq 1 \\ n+l-t_i \geq N}} u^{n+l-t_i} - C_N \sum_{\substack{i\geq 1 \\ l-t_i \geq 0}} u^n t'_{l-t_i+2} + \sum_{\substack{i\geq 1 \\ l-t_i \geq 0}} a_{l-t_i} u^N + u^N.$$

This means by (2.4), (2.5) and (1.3) that

$$t_{n+l} \leq t'_{l+2} t_n + u^{n+l} C_N (1 - \frac{t'_{l+2}}{u^l}) + a_l u^N.$$

$\square$

Lemma 2.5. *There exists $d' > 0$ such that $d' \leq \frac{t'_n}{u^n}$ for every $n \geq 2$.*

Proof. Suppose that $d'_n \leq \frac{t'_k}{u^k}$ for every $2 \leq k \leq n$. Then

$$t'_{n+1} = \sum_{i=1}^{\infty} t'_{n+1-t_i} \geq \sum_{i=1,\, n+1-t_i > 1}^{\infty} d'_n u^{n+1-t_i}.$$

So we proved $\frac{t'_{n+1}}{u^{n+1}} \geq d'_n(1 - \sum_{i=1,\, t_i \geq n}^{\infty} u^{-t_i}) = d'_n(1 - v_{n-1})$, with v_n defined as in Lemma 2.2. Hence, $0 < d' = \frac{1}{u^2} \prod_{n\geq 1}(1 - v_n)$ is such that $d' \leq \frac{t'_n}{u^n}$ for every $n \geq 2$. $\square$

Lemma 2.6. *We have $a_m \leq u^m - 1$ for every integer $m > 0$.*

Proof. For $m = 1$, this is true. if $m \geq 2$, we see by induction that

$$a_m = 1 + \sum_{i=1,\, m > t_i}^{\infty} a_{m-t_i} \leq 1 + \sum_{i=1,\, m > t_i}^{\infty} (u^{m-t_i} - 1)$$

Since $\sum_{i=1,\, m>t_i} 1 \geq 2$ and by (1.3),

$$a_m \leq 1 - \sum_{i=1,\, m>t_i}^{\infty} 1 + u^m \sum_{i=1,\, m>t_i}^{\infty} u^{-t_i} < u^m - 1$$

$\square$

Proof of the Theorem. According to Lemmas (2.4), (2.5), (2.6) and the definition of C_N, we have, for $1 \le N < n$ and $k \ge 0$:

$$t_{n+k} \le C_N u^{n+k} - t'_{k+2}(C_N u^n - t_n) + a_k u^N \le C_N u^{n+k} - d' u^{k+2}(C_N u^n - t_n) + u^{N+k}$$

Let $\epsilon > 0$. There exists N such that $C \le C_N < C + \epsilon$, and $n > N$ such that $u^{N-n} < \epsilon$ and $\frac{t_n}{u^n} < D + \epsilon$. In these conditions, we have

$$\frac{t_{n+k}}{u^{n+k}} < C + \epsilon - d' u^2 (C - D - \epsilon) + \epsilon.$$

There exists k such that $\frac{t_{n+k}}{u^{n+k}} > C - \epsilon$. Then $C - \epsilon < C + \epsilon - d' u^2 (C - D - \epsilon) + \epsilon$. Letting ϵ tend to 0 gives $C \le C - d' u^2 (C - D)$. Hence $d' u^2 (C - D) \le 0$. This implies $C \le D$ and thus $C = D$.

3 Acknowledgments

I am grateful to M. Mischler for valuable help and many fruitful discussions during the course of this work.

References

[1] N. J. A. Sloane, Online Encyclopedia of Integer Sequences,
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2000 *Mathematics Subject Classification*: Primary 11Y55.
Keywords: integer sequences.

(Concerned with sequence A052109.)

Received May 16 2005; revised version received July 28 2005. Published in *Journal of Integer Sequences*, August 2 2005.

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