Journal of Integer Sequences, Vol. 7 (2004),
Article 04.2.5

On k-colored Motzkin words

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Abstract

This paper deals with the enumeration of k-colored Motzkin words according to various parameters, such as the length, the number of rises, the length of the initial rise and the number of prime components.

1 Introduction

There exists an extended literature on Dyck and Motzkin paths and their relationship with many other combinatorial objects [7, 10, 11, 15, 16, 19, 21]. It is well known that the sets of Dyck paths of length $2n$ and Motzkin paths of length n are enumerated by the Catalan numbers C_n (A000108) and the Motzkin numbers M_n (A001006), respectively. More generally, there is great interest in k-colored Motzkin paths [2], which have horizontal steps colored by means of k colors.

This paper deals with the set of k-colored Motzkin words (or equivalently paths) and with some subsets of it, defined by various parameters.

In section 2, some basic definitions and notations referring to the sets $\mathcal{M}_k$ and $\mathcal{M}_k^c$ of (k-colored) Motzkin and c-Motzkin words respectively are given.

In section 3, using the generating functions F_k and G_k of $\mathcal{M}_k$ and $\mathcal{M}_k^c$ respectively, according to the parameters "length", "number of rises" and "length of the initial rise", the cardinalities of several subsets of $\mathcal{M}_k$ are evaluated. Furthermore, using the Lagrange inversion formula, the coefficients of the powers of F_k are determined.

Finally, in section 4, the decomposition of the elements of $\mathcal{M}_k^c$ to prime words is studied. The generating function G_k of $\mathcal{M}_k^c$ according to the three previous parameters and to the parameter "number of prime components" is determined. This is used to show that the number of all $u \in \mathcal{M}_k^c$ with s prime components and length of the initial rise equal to m is equal to the number of all $u \in \mathcal{M}_k^c$ with m prime components and length of the initial rise equal to s.

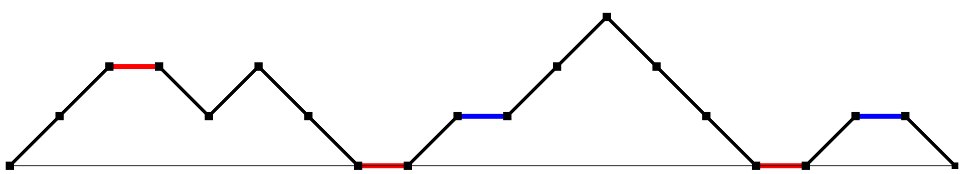
2 Preliminaries

Throughout this paper, let E be an alphabet with $k+2$ letters, where $k \in \mathbb{N}$ and $a, \bar{a}$ are two given elements of E. For $k \neq 0$, the elements of the set $E \backslash \{a, \bar{a}\} = \{\beta_1, \beta_2, \ldots, \beta_k\}$ are called *colors* of E. The number of occurrences of the letter $x \in E$ in the word u is denoted by $|u|_x$, the length of u by $l(u)$, and the number of rises of u by $r(u)$.

We denote by E^* the set which contains all the words with letters in E as well as the empty word ϵ. A word $u \in E^*$ is called *k-colored Motzkin word* if $|u|_a = |u|_{\bar{a}}$ and for every factorization $u = wv$ we have $|w|_{\bar{a}} \leq |w|_a$.

A *Motzkin path* of length n is a lattice path of $\mathbb{N}^2$ running from $(0,0)$ to $(n,0)$ that never passes below the x-axis and whose permitted steps are the up diagonal step $(1,1)$, the down diagonal step $(1,-1)$ and the horizontal step $(1,0)$, called *rise*, *fall* and *level step*, respectively. If the level steps are labelled by k colors we obtain the *k-colored Motzkin paths*.

It is clear that each k-colored Motzkin path is coded by a k-colored Motzkin word $u = u_1 u_2 \cdots u_n \in E^*$ so that every rise (resp., fall) corresponds to the letter a (resp., $\bar{a}$) and every colored level corresponds to a certain color of E; see Fig. 1.



$u = a\ a\ \beta_1\ \bar{a}\ a\ \bar{a}\ \bar{a}\ \beta_1\ a\ \beta_2\ a\ a\ \bar{a}\ \bar{a}\ \bar{a}\ \beta_1\ a\ \beta_2\ \bar{a}$

Figure 1: A 2-colored Motzkin path and its corresponding Motzkin word

We denote by $\mathcal{M}_{k,n}$ (resp., $\mathcal{M}_{k,n,r}$) the set of all $u \in \mathcal{M}_k$ with $l(u) = n$ (resp., $l(u) = n$ and $r(u) = r$) and we set $\mu_{k,n} = |\mathcal{M}_{k,n}|$ (resp., $\mu_{k,n,r} = |\mathcal{M}_{k,n,r}|$).

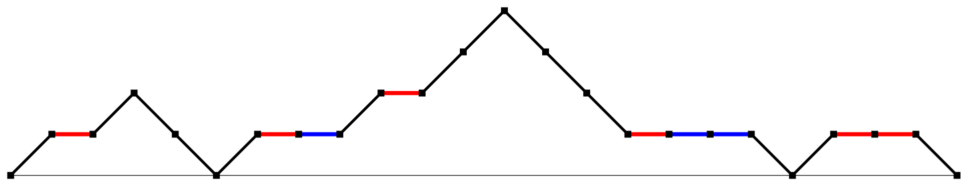
It is well known that if $k = 0, 1$ we obtain the sets of Dyck and Motzkin words, respectively. The 2-colored Motzkin words have been studied in [9]. More precisely, we have:

$$\mu_{0,n} = \begin{cases} C_{\frac{n}{2}}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd,} \end{cases} \qquad \mu_{1,n} = M_n, \qquad \mu_{2,n} = C_{n+1}.$$

The 3-colored Motzkin paths correspond to the tree-like polyhexes defined by Harary [13], as we will see in the next section.

Let $u = u_1u_2\cdots u_n \in \mathcal{M}_{k,n}$. Two indices $i, j \in [n] = \{1, 2, \ldots, n\}$ with $i < j$ are called *conjugates* with respect to u if and only if j is the smallest number in $\{i+1, i+2, \ldots, n\}$ for which the segment $u_iu_{i+1}\cdots u_j$ of u is a k-colored Motzkin word.

A word $u \in \mathcal{M}_{k,n}$ is called (k-colored) *c-Motzkin word* if and only if every $i \in [n]$ with $u_i \notin \{a, \bar{a}\}$, lies between two conjugate indices. It is clear that the c-Motzkin words code exactly those k-colored paths that have no level steps on the x-axis; see Fig. 2.



$u = a\ \beta_1\ a\ \bar{a}\ \bar{a}\ a\ \beta_1\ \beta_2\ a\ \beta_1\ a\ a\ \bar{a}\ \bar{a}\ \bar{a}\ \beta_1\ \beta_2\ \beta_2\ \bar{a}\ a\ \beta_1\ \beta_1\ \bar{a}$

Figure 2: A 2-colored Motzkin path and its corresponding c-Motzkin word

The c-Motzkin words have been introduced and studied in the case $k = 1$, [18].

In the following sections we will refer to the sets $\mathcal{M}^c_{k,n} = \mathcal{M}^c_k \cap \mathcal{M}_{k,n}$ and $\mathcal{M}^c_{k,n,r} = \mathcal{M}^c_k \cap \mathcal{M}_{k,n,r}$ with cardinalities $\mu^c_{k,n}$ and $\mu^c_{k,n,r}$, respectively.

3 Enumeration of sets of k-colored Motzkin words

In this section we evaluate the cardinal number of several subsets of $\mathcal{M}_k$ defined by various parameters. We first need the following definition.

The *initial rise* of a non-empty word $u = u_1u_2\cdots u_n \in \mathcal{M}_k$ with $u_1 = a$ is the segment $u_1u_2\cdots u_j$ where $u_\nu = a$ for every $\nu \in [j]$ and $u_{j+1} \neq a$. If $u = \epsilon$ or $u_1 \neq a$, the initial rise of u is the empty word. We denote by $p(u)$ the length of the initial rise of u.

Let F_k and G_k be the generating functions of $\mathcal{M}_k$ and $\mathcal{M}^c_k$, respectively, according to the parameters l, r, p (coded by x, y, z), i.e.,

$$F_k(x, y, z) = \sum_{u \in \mathcal{M}_k} x^{l(u)} y^{r(u)} z^{p(u)}$$

and

$$G_k(x, y, z) = \sum_{u \in \mathcal{M}^c_k} x^{l(u)} y^{r(u)} z^{p(u)}.$$

Proposition 3.1 *The generating functions F_k, G_k are given by the formulae*

$$F_k(x, y, z) = \frac{1 + kxF_k(x, y)}{1 - x^2yzF_k(x, y)} \tag{1}$$

and

$$G_k(x, y, z) = \frac{1}{1 - x^2yzF_k(x, y)}, \tag{2}$$

where the generating function $F_k(x,y) = F_k(x,y,1)$ satisfies the equation

$$x^2yF_k^2(x,y) + (kx-1)F_k(x,y) + 1 = 0 \tag{3}$$

and hence

$$F_k(x,y) = \frac{1-kx-\sqrt{(1-kx)^2-4x^2y}}{2x^2y}. \tag{4}$$

Proof: We can easily verify that for $k \neq 0$ each nonempty $u \in \mathcal{M}_k$ can be uniquely written in either of the forms $u = \beta_\nu v$ for some $v \in \mathcal{M}_k$ and $\nu \in [k]$, or $u = aw\bar{a}v$ for some $v, w \in \mathcal{M}_k$, where indices 1, $l(w)+2$ are conjugates with respect to u.

Obviously, since in the first case $p(u) = 0$, $r(u) = r(v)$ and in the second case $r(u) = r(w) + r(v) + 1$, $p(u) = p(w) + 1$, we obtain that

$$\begin{aligned} F_k(x,y,z) &= 1 + \sum_{\nu=1}^{k}\sum_{v} x^{l(\beta_\nu v)}y^{r(v)} + \sum_{w,v} x^{l(w)+l(v)+2}y^{r(w)+r(v)+1}z^{p(w)+1} \\ &= 1 + kxF_k(x,y) + x^2yzF_k(x,y,z)F_k(x,y). \end{aligned}$$

Thus,

$$F_k(x,y,z) = \frac{1+kxF_k(x,y)}{1-x^2yzF_k(x,y)}.$$

Moreover, applying the above equality for $z = 1$ we deduce that

$$x^2yF_k^2(x,y) + (kx-1)F_k(x,y) + 1 = 0.$$

The proof of (1) for $k = 0$ follows as above with some simple modifications.

The proof of (2) is similar and it is omitted. □

Remark The generating function F_k can be obtained as an application of a continued fraction result [12]. More precisely if we apply theorem 1 of [12] by counting the rises by xy, the falls by x and the level steps by kx we conclude that

$$F_k(x,y) = \cfrac{1}{1-kx-\cfrac{x^2y}{1-kx-\cfrac{x^2y}{1-kx-\cfrac{x^2y}{\cdots}}}}$$

which easily leads to equation (3).

Example We compute the number of k-colored c-Motzkin words of length n, for $k = 1$ and $k = 2$, using the generating functions $C(x)$ and $M(x)$ of Catalan and Motzkin numbers, respectively. For this we use formula (2) for the generating function $G_k(x) = G_k(x,1,1)$ of $\mathcal{M}_k^c$ according to the length.

1) For $k = 1$, we have that

$$\begin{aligned} G_1(x) &= \frac{1}{1 - x^2 F_1(x)} = \frac{1}{1 - x^2 M(x)} = \frac{1 + xM(x)}{1 + x} \\ &= (\sum_{n=0}^{\infty} (-1)^n x^n)(\sum_{n=0}^{\infty} \gamma_n x^n) \\ &= \sum_{n=0}^{\infty} (\sum_{i=0}^{n} (-1)^i \gamma_{n-i}) x^n, \end{aligned}$$

where

$$\gamma_n = \begin{cases} M_{n-1}, & \text{if } n \geq 1; \\ 0, & \text{if } n = 0. \end{cases}$$

Thus,

$$\mu^c_{1,n} = \sum_{i=0}^{n} (-1)^i \gamma_{n-i} = \sum_{i=0}^{n-2} (-1)^i M_{n-i-1},$$

for every $n \geq 2$.

We note that from the above formula we deduce that for every $n \geq 2$,

$$\mu^c_{1,n} + \mu^c_{1,n-1} = M_{n-1}$$

which implies that the number of c-Motzkin paths of length n is equal to the number of Motzkin paths of length $n - 1$ with at least one level step on the x-axis [14].

2) For $k = 2$ and since

$$F_2(x) = \sum_{n=0}^{\infty} \mu_{2,n} x^n = \sum_{n=0}^{\infty} C_{n+1} x^n = \frac{1}{x}[C(x) - 1] = C^2(x),$$

we obtain that

$$G_2(x) = \frac{1}{1 - x^2 C^2(x)}.$$

So, the generating function $G_2(x)$ coincides with the generating function of Fine numbers f_n [8] and hence we conclude that $\mu^c_{2,n} = f_n$.

In the following result we give recursive formulae for the sequences $\mu_{k,n,r}$ and $\mu_{k,n}$.

Proposition 3.2 *For every $k, \nu, n, r \in \mathbb{N}$ with $r \leq [\frac{n}{2}]$ we have that*

$$\mu_{k+\nu,n,r} = \sum_{m=2r}^{n} \binom{n}{m} \mu_{k,m,r} \nu^{n-m} = \sum_{m=2r}^{n} \binom{n}{m} \mu_{\nu,m,r} k^{n-m} \tag{5}$$

and

$$\mu_{k+\nu,n} = \sum_{m=0}^{n} \binom{n}{m} \mu_{k,m} \nu^{n-m} = \sum_{m=0}^{n} \binom{n}{m} \mu_{\nu,m} k^{n-m}. \tag{6}$$

Proof: From relation (4) we easily obtain that

$$F_{k+\nu}(x,y) = \frac{F_k(\frac{x}{1-\nu x}, y)}{1-\nu x} = \frac{F_n(\frac{x}{1-kx}, y)}{1-kx}$$

for every $k, \nu \in \mathbb{N}$.

On the other hand, we have that

$$\begin{aligned}
\frac{F_k(\frac{x}{1-\nu x}, y)}{1-\nu x} &= \sum_{m=0}^{\infty}\sum_{r=0}^{[\frac{m}{2}]} \mu_{k,m,r} x^m y^r \frac{1}{(1-\nu x)^{m+1}} \\
&= \sum_{m=0}^{\infty}\sum_{r=0}^{[\frac{m}{2}]} \mu_{k,m,r} x^m y^r \sum_{j=0}^{\infty} \binom{-m-1}{j} (-\nu x)^j \\
&= \sum_{m=0}^{\infty}\sum_{r=0}^{[\frac{m}{2}]}\sum_{j=0}^{\infty} \mu_{k,m,r} \binom{m+j}{j} \nu^j x^{j+m} y^r \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n}{2}]} \left[\sum_{m=2r}^{n} \mu_{k,m,r} \binom{n}{m} \nu^{n-m}\right] x^n y^r.
\end{aligned}$$

It follows that

$$\mu_{k+\nu,n,r} = \sum_{m=2r}^{n} \binom{n}{m} \mu_{k,m,r} \nu^{n-m}.$$

Moreover, using the above relations we obtain that

$$\mu_{k+\nu,n} = \sum_{r=0}^{[\frac{n}{2}]} \mu_{k+\nu,n,r} = \sum_{m=0}^{n} \binom{n}{m} \nu^{n-m} \sum_{r=0}^{[\frac{m}{2}]} \mu_{k,m,r} = \sum_{m=0}^{n} \binom{n}{m} \mu_{k,m} \nu^{n-m}.$$

The proofs of the second parts of relations (5) and (6) are similar and they are omitted. □

Remark 1 Since

$$\mu_{0,m,r} = \begin{cases} C_r, & \text{if } m = 2r; \\ 0, & \text{if } m \neq 2r \end{cases}$$

and

$$\mu_{0,m} = \begin{cases} C_{\frac{m}{2}}, & \text{if } m \text{ is even}; \\ 0, & \text{if } m \text{ is odd}, \end{cases}$$

setting $\nu = 0$ in relations (5) and (6) we obtain that

$$\mu_{k,n,r} = \binom{n}{2r} C_r k^{n-2r} = \frac{1}{n+1} \binom{n+1}{r+1, r, n-2r} k^{n-2r} \tag{7}$$

and

$$\mu_{k,n} = \sum_{r=0}^{[\frac{n}{2}]} \binom{n}{2r} C_r k^{n-2r} \tag{8}$$

which give (for $k=1$) the well-known corresponding relations for Motzkin words [1].

Furthermore, for $k=2$, relation (8) gives the well-known relation of Touchard

$$C_{n+1} = \sum_{r=0}^{[\frac{n}{2}]} \binom{n}{2r} 2^{n-2r} C_r.$$

Remark 2 From relation (6) we can easily deduce relations

$$\mu_{k+1,n} = \sum_{m=0}^{n} \binom{n}{m} \mu_{k,m} \tag{9}$$

and

$$\mu_{k+1,n+1} = \sum_{m=0}^{n} \binom{n}{m} (\mu_{k,m} + \mu_{k,m+1}). \tag{10}$$

It is easy to check that from the above two relations, for $k=0$ and $k=1$, relations (1), (2), (3) and (4) of [10] follow.

Remark 3 Applying relation (9) for $k=2$, we obtain the number of all 3-colored Motzkin words of length n:

$$\mu_{3,n} = \sum_{m=0}^{n} \binom{n}{m} C_{m+1}.$$

This number also gives the cardinality of the set of all tree-like polyhexes with $n+1$ hexagons (A002212) (for detailed definitions see [13]), which can be coded by the 3-colored Motzkin words in the following, recursive way:

If the polyhex consists of the root hexagon $ABCDEF$ only (with root edge AB), then the corresponding 3-colored Motzkin word is ϵ. If the polyhex consists of $n+1$ hexagons, then we have the following cases: If the only points of $ABCDE$ with degree 3 are C, D (D, E or E, F, respectively) then the corresponding $u \in \mathcal{M}_{3,n}$ is $\beta_1 w$ ($\beta_2 w$ or $\beta_3 w$, respectively), where the word $w \in \mathcal{M}_{3,n-1}$ corresponds to the polyhex with n hexagons and root edge CD (DE or EF, respectively) that we obtain if we delete the points of the root hexagon that have degree 2, as well as the edges incident with these points; see Fig. 3 a,b,c.

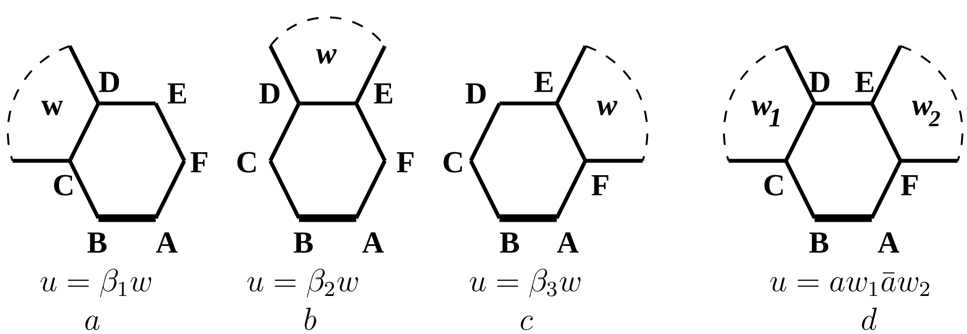


Figure 3: The recursive coding of polyhexes

If on the other hand the only points of the root hexagon with degree 3 are C, D, E, F then the corresponding $u \in \mathcal{M}_{3,n}$ is the word $aw_1\bar{a}w_2$, where w_1 (resp., w_2) is the 3-colored Motzkin word which corresponds to the polyhex with less than n-hexagons and root edge CD (resp., EF) that we obtain if we delete the points A, B as well as the edges AB, BC, DE and FA; see Fig. 3 d.

We continue by evaluating the coefficients of the powers of $F_k(x, y)$.

Proposition 3.3 *The coefficients of $F_k^s(x, y)$, with $s \in \mathbb{N}^*$, are given by the formula*

$$[x^n y^r]F_k^s = \frac{s}{n+s}\binom{n+s}{s+r, r, n-2r}k^{n-2r}, \tag{11}$$

where $n, r \in \mathbb{N}$, with $r \leq [\frac{n}{2}]$.

Proof: We define the function $H(x) = xF_k(x, y)$. It follows easily by equation (3) that

$$H(x) = x[yH^2(x) + kH(x) + 1].$$

Thus, if we set $P(\lambda) = y\lambda^2 + k\lambda + 1$ we obtain that $H(x) = xP(H(x))$ and $P(0) = 1$. Using Lagrange inversion formula [20] we obtain

$$[x^n]H^s = \frac{1}{n}[\lambda^{n-1}]\{s\lambda^{s-1}(P(\lambda))^n\}.$$

Moreover, we have

$$\begin{aligned}
\frac{s}{n}\lambda^{s-1}(P(\lambda))^n &= \frac{s}{n}\lambda^{s-1}\sum_{i=0}^{n}\binom{n}{i}\lambda^i(y\lambda + k)^i \\
&= \frac{s}{n}\lambda^{s-1}\sum_{i=0}^{n}\binom{n}{i}\lambda^i\sum_{\nu=0}^{i}\binom{i}{\nu}y^\nu\lambda^\nu k^{i-\nu} \\
&= \frac{s}{n}\sum_{m=0}^{2n}\sum_{\nu=(m-n)^+}^{[\frac{m}{2}]}\binom{n}{m-\nu}\binom{m-\nu}{\nu}k^{m-2\nu}y^\nu\lambda^{m+s-1},
\end{aligned}$$

where $(m-n)^+ = \max\{0, m-n\}$.

Thus, for $m = n - s$ we deduce that

$$[x^n]H^s = \frac{s}{n}\sum_{\nu=0}^{[\frac{n-s}{2}]}\binom{n}{n-s-\nu}\binom{n-s-\nu}{\nu}k^{n-s-2\nu}y^\nu$$

for every $n \geq s$.

Finally, applying the above equality for $n+s$ instead of s and setting $\nu = r$, we conclude that

$$\begin{aligned}
[x^n y^r]F_k^s &= \frac{s}{n+s}\binom{n+s}{n-r}\binom{n-r}{r}k^{n-2r} \\
&= \frac{s}{n+s}\binom{n+s}{s+r, r, n-2r}k^{n-2r}.
\end{aligned}$$

□

We note that relation (7) is a special case of relation (11), for $s = 1$.

We use the last proposition in order to prove the following result:

Proposition 3.4 *The number of all $u \in \mathcal{M}^c_{k,n,r}$ that have initial rise of length s is equal to*

$$[x^n y^r z^s]G_k = \frac{s}{n-s}\binom{n-s}{r, r-s, n-2r}k^{n-2r},$$

where $1 \leq s \leq r \leq [\frac{n}{2}]$.

Proof: By relation (2) and proposition 3.3 we obtain that

$$\begin{aligned}[x^n y^r z^s]G_k &= [x^n y^r z^s]\{\sum_{s=0}^{\infty} x^{2s} y^s F^s_k(x,y) z^s\} \\ &= [x^n y^r]\{x^{2s} y^s F^s_k(x,y)\} \\ &= [x^{n-2s} y^{r-s}]F^s_k \\ &= \frac{s}{n-s}\binom{n-s}{r, r-s, n-2r}k^{n-2r}.\end{aligned}$$

□

Using proposition 3.1 and the same arguments as in the proof of proposition 3.4 we obtain the following result:

Proposition 3.5 *The number of all $u \in \mathcal{M}_{k,n,r}$ that have initial rise of length s is equal to*

$$[x^n y^r z^s]F_k = \frac{ns - rs + n + s - 2r}{(n-s)(n-s+1)}\binom{n-s+1}{r+1, r-s, n-2r}k^{n-2r},$$

where $1 \leq s \leq r \leq [\frac{n}{2}]$.

Notice that if $n = 2r$ then both propositions 3.4 and 3.5 give the number of Dyck words with prescribed height of the first peak [6].

4 Decomposition into prime words

A non-empty word $u \in \mathcal{M}^c_k$ is called *prime* if and only if it is not the product of two non-empty c-Motzkin words. It is clear that the k-colored Motzkin paths coded by a prime word are the paths whose only intersections with the x-axis are their initial and final points. It is evident that the word $u \in \mathcal{M}_k$ is prime if and only if the indices $1, l(u)$ are conjugates with respect to u.

The following result, known for Dyck [17] and c-Motzkin [18] words is naturally extended to k-colored c-Motzkin words.

Proposition 4.1 *Every $u \in \mathcal{M}_k^c$ is uniquely decomposed into a product of prime words.*

It is clear that the words $u \in \mathcal{M}_{k,n}^c$ which are decomposed into s prime words (components) are the ones whose corresponding k-colored Motzkin paths meet the x-axis at exactly $s-1$ points, in addition to the points $(0,0)$ and $(n,0)$.

In this section, among others, the number of all $u \in \mathcal{M}_{k,n}^c$ with a fixed number of prime components is evaluated. This is a well-known result in the case of $k=0$ (i.e., for Dyck words, [7, 17]) and it is extended here for arbitrary k. For this, we consider one more parameter d of $\mathcal{M}_k^c$, defined by the number of prime components. Let G_k be the generating function of $\mathcal{M}_k^c$ according to the parameters l, r, p, d (coded by x, y, z, ϕ), i.e.,

$$G_k(x,y,z,\phi) = \sum_u x^{l(u)} y^{r(u)} z^{p(u)} \phi^{d(u)}.$$

Proposition 4.2 *The generating function $G_k(x,y,z,\phi)$ is given by the formula*

$$G_k(x,y,z,\phi) = 1 + \frac{x^2 yz\phi(1+kxF_k(x,y))}{(1-x^2yzF_k(x,y))(1-x^2y\phi F_k(x,y))}.$$

Proof: Every non-empty $u \in \mathcal{M}_k^c$ can be uniquely written in the form $u = aw\bar{a}v$, where $w \in \mathcal{M}_k$, $v \in \mathcal{M}_k^c$, $r(u) = r(w)+r(v)+1$, $p(u) = p(w)+1$ and $d(u) = d(v)+1$. Thus, by proposition 3.1 follows that

$$\begin{aligned}
G_k(x,y,z,\phi) &= 1 + \sum_{w,v} x^{l(w)+l(v)+2} y^{r(w)+r(v)+1} z^{p(w)+1} \phi^{d(w)+1} \\
&= 1 + x^2yz\phi(\sum_w x^{l(w)} y^{r(w)} z^{p(w)})(\sum_v x^{l(v)} y^{r(v)} \phi^{d(v)}) \\
&= 1 + x^2yz\phi F_k(x,y,z) G_k(x,y,1,\phi) \\
&= 1 + x^2yz\phi \frac{1+kxF_k(x,y)}{1-x^2yzF_k(x,y)} G_k(x,y,1,\phi).
\end{aligned}$$

Further, applying the previous equality for $z=1$ and using relation (3) we conclude that

$$G_k(x,y,1,\phi) = \frac{1}{1-x^2y\phi F_k(x,y)}$$

which implies the required formula. □

Remark Since $G_k(x,y,1,\phi) = G_k(x,y,z,1)$, we obtain that the parameters p and d are equidistributed. This is a well-known result for Dyck paths, i.e., for the case $k=0$, see [4, 5, 6, 7].

Furthermore, since $G_k(x,y,z,\phi) = G_k(x,y,\phi,z)$ we obtain the following result.

Proposition 4.3 *The number of all $u \in \mathcal{M}_{k,n,r}^c$ with s prime components and length of the initial rise equal to m, is equal to the number of all $u \in \mathcal{M}_{k,n,r}^c$ with m prime components and length of the initial rise equal to s.*

Proposition 4.3 can also be proved directly, by constructing an involution of $\mathcal{M}_k^c$ as follows:

We first define the mapping

$$\phi : \{u \in \mathcal{M}_k^c : p(u) \geq 2\} \to \{u \in \mathcal{M}_k^c : d(u) \geq 2\}$$

such that if $u = aaw\bar{a}v\bar{a}z$ with $w, v \in \mathcal{M}_k$, $z \in \mathcal{M}_k^c$, $l(w)+3$ conjugate of 2 and $l(w)+l(v)+4$ conjugate of 1, then $\phi(u) = aw\bar{a}av\bar{a}z$. Obviously, ϕ is a bijection. Next we define the mapping

$$\theta : \mathcal{M}_k^c \to \mathcal{M}_k^c$$

with $\theta(u) = \phi^{p(u)-d(u)}(u)$, for every $u \in \mathcal{M}_k^c$; (here ϕ^j stands for $\phi \circ \cdots \circ \phi$). This mapping is well defined, with $l(\theta(u)) = l(u)$ and $r(\theta(u)) = r(u)$.

It is easy to check, by induction on the number $\nu(u) = |p(u) - d(u)|$ that $p(\theta(u)) = d(u)$ and $d(\theta(u)) = p(u)$ for every $u \in \mathcal{M}_k^c$. It follows that θ is the required involution of $\mathcal{M}_k^c$.

In order to construct $\theta(u)$ from $u \in \mathcal{M}_k^c$, we note that if $p(u) = d(u)$ then $\theta(u) = u$. If $p(u) > d(u)$, we delete the first $\nu(u)$ a's of u and we insert one a after each $\bar{a}$ of u which corresponds to a conjugate of $2, 3, \ldots, \nu(u) + 1$. Finally, if $p(u) < d(u)$, we add $\nu(u)$ a's in the beginning of u, whereas we delete the initial a from each one of the 2nd, 3rd, ..., $(\nu(u) + 1)$st prime component of u.

For example, for

$$u = a\ a\ a\ a\ a\ \beta_1\ \bar{a}\ \bar{a}\ a\ \bar{a}\ \bar{a}\ \beta_2\ a\ \bar{a}\ \bar{a}\ \bar{a}\ a\ \beta_2\ a\ \bar{a}\ \bar{a}\ a\ \beta_1\ \bar{a} \in \mathcal{M}_{2,24}^c$$

we obtain

$$\theta(u) = a\ a\ a\ \beta_1\ \bar{a}\ \bar{a}\ a\ \bar{a}\ \bar{a}\ a\ \beta_2\ a\ \bar{a}\ \bar{a}\ a\ \bar{a}\ a\ \beta_2\ a\ \bar{a}\ \bar{a}\ a\ \beta_1\ \bar{a} \in \mathcal{M}_{2,24}^c.$$

This is illustrated by the corresponding 2-colored paths of u and $\theta(u)$ in Fig. 4.

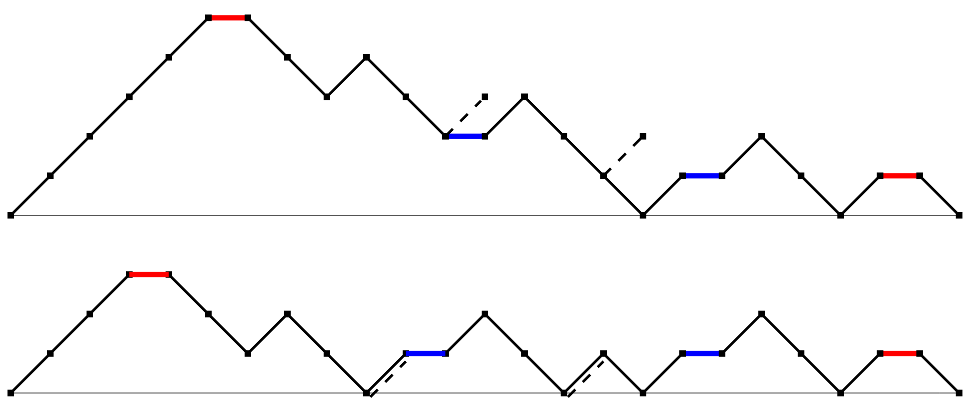


Figure 4: The 2-colored Motzkin paths corresponding to u and $\theta(u)$

Remark From propositions 3.4 and 4.3 follows that the number of all $u \in \mathcal{M}_{k,n,r}^c$ with s prime components is equal to

$$\frac{s}{n-s}\binom{n-s}{r, r-s, n-2r} k^{n-2r},$$

where $1 \leq s \leq r \leq [\frac{n}{2}]$.

This extends a well-known result on Dyck words (i.e., for $k = 0$) [7, 17], to k-colored c-Motzkin words for arbitrary k.

Furthermore, by summing the above numbers for all $s \in [r]$ we easily obtain that the number of all k-colored c-Motzkin words of length n, with r rises is given by the formula

$$\mu^c_{k,n,r} = \frac{1}{n-r+1}\binom{n}{r}\binom{n-r-1}{r-1}k^{n-2r}.$$

This formula has been proved for $k = 1$ in a different way [18].

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2000 *Mathematics Subject Classification*: Primary 05A15; Secondary 05A19.
Keywords: Motzkin words, Motzkin numbers, Dyck words, Catalan numbers, Polyhexes.

(Concerned with sequences A000108, A001006 and A002212.)

Received April 1 2004; revised version received June 8 2004. Published in *Journal of Integer Sequences*, June 8 2004.

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