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An Asymptotic Expansion for the Catalan-Larcombe-French Sequence

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Abstract

We give an elementary development of a complete asymptotic expansion for the Catalan-Larcombe-French sequence.

1 Introduction

In their delightful paper, Larcombe and French [3] developed a number of properties of the sequence (A053175) $P_0 = 1$, $P_1 = 8$, $P_2 = 80$, $P_3 = 896$, $P_4 = 10816$, ... originally discussed by Catalan [1]. In addition to a generating function, the following formula for P_n was derived

$$P_n = \frac{1}{n!} \sum_{p+q=n} \binom{2p}{p} \binom{2q}{q} \frac{(2p)!\,(2q)!}{p!\,q!} \qquad (n \in \mathbb{N})\,. \tag{1}$$

Recently, Larcombe et al. [4] showed that $P_n/2\binom{2n}{n}^2 \to 1$ as $n \to \infty$ by a rather lengthy analysis. In this short paper, we give an elementary development of a complete asymptotic expansion for $P_n/2\binom{2n}{n}^2$. We conclude with a table of numerical calculations as a companion of the theoretical results.

2 Main result

The positive integers are denoted by $\mathbb{P}$; the nonnegative integers by $\mathbb{N}$; the nonnegative rational numbers by $\mathbb{Q}_0$; and the complex numbers by $\mathbb{C}$. Let $z^0 = (z)_0 \equiv 1$ and $(z)_p =$

$(z)\cdots(z-p+1)$ when $z \in \mathbb{C}$ and $p \in \mathbb{P}$. For $z \in \mathbb{C}$ and $p \in \mathbb{P}$, $(z)_p = \sum_{q=0}^{p} s(p,q) z^q$ where the $s(p,q)$ are the Stirling numbers of the first kind (see [2; pp. 212–214]). We write $f(z) = O(g(z))$ provided there exist real constants C, D with $|f(z)| \le C|g(z)|$ for all $|z| \ge D$.

For $0 \le p \le n$, let

$$a_p = \binom{2p}{p}^2 \binom{2n-2p}{n-p}^2 p!\,(n-p)! \in \mathbb{P}$$

hence, $a_p = a_{n-p}$ and, let

$$b_p = \binom{2p}{p}\binom{2n-2p}{n-p} \in \mathbb{P}$$

hence, $b_p = b_{n-p}$ and $a_p = n!\, b_p^2 \binom{n}{p}^{-1}$. For $0 \le p \le n-1$,

$$\frac{b_{p+1}}{b_p} = \frac{(2p+1)(n-p)}{(p+1)(2n-2p+1)}$$

then, for $0 \le p \le (n-1)/2$,

$$0 < \frac{b_{p+1}}{b_p} \le 1$$

hence, for $1 \le p \le (n-1)/2$

$$0 < \frac{b_p}{b_0} = \frac{b_p}{b_{p-1}} \cdots \frac{b_1}{b_0} \le 1 \tag{2}$$

which is correct for $p = 0$ also.

Fix $s \ge 1$. For $n = 2m+1 \ge 2s+3$, (1) and symmetry give

$$\begin{aligned} P_n &= \frac{1}{n!}\sum_{p=0}^{n} \binom{2p}{p}\binom{2n-2p}{n-p} \frac{(2p)!\,(2n-2p)!}{p!\,(n-p)!} \\ &= \frac{1}{n!}\sum_{p=0}^{n} a_p = \frac{2}{n!}\sum_{p=0}^{m} a_p = 2\sum_{p=0}^{m} b_p^2 \binom{n}{p}^{-1} \\ &= 2b_0^2 \left\{ \sum_{p=0}^{s} \left(\frac{b_p}{b_0}\right)^2 \binom{n}{p}^{-1} + \sum_{p=s+1}^{m} \left(\frac{b_p}{b_0}\right)^2 \binom{n}{p}^{-1} \right\}. \end{aligned} \tag{3}$$

Now $\binom{n}{p}$ is an increasing sequence of positive integers for $0 \le p \le (n-1)/2$, so (2) gives

$$0 < \sum_{p=s+1}^{m} \left(\frac{b_p}{b_0}\right)^2 \binom{n}{p}^{-1} \le n\binom{n}{s+1}^{-1} = O(n^{-s}) \quad \text{as } n \to \infty. \tag{4}$$

For $0 \le p \le (n-1)/2$,

$$\frac{b_p}{b_0} = \binom{2p}{p} \frac{(n)_p^2}{(2n)_{2p}}$$

hence,

$$\sum_{p=0}^{s} \left(\frac{b_p}{b_0}\right)^2 \binom{n}{p}^{-1} = \sum_{p=0}^{s} p! \binom{2p}{p}^2 \frac{(n)_p^3}{(2n)_{2p}^2}\,. \tag{5}$$

For $z \in \mathbb{C}$, let $f_0(z) \equiv 1$ and, for $1 \le p \le s$, let

$$\begin{aligned} f_p(z) = p!\binom{2p}{p}^2 \frac{(z)_p^3}{(2z)_{2p}^2} &= \frac{p!\binom{2p}{p}^2 (z)_p}{2^{4p} z^{2p}} \prod_{j=1}^{p} \left(1 - \frac{2j-1}{2z}\right)^{-2} \\ &= \frac{p!\binom{2p}{p}^2}{2^{4p}} \sum_{q=0}^{p} s(p,q) z^{q-2p} \left\{ \prod_{j=1}^{p} \left(\sum_{r=0}^{\infty} (r+1)(j-0.5)^r z^{-r} \right) \right\} \\ &= \sum_{r=p}^{\infty} b(p,r) z^{-r} \qquad\qquad (z \in \mathbb{C};\ |z| \ge s) \end{aligned} \tag{6}$$

where $b(p,r) \in \mathbb{Q}_0$ for $r \ge p$ and $b(p,p) = p!\binom{2p}{p}^2/2^{4p}$. For $1 \le p \le s$,

$$\left.\begin{aligned} &|(z)_p| \le (|z| + s)^p \le \left(\tfrac{3}{2}|z|\right)^p \\ &|(2z)_{2p}| \ge (2|z| - 2s)^{2p} \ge |z|^{2p}\,, \end{aligned}\right\} \qquad (|z| \ge 2s)$$

hence,

$$\left|\frac{(z)_p^3}{(2z)_{2p}^2}\right| \le \left(\frac{4}{|z|}\right)^p \qquad\qquad (|z| \ge 2s). \tag{7}$$

For $r, s \ge p \ge 1$, Laurent's Theorem (see [5; V. 2, p. 6]), standard estimates for the integral and (7) give

$$\begin{aligned} \big|b(p,r)\big| &= \left| \frac{1}{2\pi i} \oint_{|z|=2s} \frac{f_p(z)}{z^{r+1}}\,\mathrm{dz} \right| \\ &\le p!\binom{2p}{p}^2 \left(\frac{2}{s}\right)^p \frac{1}{(2s)^r} \le s!\binom{2s}{s}^2 . \end{aligned} \tag{8}$$

Then (8) gives

$$\left|\sum_{r=s}^{\infty} b(p,r) z^{-r}\right| \le |z|^{-s} s! \binom{2s}{s}^2 \sum_{t=0}^{\infty} |z|^{-t} = O\big(|z|^{-s}\big)\,, \tag{9}$$

hence, (6,9) give

$$f_p(z) = \sum_{r=p}^{s-1} b(p,r) z^{-r} + O\big(|z|^{-s}\big) \tag{10}$$

where $f_s(z) = O\big(|z|^{-s}\big)$. For $s \ge 1$, (10) gives

$$g_s(z) := \sum_{p=0}^{s-1} f_p(z) \qquad\qquad (z \in \mathbb{C};\ |z| \ge s)$$

$$= \sum_{r=0}^{s-1} c(s,r) z^{-r} + O(|z|^{-s}) \tag{11}$$

where $c(s,0) = 1$ for $s \ge 1$ and $c(s,r) = \sum_{p=1}^{r} b(p,r) \in \mathbb{Q}_0$ for $1 \le r \le s-1$. Observe that $c(s+1,r) = c(s,r)$ for $0 \le r \le s-1$. The analysis for $n = 2m \ge 2s+2$ is identical except that P_n includes $b_m^2 \binom{n}{m}^{-1}$ not $2b_m^2 \binom{n}{m}^{-1}$. Then (3–5,11) give the following complete asymptotic expansion for $P_n/2\binom{2n}{n}^2$.

Theorem. Fix $s \ge 1$. There exist effectively calculable nonnegative rational numbers $c(s,0) = 1$, $c(s,1)$, ... , $c(s,s-1)$ so that

$$P_n/2\binom{2n}{n}^2 = \sum_{r=0}^{s-1} c(s,r) n^{-r} + O(n^{-s}) \ \text{ as } n \to \infty. \qquad \blacksquare$$

Let

$$h_s(n) = \sum_{r=0}^{s-1} c(s,r) n^{-r}$$

hence,

$$h_1(n) = 1, \quad h_2(n) = 1 + \frac{1}{4n}, \quad h_3(n) = 1 + \frac{1}{4n} + \frac{17}{32n^2},$$

$$h_4(n) = 1 + \frac{1}{4n} + \frac{17}{32n^2} + \frac{207}{128n^3} \quad \text{and}$$

$$h_5(n) = 1 + \frac{1}{4n} + \frac{17}{32n^2} + \frac{207}{128n^3} + \frac{14875}{2048n^4}.$$

Let $Q_s(n) = \big[P_n/2\binom{2n}{n}^2 - h_s(n)\big] n^s$. Our theorem shows $Q_s(n) = O(1)$ as $n \to \infty$. The table below which gives the first 10 digits of $Q_2(n)$ and $Q_3(n)$ for several values of n was found using *Mathematica*. This provides numerical evidence for our theorem.

n	$Q_2(n)$	$Q_3(n)$
100	.5481946735	1.6944673581
200	.5395231003	1.6546200668
300	.5367229603	1.6418880975
400	.5353390483	1.6356193435
500	.5345137769	1.6318884961
600	.5339656896	1.6294137740
700	.5335752174	1.6276521934
800	.5332829178	1.6263343131
900	.5330559013	1.6253112456
1000	.5328744940	1.6244940166
2000	.5320604149	1.6208298850
3000	.5317898711	1.6196133523

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