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# Partition Statistics and $q$-Bell Numbers ($q = -1$)

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**Abstract**

We study three partition statistics and the $q$-Stirling and $q$-Bell numbers that serve as their generating functions, evaluating these numbers when $q = -1$. Among the numbers that arise in this way are (1) Fibonacci numbers and (2) numbers occurring in the study of fermionic oscillators.

## 1 Introduction

The notational conventions of this paper are as follows: $\mathbb{N} := \{0, 1, 2, \ldots\}$, $\mathbb{P} := \{1, 2, \ldots\}$, $[0] := \varnothing$, and $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{P}$. Empty sums take the value 0 and empty products the value 1, with $0^0 := 1$. The letter $q$ denotes an indeterminate, with $0_q := 0$, $n_q := 1 + q + \cdots + q^{n-1}$ for $n \in \mathbb{P}$, $0\underset{q}{!} := 1$, and $n\underset{q}{!} := 1_q 2_q \cdots n_q$ for $n \in \mathbb{P}$. The binomial coefficient $\binom{n}{k}$ is equal to zero if $k$ is a negative integer or if $0 \leqslant n < k$.

Let $\Delta$ be a finite set of discrete structures, with $I : \Delta \to \mathbb{N}$. The generating function

$$G(I, \Delta; q) := \sum_{\delta \in \Delta} q^{I(\delta)} = \sum_{k} |\{\delta \in \Delta : \ I(\delta) = k\}| \, q^k$$

is a useful tool for studying the statistic $I$. Elementary examples include the binomial theorem,

$$(1 + q)^n = \sum_{S \subset [n]} q^{|S|} = \sum_{k=0}^{n} \binom{n}{k} q^k, \tag{1}$$

and

$$n\underset{q}{!} = \sum_{\sigma \in \mathcal{S}_n} q^{i(\sigma)}, \tag{2}$$

where $\mathcal{S}_n$ is the set of permutations of $[n]$ and $i(\sigma)$ is the number of inversions in the permutation $\sigma = i_1 i_2 \dots i_n$, i.e., the number of pairs $(r, s)$ with $1 \leqslant r < s \leqslant n$ and $i_r > i_s$ [7, Corollary 1.3.10].

Of course, $G(I, \Delta; 1) = |\Delta|$. On the other hand,

$$G(I, \Delta; -1) = |\{\delta \in \Delta : \ I(\delta) \text{ is even}\}| - |\{\delta \in \Delta : \ I(\delta) \text{ is odd}\}| . \tag{3}$$

Hence if $G(I, \Delta; -1) = 0$, the set $\Delta$ is "balanced" with respect to the parity of $I$. In particular, setting $q = -1$ in (1) yields the familiar result that a finite nonempty set has as many subsets of odd cardinality as it has subsets of even cardinality. Setting $q = -1$ in (2) reveals that if $n \geqslant 2$, then among the permutations of $[n]$ there are as many with an odd number of inversions as there are with an even number of inversions.

In this note we consider three $q$-generalizations of Stirling numbers of the second kind, denoted $S_q^*(n, k)$, $S_q(n, k)$, and $\tilde{S}_q(n, k)$. These polynomials are generating functions for three closely related statistics on the set of partitions of $[n]$ with $k$ blocks. Most of the properties of these $q$-Stirling numbers, to be established below in § 3, have appeared in the literature in various contexts, Carlitz [1] having apparently been the first to construe these numbers as generating functions for partition statistics. See also [3], [4], [8], and [9]. Our aim here is to offer a compact, unified treatment of these numbers. Our analysis is facilitated by a powerful formal algebraic result of Comtet [2].

We derive in § 4 new results on the evaluation of $S_q^*(n, k)$, $S_q(n, k)$, and $\tilde{S}_q(n, k)$ and their associated $q$-Bell numbers (gotten by summing $q$-Stirling numbers over $k$ for fixed $n$) when $q = -1$. Apart from the interpretation of these results in terms of (3), the evaluation of $S_{-1}(n, k)$ and its associated Bell numbers may be of additional interest, since these numbers arise in the study of fermionic oscillators [6].

In § 5 we discuss an alternative approach to establishing our results by means of bijective proofs.

In § 6 the numbers $S_{-1}^*(n, k)$, $S_{-1}(n, k)$, and $\tilde{S}_{-1}(n, k)$ are displayed as triangular arrays for $1 \leqslant k \leqslant n \leqslant 8$. Here, for immediate reference, we record these arrays in linearized form:

$$\begin{aligned}
S_{-1}^*(n,k) = \ & -1,\ 1,\ -1,\ -1,\ 1,\ 1,\ 1,\ -1,\ -2,\ 1,\ -1,\ 1,\\
& 3,\ -2,\ -1,\ 1,\ -1,\ -4,\ 3,\ 3,\ -1,\ -1,\ 1,\ 5,\\
& -4,\ -6,\ 3,\ 1,\ 1,\ -1,\ -6,\ 5,\ 10,\ -6,\ -4,\ 1,\ \dots\\
S_{-1}(n,k) = \ & 1,\ 1,\ -1,\ 1,\ -1,\ -1,\ 1,\ -1,\ -2,\ 1,\ 1,\ -1,\\
& -3,\ 2,\ 1,\ 1,\ -1,\ -4,\ 3,\ 3,\ -1,\ 1,\ -1,\ -5,\\
& 4,\ 6,\ -3,\ -1,\ 1,\ -1,\ -6,\ 5,\ 10,\ -6,\ -4,\ 1,\ \dots\\
\tilde{S}_{-1}(n,k) = \ & 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 2,\ 1,\ 1,\ 1,\\
& 3,\ 2,\ 1,\ 1,\ 1,\ 4,\ 3,\ 3,\ 1,\ 1,\ 1,\ 5,\\
& 4,\ 6,\ 3,\ 1,\ 1,\ 1,\ 6,\ 5,\ 10,\ 6,\ 4,\ 1,\ \dots
\end{aligned}$$

# 2 Preliminaries

This section reviews some material to be used later in the paper.

2.1. *Comtet Numbers*. The following theorem, due to Comtet [2] greatly facilitates the analysis of many combinatorial arrays:

**Theorem 2.1.** *Let $D$ be an integral domain. If $(u_n)_{n\geqslant 0}$ is a sequence in $D$ and $x$ is an indeterminate over $D$, then the following are equivalent characterizations of an array $(U(n,k))_{n,k\geqslant 0}$:*

$$U(n,k) = U(n-1,k-1) + u_k U(n-1,k), \qquad \forall\ n,k \in \mathbb{P}, \tag{4}$$

*with $U(n,0) = u_0^n$ and $U(0,k) = \delta_{0,k}$ $\forall\ n,k \in \mathbb{N}$,*

$$U(n,k) = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i\in\mathbb{N}}} u_0^{d_0} u_1^{d_1} \cdots u_k^{d_k}, \qquad \forall\ n,k \in \mathbb{N}, \tag{5}$$

$$\sum_{n\geqslant 0} U(n,k)x^n = \frac{x^k}{(1-u_0x)(1-u_1x)\cdots(1-u_kx)}, \qquad \forall\ k \in \mathbb{N}, \tag{6}$$

*and*

$$x^n = \sum_{k=0}^{n} U(n,k)p_k(x), \qquad \forall\ n \in \mathbb{N}, \tag{7}$$

*where $p_0(x) := 1$ and $p_k(x) := (x-u_0)\cdots(x-u_{k-1})$ for $k \in \mathbb{P}$.*

**Proof.** Straightforward algebraic exercise. □

In what follows, we call the numbers $U(n,k)$ the *Comtet numbers associated with the sequence* $(u_n)_{n\geqslant 0}$.

2.2 *Partitions of a Set.* A *partition* of a set $S$ is a set of nonempty, pairwise disjoint subsets (called *blocks*) of $S$, with union $S$. For all $n,k \in \mathbb{N}$, let $S(n,k)$ denote the number of partitions of $[n]$ with $k$ blocks. Then $S(0,0) = 1$, $S(n,0) = S(0,k) = 0$, $\forall\ n,k \in \mathbb{P}$, and

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \qquad \forall\ n,k \in \mathbb{P}, \tag{8}$$

$S(n-1,k-1)$ enumerating those partitions in which $n$ is the sole element of one of the blocks, and $kS(n-1,k)$ those in which the block containing $n$ contains at least one other element of $[n]$. From (8) it follows that the numbers $S(n,k)$, called *Stirling numbers of the second kind*, are the Comtet numbers associated with the sequence $(0,1,2,\dots)$. Hence by Theorem 2.1

$$S(n,k) = \sum_{\substack{d_1+\cdots+d_k=n-k \\ d_i\in\mathbb{N}}} 1^{d_1} 2^{d_2} \cdots k^{d_k}, \qquad \forall\ n,k \in \mathbb{N},$$

$$\sum_{n\geqslant 0} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}, \qquad \forall\ k \in \mathbb{N},$$

and

$$x^n = \sum_{k=0}^{n} S(n,k)x^{\underline{k}}, \qquad \forall\ n \in \mathbb{N},$$

where $x^{\underline{0}} := 1$ and $x^{\underline{k}} := x(x-1)\cdots(x-k+1)$ for $k \in \mathbb{P}$.

The total number of partitions of $[n]$ is given by the *Bell number* $B_n$, where

$$B_n = \sum_{k=0}^{n} S(n,k).$$

Clearly, $B_0 = 1$, and

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k,$$

since $\binom{n}{k} B_k$ enumerates those partitions of $[n+1]$ for which the size of the block containing the element $n+1$ is $n-k+1$.

2.3 *Restricted Sums of Binomial Coefficients.* As we have already noted in § 1, setting $q = 1$ and $q = -1$ in (1) yields the well known result

$$\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k} = 2^{n-1}, \qquad \forall\, n \in \mathbb{P}.$$

Here we recall a method for evaluating sums such as

$$\sum_{k \equiv 0 \pmod 3} \binom{n}{k}.$$

Let $\omega$ be either of the two complex cube roots of 1, e.g., $\omega = (-1 + i\sqrt{3})/2$. Then

$$\begin{aligned}(1+x)^n + (1+\omega x)^n + (1+\omega^2 x)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k \left(1 + \omega^k + \omega^{2k}\right) \\ &= 3 \sum_{k \equiv 0 \pmod 3} \binom{n}{k} x^k, \quad (9)\end{aligned}$$

since $k \equiv 0 \pmod 3$ implies that $1 + \omega^k + \omega^{2k} = 3$ and $k \equiv 1$ or $2 \pmod 3$ implies that $1 + \omega^k + \omega^{2k} = 1 + \omega + \omega^2 = 0$. Setting $x = 1$ in (9) yields

$$\sum_{k \equiv 0 \pmod 3} \binom{n}{k} = \frac{1}{3}\left(2^n + (1+\omega)^n + (1+\omega^2)^n\right). \qquad (10)$$

# 3 Partition Statistics and $q$-Stirling Numbers

Let $\Pi(n,k)$ denote the set of all partitions of $[n]$ with $k$ blocks. Given a partition $\pi \in \Pi(n,k)$, let $(E_1, \ldots, E_k)$ be the unique *ordered* partition of $[n]$ comprising the same blocks as $\pi$, arranged in increasing order of their smallest elements, and define statistics $w^*$, $w$, and $\tilde{w}$ by

$$\begin{aligned} w^*(\pi) &:= \sum_{i=1}^{k} i|E_i|, \\ w(\pi) &:= \sum_{i=1}^{k} (i-1)|E_i| = w^*(\pi) - n, \end{aligned}$$

and

$$\tilde{w}(\pi) := \sum_{i=1}^{k}(i-1)(|E_i|-1) = w^*(\pi) - n - \binom{k}{2}.$$

If elements of $[n]$ are regarded as labels on $n$ unit masses, then $w^*(\pi)$ is the moment about $x=0$ of the mass configuration in which the masses with labels in $E_i$ are placed at $x=i$. The statistics $w(\pi)$ and $\tilde{w}(\pi)$ admit of similar interpretations.

We wish to study the generating functions

$$S_q^*(n,k) := \sum_{\pi \in \Pi(n,k)} q^{w^*(\pi)}, \tag{11}$$

$$S_q(n,k) := \sum_{\pi \in \Pi(n,k)} q^{w(\pi)} = q^{-n} S_q^*(n,k), \tag{12}$$

and

$$\tilde{S}_q(n,k) := \sum_{\pi \in \Pi(n,k)} q^{\tilde{w}(\pi)} = q^{-n-\binom{k}{2}} S_q^*(n,k). \tag{13}$$

Each of these polynomials furnishes a $q$-generalization of $S(n,k)$, reducing to the latter when $q=1$. As closely related as these $q$-Stirling numbers appear to be, it might be thought that one could carry out an analysis of any one of them, chosen arbitrarily, with properties of the others derived as easy corollaries. Interestingly, it turns out that each is best suited for elucidating a particular subset of their more or less common properties. We consider first the matter of recursive formulas.

**Theorem 3.1.** *The $q$-Stirling numbers $S_q^*(n,k)$ are generated by the recurrence relation*

$$S_q^*(n,k) = q^k S_q^*(n-1,k-1) + qk_q S_q^*(n-1,k), \qquad \forall\ n,k \in \mathbb{P}, \tag{14}$$

*with $S_q^*(0,0)=1$ and $S_q^*(n,0) = S_q^*(0,k) = 0$, $\forall\ n,k \in \mathbb{P}$.*

**Proof.** The boundary conditions are obvious. To establish the recurrence (14), let

$$c(n,k,t) := |\{\pi \in \Pi(n,k) :\ w^*(\pi) = t\}|.$$

Then,

$$c(n,k,t) = c(n-1,k-1,t-k) + \sum_{i=1}^{k} c(n-1,k,t-i), \qquad \forall\ n,k \in \mathbb{P}. \tag{15}$$

For if $w^*(\pi)=t$, with $(E_1,\ldots,E_k)$ being the ordered partition associated with $\pi$, then the number $n \in [n]$ is either (i) in $E_k$ alone (there are clearly $c(n-1,k-1,t-k)$ such $\pi$'s) or (ii) in some $E_i$, where $1 \leqslant i \leqslant k$, with at least one element of $[n-1]$ (there are clearly $c(n-1,k,t-i)$ such $\pi$'s). From (15) it follows that

$$\begin{aligned} S_q^*(n,k) &= \sum_t c(n,k,t)q^t \\ &= \sum_r c(n-1,k-1,r)q^{r+k} + \sum_{i=1}^{k} q^i \sum_r c(n-1,k,r)q^r \\ &= q^k S_q^*(n-1,k-1) + qk_q S_q^*(n-1,k). \end{aligned}$$

□

Recurrence relations for $S_q(n,k)$ and $\tilde{S}_q(n,k)$ follow immediately from (14), along with (12) and (13), respectively. We have

$$S_q(n,k) = q^{k-1} S_q(n-1,k-1) + k_q S_q(n-1,k), \qquad \forall\, n,k \in \mathbb{P}, \tag{16}$$

and

$$\tilde{S}_q(n,k) = \tilde{S}_q(n-1,k-1) + k_q \tilde{S}_q(n-1,k), \qquad \forall\, n,k \in \mathbb{P}. \tag{17}$$

By (17), the numbers $\tilde{S}_q(n,k)$ are the Comtet numbers associated with the sequence $(n_q)_{n \geqslant 0}$. By Theorem 2.1 it follows immediately that

$$\tilde{S}_q(n,k) = \sum_{\substack{d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} (1_q)^{d_1}(2_q)^{d_2}\cdots(k_q)^{d_k}, \qquad \forall\, n,k \in \mathbb{N}, \tag{18}$$

$$\sum_{n \geqslant 0} \tilde{S}_q(n,k) x^n = \frac{x^k}{(1-1_q x)(1-2_q x)\cdots(1-k_q x)}, \qquad \forall\, k \in \mathbb{N}, \tag{19}$$

and

$$x^n = \sum_{k=0}^{n} \tilde{S}_q(n,k)\phi_k(x), \qquad \forall\, n \in \mathbb{N}, \tag{20}$$

where $\phi_0(x) := 1$ and $\phi_k(x) := x(x-1_q)\cdots(x-(k-1)_q)$, $\forall\, k \in \mathbb{P}$.

Variants of (18)–(20) that hold for $S_q(n,k)$ and $S_q^*(n,k)$ follow immediately from the relations $S_q(n,k) = q^{\binom{k}{2}} \tilde{S}_q(n,k)$ and $S_q^*(n,k) = q^n S_q(n,k)$. To cite a few examples, we have

$$\sum_{n \geqslant 0} S^*(n,k) x^n = \frac{q^{\binom{k+1}{2}} x^k}{(1-qx)(1-qx-q^2x)\cdots(1-qx-\cdots-q^k x)}, \qquad \forall\, k \in \mathbb{N},$$

and

$$x^n = \sum_{k=0}^{n} S_q(n,k)\psi_k(x) = \sum_{k=0}^{n} S_q^*(n,k)\psi_k\left(\frac{x}{q}\right), \tag{21}$$

where $\psi_k(x) := q^{-\binom{k}{2}}\phi_k(x)$.

Using the method of linear functionals [5, pp. 89–90] one can derive from (21) the recurrence [8, Theorem 5.4]

$$S_q(n+1,k) = \sum_{j=0}^{n} \binom{n}{j} q^j S_q(j,k-1), \qquad \forall\, n \in \mathbb{N},\ k \in \mathbb{P}, \tag{22}$$

from which the variant recurrences

$$S_q^*(n+1,k) = q^{n+1} \sum_{j=0}^{n} \binom{n}{j} S_q^*(j,k-1), \qquad \forall\, n \in \mathbb{N},\ k \in \mathbb{P},$$

and

$$\tilde{S}_q(n+1,k) = \sum_{j=0}^{n} \binom{n}{j} q^{j-k+1} \tilde{S}_q(j,k-1), \qquad \forall\, n \in \mathbb{N},\ k \in \mathbb{P} \tag{23}$$

follow immediately.

Summing the $q$-Stirling numbers $S_q^*(n,k)$, $S_q(n,k)$ and $\tilde{S}_q(n,k)$ over $k$ yields the respective $q$-Bell numbers $B_q^*(n)$, $B_q(n)$, and $\tilde{B}_q(n)$. From (22) it follows that

$$B_q(n+1) = \sum_{j=0}^{n} \binom{n}{j} q^j B_q(j), \qquad \forall\, n \in \mathbb{N}. \tag{24}$$

Since $B_q^*(n) = q^n B_q(n)$, the recurrence (24) yields

$$B_q^*(n+1) = q^{n+1} \sum_{j=0}^{n} \binom{n}{j} B_q^*(j), \qquad \forall\, n \in \mathbb{N}. \tag{25}$$

Due to the factor $q^{-k}$ in (23), we do not get any recurrence for $\tilde{B}_q(n)$ analogous to (24) and (25), this being the single exception to the general parallelism between properties of the three $q$-Stirling numbers under consideration. The uniqueness of $\tilde{B}_q(n)$ is further manifested when $q = -1$, as we shall see in the next section.

# 4 The Case $q = -1$

In this section we derive simple expressions for the foregoing $q$-Stirling and $q$-Bell numbers when $q = -1$.

**Theorem 4.1.** *The number $\tilde{S}_{-1}(n,k)$ is given by the formula*

$$\tilde{S}_{-1}(n,k) = \binom{n - \lfloor \frac{k}{2} \rfloor - 1}{\lfloor \frac{k-1}{2} \rfloor}, \qquad 1 \leqslant k \leqslant n. \tag{26}$$

**Proof.** Note that

$$i_q\big|_{q=-1} = \omega_i := \begin{cases} 1, & \text{if } i \text{ is odd;} \\ 0, & \text{if } i \text{ is even.} \end{cases}$$

Hence by (18), if $1 \leqslant m \leqslant \lfloor n/2 \rfloor$,

$$\tilde{S}_{-1}(n, 2m) = \sum_{\substack{d_1+d_3+\cdots+d_{2m-1}=n-2m \\ d_i \in \mathbb{N}}} 1 = \binom{n-m-1}{m-1}, \tag{27}$$

since the number of sequences $(t_1, \ldots, t_m)$ of nonnegative integers summing to $s$ is $\binom{s+m-1}{m-1}$ [7, p. 15]. Similarly, if $0 \leqslant m \leqslant \lfloor (n-1)/2 \rfloor$,

$$\tilde{S}_{-1}(n, 2m+1) = \binom{n-m-1}{m}. \tag{28}$$

Formula (26) incorporates (27) and (28). □

In tabulating the numbers $\tilde{S}_{-1}(n,k)$ it is of course more efficient to use the recurrence

$$\tilde{S}_{-1}(n,k) = \tilde{S}_{-1}(n-1,k-1) + \omega_k \tilde{S}_{-1}(n-1,k),$$

representing the case $q = -1$ of (17).

Let $F_0 = F_1 = 1$, with $F_n = F_{n-1} + F_{n-2}$ if $n \geqslant 2$. As is well known,

$$F_n = \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n-m}{m}, \qquad \forall n \in \mathbb{N}. \tag{29}$$

**Theorem 4.2.** *For all $n \in \mathbb{N}$,*

$$\tilde{B}_{-1}(n) := \sum_{k=0}^{n} \tilde{S}_{-1}(n,k) = F_n. \tag{30}$$

**Proof.** It is easy to check that (30) holds for $n = 0, 1$. If $n \geqslant 2$, then by (27), (28), and (29),

$$\begin{aligned} \tilde{B}_{-1}(n) &= \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-m-1}{m} + \sum_{m=1}^{\lfloor n/2 \rfloor} \binom{n-m-1}{m-1} \\ &= \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \binom{(n-1)-m}{m} + \sum_{m=0}^{\lfloor (n-2)/2 \rfloor} \binom{(n-2)-m}{m} \\ &= F_{n-1} + F_{n-2} = F_n. \end{aligned}$$

□

From (26) and the fact that $S^*_q(n,k) = q^{\binom{k}{2}+n}\tilde{S}_q(n,k)$, we have

$$S^*_{-1}(n,k) = (-1)^{\binom{k}{2}+n}\binom{n - \lfloor \frac{k}{2} \rfloor - 1}{\lfloor \frac{k-1}{2} \rfloor}, \qquad 1 \leqslant k \leqslant n.$$

On the other hand, the Bell numbers $B^*_{-1}(n)$ are quite different from the numbers $\tilde{B}_{-1}(n)$.

**Theorem 4.3.** *For all $n \in \mathbb{N}$,*

$$B^*_{-1}(n) := \sum_{k=0}^{n} S^*_{-1}(n,k) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 3; \\ -1, & \text{if } n \equiv 1 \pmod 3; \\ 0, & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{31}$$

**Proof.** Noting that $B^*_{-1}(0) = 1$, we prove (31) by induction on $n$. In what follows

$$b_r(n) := \sum_{j \equiv r \pmod 3} \binom{n}{j}.$$

From (25) with $q = -1$, we have

$$\begin{aligned} B^*_{-1}(n+1) = (-1)^{n+1}\sum_{j=0}^{n}\binom{n}{j}B^*_{-1}(j) &= (-1)^{n+1}b_0(n) + (-1)^n b_1(n) \\ &= (-1)^{n+1}b_0(n) + (-1)^n b_0(n-1) + (-1)^n b_1(n-1). \end{aligned}$$

Similarly, $B^*_{-1}(n) = (-1)^n b_0(n-1) + (-1)^{n-1} b_1(n-1)$, and so

$$\begin{aligned} B^*_{-1}(n+1) &= (-1)^{n+1}b_0(n) + 2(-1)^n b_0(n-1) - B^*_{-1}(n) \\ &= \frac{1}{3}\left[\omega^{2n-1} - \omega^{2n-2} + \omega^{n+1} - \omega^{n-1}\right] - B^*_{-1}(n), \end{aligned} \tag{32}$$

by (10), where $\omega$ is either of the two complex cube roots of 1. Taking $n+1 = 3m$, $3m+1$, and $3m+2$, respectively, in (32) yields

$$\begin{aligned} B^*_{-1}(3m) &= 1 - B^*_{-1}(3m-1) = 1, \\ B^*_{-1}(3m+1) &= 0 - B^*_{-1}(3m) = -1, \qquad \text{and} \\ B^*_{-1}(3m+2) &= -1 - B^*_{-1}(3m+1) = 0. \end{aligned}$$

□

It is easy to check that one can write (31) more compactly as

$$B^*_{-1}(n) = \frac{1}{1-\omega}\omega^n - \frac{\omega}{1-\omega}\omega^{2n},$$

from which we get the nice exponential generating function

$$\sum_{n=0}^{\infty} B^*_{-1}(n)\frac{x^n}{n!} = \frac{1}{1-\omega}e^{\omega x} - \frac{\omega}{1-\omega}e^{\omega^2 x}. \tag{33}$$

From (26) and the fact that $S_q(n,k) = q^{\binom{k}{2}}\tilde{S}_q(n,k)$, we have

$$S_{-1}(n,k) = (-1)^{\binom{k}{2}}\binom{n - \lfloor \frac{k}{2} \rfloor - 1}{\lfloor \frac{k-1}{2} \rfloor}, \qquad 1 \leqslant k \leqslant n.$$

By (12),

$$B_{-1}(n) := \sum_{k=0}^{n} S_{-1}(n,k) = (-1)^n B^*_{-1}(n),$$

and so by (31)

$$B_{-1}(n) = \begin{cases} (-1)^n, & \text{if } n \equiv 0 \pmod 3; \\ (-1)^{n+1}, & \text{if } n \equiv 1 \pmod 3; \\ 0, & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{34}$$

and by (33)

$$\sum_{n=0}^{\infty} B_{-1}(n)\frac{x^n}{n!} = \frac{1}{1-\omega}e^{-\omega x} - \frac{\omega}{1-\omega}e^{-\omega^2 x}. \tag{35}$$

In a paper which showed how the numbers $S_{-1}(n,k)$ and $B_{-1}(n)$ arise in the study of fermionic oscillators, Schork [6] posed the problem of finding a closed form and a generating function for the numbers $B_{-1}(n)$. Formulas (34) and (35) furnish solutions to Schork's problem.

To conclude this section we remark that our list of partition statistics might have been rounded out to include the statistic

$$\hat{w}(\pi) := \sum_{i=1}^{k} i(|E_i| - 1) = \tilde{w}(\pi) + n - k,$$

with generating function

$$\hat{S}_q(n,k) := \sum_{\pi \in \Pi(n,k)} q^{\hat{w}(\pi)} = q^{(n-k)}\tilde{S}_q(n,k). \tag{36}$$

Formula (36) and Theorem 4.1 yield an easy evaluation of $\hat{S}_{-1}(n,k)$. As for

$$\hat{B}_{-1}(n) := \sum_{k=0}^{n} \hat{S}_{-1}(n,k),$$

we have $\hat{B}_{-1}(0) = \hat{B}_{-1}(1) = 1$, $\hat{B}_{-1}(2) = 0$, and

$$\hat{B}_{-1}(n) = (-1)^{n-1}F_{n-3}, \qquad \forall n \geqslant 3, \tag{37}$$

the proof of which we leave to interested readers.

## 5 Bijective Proofs

We conclude by returning to the opening theme of this paper. If $G(I, \Delta; -1) = 0$, then, as already noted, $|\Delta_0| = |\Delta_1|$, where $\Delta_i = \{\delta \in \Delta : \; I(\delta) \equiv i \pmod 2\}$. One gains a deeper understanding (and bijective proof) of such results by identifying an $I$-parity changing involution of $\Delta$. For the statistic $|S|$ in (1), the map

$$S \mapsto \begin{cases} S \cup \{1\}, & \text{if } 1 \notin S; \\ S - \{1\}, & \text{if } 1 \in S, \end{cases}$$

furnishes such an involution. For the statistic $i(\sigma)$ in (2), switching positions of the elements 1 and 2 (or of $k$ and $k+1$, for any fixed $k$) in a permutation furnishes such an involution.

A similar task arises when (3) is nonzero. Suppose, for example, that $G(I, \Delta; -1) = c > 0$. Here one wishes to identify a subset $\Delta^+$ of $\Delta_0$, with $|\Delta^+| = c$, and an $I$-parity changing involution of $\Delta - \Delta^+$.

My student, Mark Shattuck, has recently succeeded in finding such bijective proofs of formulas (26), (30), (31), (34), and (37). Details will appear in a forthcoming paper.

## 6 Tables

Table 1: The numbers $S^*_{-1}(n,k)$ for $1 \leqslant k \leqslant n \leqslant 8$.

| | $k=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $n=1$ | $-1$ | | | | | | | |
| 2 | 1 | $-1$ | | | | | | |
| 3 | $-1$ | 1 | 1 | | | | | |
| 4 | 1 | $-1$ | $-2$ | 1 | | | | |
| 5 | $-1$ | 1 | 3 | $-2$ | $-1$ | | | |
| 6 | 1 | $-1$ | $-4$ | 3 | 3 | $-1$ | | |
| 7 | $-1$ | 1 | 5 | $-4$ | $-6$ | 3 | 1 | |
| 8 | 1 | $-1$ | $-6$ | 5 | 10 | $-6$ | $-4$ | 1 |

Table 2: The numbers $S_{-1}(n,k)$ for $1 \leqslant k \leqslant n \leqslant 8$.

| | $k=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $n=1$ | 1 | | | | | | | |
| 2 | 1 | $-1$ | | | | | | |
| 3 | 1 | $-1$ | $-1$ | | | | | |
| 4 | 1 | $-1$ | $-2$ | 1 | | | | |
| 5 | 1 | $-1$ | $-3$ | 2 | 1 | | | |
| 6 | 1 | $-1$ | $-4$ | 3 | 3 | $-1$ | | |
| 7 | 1 | $-1$ | $-5$ | 4 | 6 | $-3$ | $-1$ | |
| 8 | 1 | $-1$ | $-6$ | 5 | 10 | $-6$ | $-4$ | 1 |

Table 3: The numbers $\tilde{S}_{-1}(n,k)$ for $1 \leqslant k \leqslant n \leqslant 8$.

| | $k=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $n=1$ | 1 | | | | | | | |
| 2 | 1 | 1 | | | | | | |
| 3 | 1 | 1 | 1 | | | | | |
| 4 | 1 | 1 | 2 | 1 | | | | |
| 5 | 1 | 1 | 3 | 2 | 1 | | | |
| 6 | 1 | 1 | 4 | 3 | 3 | 1 | | |
| 7 | 1 | 1 | 5 | 4 | 6 | 3 | 1 | |
| 8 | 1 | 1 | 6 | 5 | 10 | 6 | 4 | 1 |

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(Concerned with sequence A000045.)

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