Sequences of Definite Integrals, Factorials and Double Factorials

Thierry Dana-Picard

Department of Applied Mathematics

Jerusalem College of Technology

Havaad Haleumi Str. 21

POB 16031

Jerusalem 91160

Israel

dana@jct.ac.il

Abstract

We study sequences of definite integrals. Some of them provide closed forms involving factorials and/or double factorials. Other examples are associated with either sequences or pairs of sequences of rational numbers, for which summations are found.

1   Introduction.

The study of sequences of either definite or improper integrals has connections with various fields, such as combinatorics (see [6], [2]), infinite series (see [4]), and others. In this paper, we study sequences of integrals, depending mostly on one parameter, sometimes on two parameters. The method generally used is a telescopic method (see [5, p. 579]); when a recurrence relation based on multiplication by a homographic function exists, the implementation of the method is easy. Otherwise it can be very hard, sometimes impossible.

In Section 2, improper integrals depending either on one or two parameters (that are non-negative integers) are considered, where the integrand involves a logarithm. In these examples, various situations are described (different roles and influences of the parameters, existence or non-existence of a closed expression for the general term of the sequence, etc.).

In Section 3, three sequences of integrals are studied; the main one is the sum of a sequence of rational numbers and a sequence of rational multiples of $\sqrt{3}$. Then formulas for related integrals are derived.

1

2 Logarithmic integrals with a parameter.

For every natural number n, we define the improper integral

$$I_n = \int_0^1 x \, (\ln x)^n \, dx. \tag{1}$$

As

$$\lim_{x \to 0^+} x^p \, (\ln x)^q = 0$$

holds for any two positive integers p and q, we can work with I_n as if it is a definite integral, i.e., by writing "ordinary" expressions for the integrals and not writing limits for λ arbitrarily close to 0 of $\int_\lambda^1 x \, (\ln x)^n \, dx$.

We take $u(x) = (\ln x)^n$ and $v(x) = x^2/2$ in order to perform an integration by parts; we have

$$I_n = \left[\frac{1}{2} x^2 \, (\ln x)^n \right]_0^1 - \frac{n}{2} \int_0^1 x \, (\ln x)^{n-1} \, dx,$$

thus,

$$I_n = -\frac{n}{2} I_{n-1}. \tag{2}$$

By telescoping, we have

$$I_n = -\frac{n}{2} I_{n-1} = \left(-\frac{n}{2} \right) \left(-\frac{n-1}{2} \right) I_{n-2} = \cdots = (-1)^n \, \frac{n!}{2^n} \, I_0.$$

A straightforward computation provides $I_0 = 1/2$, and we have proved the following proposition:

Proposition 2.1 *For any natural number n,*

$$\int_0^1 x \, (\ln x)^n \, dx = (-1)^n \, \frac{n!}{2^{n+1}}.$$

Note that the sequence whose general term is $(-1)^n \, 2^{n+1} I_n$ provides an integral representation of factorials.

Now consider the sequence of integrals defined by

$$I_n = \int_0^1 x \, (\ln x)^{n+1/2} \, dx. \tag{3}$$

The natural logarithm is a negative function over the interval $(0, 1)$, thus the value of this integral, if it exists, is a pure imaginary complex number. Actually, the square root function is a multi-valued function with two branches. Each branch is analytic, thus an integral of the form

$$\int_\varepsilon^1 x \, (\ln x)^{n+1/2} \, dx \qquad \text{where} \quad \varepsilon \neq 0,$$

2

is path-independent (in fact, as x is a positive real number, $\ln x < 0$ and the involved path can be a segment on the $y-$axis), when computed along a path which does not intersect the standard branch cut (see [10]). Moreover, for any non-negative integer n, we have

$$\lim_{x \to 0^+} x \, (\ln x)^{n+1/2} = 0.$$

Therefore the given integral I_n is well-defined (we use a method as in [10, p. 362]).

In a manner similar to the method used above, we obtain the following recurrence relation:

$$I_n = -\frac{1}{2}\left(n + \frac{1}{2}\right) I_{n-1} = -\frac{2n+1}{4} I_{n-1}. \tag{4}$$

We need now to compute the first integral of the sequence:

$$I_0 = \int_0^1 x \, (\ln x)^{1/2} \, dx = \frac{i \, \sqrt{2\pi}}{8}.$$

We recall the definition of the double factorial of an odd number (see Sloane's sequence A001147 and [9]):

$$\forall n \in \mathbb{N}, \ (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1),$$

and for an even number

$$\forall n \in \mathbb{N}, \ (2n)!! = 2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n).$$

Therefore the following formula holds:

Proposition 2.2 *For any natural number n:*

$$\int_0^1 x \, (\ln x)^{n+1/2} \, dx = \frac{(-1)^n \, n!!}{2^{2n+3}} \, i\sqrt{2\pi}.$$

Now let p and q be non-negative integers. We define

$$I_{p,q} = \int_0^1 x^p \, (\ln x)^q \, dx. \tag{5}$$

We take $u(x) = (\ln x)^q$ and $v(x) = x^{p+1}/(p+1)$ in order to perform an integration by parts and get

$$I_{p,q} = \underbrace{\left[\frac{1}{p+1} x^{p+1} \, (\ln x)^q\right]_0^1}_{=0} - \frac{q}{p+1} \int_0^1 \frac{1}{x} \cdot x^{p+1} \, (\ln x)^{q-1} \, dx,$$

i.e.,

$$I_{p,q} = -\frac{q}{p+1} \, I_{p,q-1}. \tag{6}$$

3

Now we have

$$I_{p,q} = -\frac{q}{p+1}\, I_{p,q-1}$$
$$= \left(-\frac{q}{p+1}\right) \left(-\frac{q-1}{p+1}\right)\, I_{p,q-2}$$
$$= \left(-\frac{q}{p+1}\right) \left(-\frac{q-1}{p+1}\right) \left(-\frac{q-2}{p+1}\right)\, I_{p,q-3}$$
$$= \ldots$$

As $I_{p,0} = 1/(p+1)$, we have finally

Proposition 2.3 *For any pair (p,q) of natural numbers,*

$$\int_0^1 x^p \, (\ln x)^q \, dx = \frac{(-1)^q \, q!}{(p+1)^{q+1}}.$$

This example of a parametric integral depending on two parameters that are non-negative integers, together with another integral described in [3], shows the great difference between the influences of the parameters: the whole computation is concentrated on one parameter only, and the other one is "passive". Nevertheless, the final result is depends on both parameters.

Remark 2.1 Equation (2.3) is equivalent to

$$q! = (-1)^q \, (p+1)^{q+1} \int_0^1 x^p \, (\ln x)^q \, dx.$$

This integral form for a factorial is surprising, as it contains a parameter without influence.

In the examples studied above, the reason for the computation of a closed form to be so easy lies in the fact that, when performing the integration by parts, the integrated part of the result is equal to 0. This provides a recurrence relation for the sequence (I_n) of the form

$$I_n = f(n) \, I_{n-1}, \tag{7}$$

where f is a homographic function of n with integer coefficients. Other examples of this kind have been studied in [4, 1, 3]. When this situation does not occur, computations can be more complicated, as the next example shows.

For every natural number n, we define the integral

$$I_n = \int_1^e x \, (\ln x)^n \, dx. \tag{8}$$

We have

$$I_0 = \int_1^e x \, dx = \left[\frac{1}{2}x^2\right]_1^e = \frac{1}{2}(e^2 - 1).$$

4

Choosing as above $u(x) = (\ln x)^n$ and $v(x) = x^2/2$, an integration by parts yields

$$I_n = \left[\frac{1}{2} x^2 \, (\ln x)^n \right]_1^e - \frac{n}{2} \int_1^e x \, (\ln x)^{n-1} \, dx$$

and leads to the following recurrence relation:

$$I_n = \frac{1}{2} e^2 - \frac{n}{2} \, I_{n-1}. \tag{9}$$

The presence of a non-zero integrated term makes the work harder than in previous examples. We have

$$
\begin{aligned}
I_n &= \frac{1}{2} \, e^2 - \frac{n}{2} \, \left(\frac{1}{2} e^2 - \frac{n-1}{2} \, I_{n-2} \right) \\
&= \frac{1}{2} \, e^2 \, \left(1 - \frac{n}{2} \right) + \frac{n(n-1)}{4} \, I_{n-2} \\
&= \frac{1}{2} \, e^2 \, \left(1 - \frac{n}{2} \right) + \frac{n(n-1)}{4} \, \left(\frac{1}{2} e^2 - \frac{n-2}{2} \, I_{n-3} \right) \\
&= \frac{1}{2} \, e^2 \, \left(1 - \frac{n}{2} + \frac{n(n-1)}{4} \right) - \frac{n(n-1)(n-2)}{8} \, I_{n-3} \\
&= \ldots \\
&= \frac{1}{2} \, e^2 \, \left(1 - \frac{n}{2} + \frac{n(n-1)}{4} + \cdots + (-1)^{n-1} \frac{n!}{2^{n-1}} \right) + \frac{(-1)^n \, n!}{2^n} \, I_0 \\
&= \frac{1}{2} \, e^2 \, \left(1 - \frac{n}{2} + \frac{n(n-1)}{2^2} - \frac{n(n-1)(n-2)}{2^3} + \cdots + (-1)^{n-1} \frac{n!}{2^{n-1}} + (-1)^n \frac{n!}{2^n} \right) \\
&\quad + (-1)^{n+1} \frac{n!}{2^{n+1}}.
\end{aligned}
$$

Recall that for any two non-negative integers n and k such that $0 \leq k \leq n$,

$$A_n^k = \frac{n!}{(n-k)!}$$

(A_n^k is the number of arrangements without repetition of n elements by k). Hence, the following holds:

Proposition 2.4

$$\int_1^e x \, (\ln x)^n \, dx = \frac{1}{2} \, e^2 \sum_{k=0}^n (-1)^k \, \frac{A_n^k}{2^k} + (-1)^{n+1} \frac{n!}{2^{n+1}}.$$

3 Three related parametric rational integrals

For n a positive integer, we define the integrals

$$I_n = \int_0^1 \frac{1}{(x^2 + x + 1)^n} \, dx, \tag{10}$$

$$J_n = \int_0^1 \frac{x}{(x^2 + x + 1)^n} \, dx, \tag{11}$$

$$K_n = \int_0^1 \frac{x^2}{(x^2 + x + 1)^n} \, dx. \tag{12}$$

3.1 First integral: complete computations

As in the previous examples, we wish to find a recurrence relation for the sequence (I_n), then a closed form for the general term, if possible. We perform an integration by parts; let

$$u(x) = \frac{1}{(x^2 + x + 1)^n} \qquad \text{and} \qquad v(x) = x,$$

whence

$$u'(x) = \frac{-n(2x + 1)}{(x^2 + x + 1)^{n+1}} \qquad \text{and} \qquad v'(x) = 1.$$

It follows that

$$I_n = \left[\frac{x}{(x^2 + x + 1)^n} \right]_0^1 + n \int_0^1 \frac{x(2x + 1)}{(x^2 + x + 1)^{n+1}} \, dx$$

$$= \frac{1}{3^n} + n \underbrace{\int_0^1 \frac{x(2x + 1)}{(x^2 + x + 1)^{n+1}} \, dx}_{=T_n}.$$

In order to compute K_n, we decompose the integrand into partial fractions:

$$\forall x \in \mathbb{R}, \frac{x(2x + 1)}{(x^2 + x + 1)^{n+1}} = -\frac{x}{(x^2 + x + 1)^{n+1}} - \frac{2}{(x^2 + x + 1)^{n+1}} + \frac{2}{(x^2 + x + 1)^n}$$

$$= -\frac{1}{2} \cdot \frac{2x + 1}{(x^2 + x + 1)^{n+1}} - \frac{3}{2} \cdot \frac{1}{(x^2 + x + 1)^{n+1}} + \frac{2}{(x^2 + x + 1)^n}.$$

Thus,

$$T_n = -\frac{1}{2} \left[-\frac{1}{n} \cdot \frac{1}{(x^2 + x + 1)^n} \right]_0^1 - \frac{3}{2} I_{n+1} + 2 I_n$$

$$= \frac{1}{2n} \left(\frac{1}{3^n} - 1 \right) - \frac{3}{2} I_{n+1} + 2 I_n.$$

By re-arranging the terms, we obtain the following relation of recurrence:

$$I_{n+1} = \frac{1}{3n} \left(\frac{1}{3^{n-1}} - 1 \right) + \frac{2(2n - 1)}{3n} I_n. \tag{13}$$

6

We first compute the integral I_1:

$$I_1 = \int_0^1 \frac{1}{x^2 + x + 1} \, dx = \int_0^1 \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \, dx = \int_0^1 \frac{1}{\frac{3}{4}\left[\frac{4}{3}\left(x + \frac{1}{2}\right)^2 + 1\right]} \, dx.$$

Using the substitution $t = (2/\sqrt{3})(x + 1/2)$, we obtain:

$$I_1 = \frac{2}{\sqrt{3}} \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{t^2 + 1} \, dt = \frac{2}{\sqrt{3}}\left(\arctan\sqrt{3} - \arctan\frac{1}{\sqrt{3}}\right) = \frac{\pi\sqrt{3}}{9}. \tag{14}$$

From Equations (13) and (14) follows that I_n is given by a relation of the form

$$I_n = a_n + b_n \pi \sqrt{3} \tag{15}$$

where a_n and b_n are rational numbers. We study separately the sequences (a_n) and (b_n).
Consider b_n first:

$$\begin{aligned}
b_{n+1} &= \frac{2(2n-1)}{3n} \, b_n = \frac{2}{3} \cdot \frac{2n-1}{n} \, b_n \\
&= \left(\frac{2}{3}\right)^2 \cdot \frac{2n-1}{n} \cdot \frac{2n-3}{n-1} \, b_{n-1} \\
&= \left(\frac{2}{3}\right)^3 \cdot \frac{2n-1}{n} \cdot \frac{2n-3}{n-1} \cdot \frac{2n-5}{n-2} \, b_{n-2} \\
&= \ldots \\
&= \left(\frac{2}{3}\right)^n \cdot \frac{(2n-1)(2n-3)(2n-5)\ldots 3 \cdot 1}{n(n-1)(n-2)\ldots 2 \cdot 1} \, b_1.
\end{aligned}$$

Inserting suitable factors into the numerator, and dividing out by the same factors, we obtain a closed factorial form for b_{n+1}:

$$b_{n+1} = \left(\frac{2}{3}\right)^n \cdot \frac{(2n)!}{2^n \cdot (n!)^2} \, b_1 = \frac{1}{3^n} \cdot \frac{(2n)!}{(n!)^2} \, b_1 = \frac{1}{3^{n+2}} \cdot \frac{(2n)!}{(n!)^2}. \tag{16}$$

Note that

$$\frac{(2n)!}{(n!)^2} = \binom{2n}{n}.$$

The interested reader will find concrete occurences of these numbers (special paths in graphs, etc.) in Sloane's encylopedia, sequence A000984.

Another representation can be given for the sequence (b_n), using the *double factorial* (see Sloane's encyclopedia, A001147 and [9]). We have

$$\forall n \in \mathbb{N}, \ b_n = \frac{2^{n-1}}{3^{n+1}} \cdot \frac{(2n-3)!!}{(n-1)!}. \tag{17}$$

A closed form for a_n is harder to derive. From Equation (13), we derive the following recurrence relation for a_n:

$$a_{n+1} = \frac{1}{3n}\left(\frac{1}{3^{n-1}} - 1\right) + \frac{2(2n-1)}{3n}\, a_n. \tag{18}$$

By Equation (14), $a_1 = 0$, whence the sequence (a_n) is well defined by the above relation.

Let's now use a telescopic process:

$$
\begin{aligned}
a_{n+1} &= \frac{1}{3n}\left(\frac{1}{3^{n-1}} - 1\right) + \frac{2(2n-1)}{3n}\, a_n \\
&= \frac{1}{3n}\left(\frac{1}{3^{n-1}} - 1\right) + \frac{2(2n-1)}{3n}\left[\frac{1}{3(n-1)}\left(\frac{1}{3^{n-2}} - 1\right) + \frac{2(2n-3)}{3(n-1)}a_{n-1}\right] \\
&= \frac{1}{3n}\left(\frac{1}{3^{n-1}} - 1\right) + \frac{2}{3^2}\frac{(2n-1)}{n(n-1)}\left(\frac{1}{3^{n-2}} - 1\right) + \left(\frac{2}{3}\right)^2\frac{(2n-1)(2n-3)}{n(n-1)}a_{n-1} \\
&= \frac{1}{3n}\left(\frac{1}{3^{n-1}} - 1\right) + \frac{2}{3^2}\frac{(2n-1)}{n(n-1)}\left(\frac{1}{3^{n-2}} - 1\right) \\
&\quad + \left(\frac{2}{3}\right)^2\frac{(2n-1)(2n-3)}{n(n-1)}\left[\frac{1}{3(n-2)}\left(\frac{1}{3^{n-3}} - 1\right) + \frac{2(2n-5)}{3(n-2)}a_{n-2}\right] \\
&= \dots \quad .
\end{aligned}
$$

Iterations are needed until a_2 is reached, because $a_2 = a_1 = 0$. Finally, the following formula is derived:

$$a_{n+1} = \frac{1}{3n}\left(\frac{1}{3^{n-1}} - 1\right) + \sum_{k=2}^{n-2} 2^{k-1}\left(3^{-n} - 3^{-k}\right)\frac{(2n-1)(2n-3)\dots(2n-2k+3)}{n(n-1)(n-2)\dots(n-k+1)}.$$

The rational fraction on the right can be turned into a closed factorial formula. Shifting the index $n+1$ to n, we obtain

$$a_n = \frac{1}{3(n-1)}\left(\frac{1}{3^{n-2}} - 1\right) + \frac{1}{2}\sum_{k=2}^{n-3}\left(3^{-n+1} - 3^{-k}\right)\frac{(2(n-1))!}{(2n-2k-1)!}\left(\frac{(n-k-1)!}{(n-1)!}\right)^2. \tag{19}$$

A formula involving double factorials looks a little more compact (see Sloane's sequence A001147 and [9]):

$$a_n = \frac{1}{3(n-1)}\left(\frac{1}{3^{n-2}} - 1\right) + \frac{1}{2}\sum_{k=2}^{n-3} 2^k\left(3^{-n+1} - 3^{-k}\right)\frac{(2n-3)!!\,(n-k-1)!}{(2n-2k-1)!!\,(n-1)!}. \tag{20}$$

In conclusion, we have

Proposition 3.1 *For any non-negative integer n,*

$$\int_0^1 \frac{dx}{(x^2+x+1)^n} = a_n + b_n\,\pi\,\sqrt{3},$$

8

with

$$a_n = \frac{1}{3(n-1)} \left(\frac{1}{3^{n-2}} - 1 \right) + \frac{1}{2} \sum_{k=2}^{n-3} 2^k \left(3^{-n+1} - 3^{-k} \right) \frac{(2n-3)!! \, (n-k-1)!}{(2n-2k-1)!! \, (n-1)!}$$

and

$$b_n = \frac{2^{n-1}}{3n+1} \cdot \frac{(2n-3)!!}{(n-1)!}.$$

3.2   Extensions

From the results above, the two related parametric integrals J_n and K_n can be computed:

We have

$$\begin{aligned} I_n + 2J_n &= \int_0^1 \frac{1+2x}{(x^2+x+1)^n} \, dx \\ &= \frac{1}{1-n} \left[\frac{1}{(x^2+x+1)^{n+1}} \right]_0^1 \\ &= \frac{1}{1-n} \left(\frac{1}{3^{n-1}} - 1 \right), \end{aligned}$$

i.e.,

$$J_n = \frac{1}{2(1-n)} \left(\frac{1}{3^{n-1}} - 1 \right) - \frac{1}{2} \, I_n. \tag{21}$$

An expression of J_n as a function of n is obtained by substitution.

A closed form for K_n is obtained by substitution, according to the following remark:

$$I_n + J_n + K_n = \int_0^1 \frac{1+x+x^2}{(x^2+x+1)^n} \, dx = \int_0^1 \frac{1}{(x^2+x+1)^{n-1}} \, dx = I_{n-1}. \tag{22}$$

4   Acknowledgements

The author wishes to thank the referee for his remarks and appreciations, and the editor for his care, including long-distance phone calls.

References

[1]   Th. Dana-Picard, Explicit closed forms for parametric integrals, *Internat. J. Math. Educ. in Science and Technology* **35** (2004), 456–467.

[2]   Th. Dana-Picard, Parametric integrals and Catalan numbers, *Internat. J. Math. Educ. in Science and Technology* **36** (2005), 410–414.

[3]   Th. Dana-Picard, Sequences of definite integrals, preprint, 2004.

[4]   P. Glaister, Factorial sums, *Internat. J. Math. Educ. in Science and Technology* **34** (2003), 250–257.

[5]   H. Johnston and J. Mathews, *Calculus*, Addison-Wesley, 2002.

[6]   B. Sury, T. Wang and F.-Z. Zhao, Identities involving reciprocal of binomial coefficients *J. Integer Sequences* **7** (2004), Article 04.2.8.

[7]   N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences.

[8]   I. M. Vinogradov, *Elements of Number Theory*, 5th rev. ed., Dover, New York, 1954.

[9]   E. W. Weisstein, *Double Factorial*, MathWorld.

[10] A. David Wunsch, Complex Variables with Applications, 2nd edition, Addison-Wesley, Reading, Massachussetts, 1994.

(Concerned with sequences A000984 and A001147.)

Return to Journal of Integer Sequences home page.