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On Obláth's problem

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Abstract. In this paper we determine those squares whose decimal representation consists of $k \geq 2$ digits such that $k - 1$ of them equal.

1 Introduction

R. Obláth [5] succeeded in almost entirely solving the problem of finding all the numbers n^m ($n, m \in \mathbb{N}$, $n \geq 2$, $m \geq 2$) that have equal digits. The special case $m = 2$ is a very well known result, although its proof involves no difficulty. In this connection, the following question naturally arises: is it possible to determine all of the squares having all digits but one equal?

The answer is given by

Theorem 1.1 *The squares whose decimal representation makes use of $k \geq 2$ digits, such that $k - 1$ of these digits are equal, are precisely* 16, 25, 36, 49, 64, 81, 121, 144, 225, 441, 484, 676, 1444, 44944, 10^{2i}, $4 \cdot 10^{2i}$ *and* $9 \cdot 10^{2i}$ *with* $i \geq 1$.

When we are looking for the squares with k digits among which $k - 1$ digits equal 0, we immediately get that the corresponding numbers are 10^{2i}, $4 \cdot 10^{2i}$ and $9 \cdot 10^{2i}$ with $i \geq 1$.

A simple computation shows that the numbers with at most 4 digits verifying the condition in the statement are just the ones listed above.

Since every natural number can be written in the form $50000k \pm r$ with $0 \leq r \leq 25000$, and $(50000k \pm r)^2 \equiv r^2 \pmod{100000}$, we compute r^2 for $r \leq 25000$ and find that the last 4 digits of any square can be equal only when all of them equal 0, which solves Obláth's problem for squares having $k \geq 4$ digits.

We select the squares such that 4 of the last 5 digits are equal, because these point out the possible squares with $k \geq 5$ digits, $k-1$ digits of them being equal. If one excludes the numbers for which there are $k-1$ digits equal to 0, then there still remain 22 types of numbers, namely:

$a_1 = 1\cdots 121$	$a_7 = 4\cdots 441$	$a_{13} = 4\cdots 4944$	$a_{18} = 7\cdots 76$
$a_2 = 1\cdots 161$	$a_8 = 4\cdots 449$	$a_{14} = 4\cdots 45444$	$a_{19} = 8\cdots 81$
$a_3 = 2\cdots 224$	$a_9 = 4\cdots 464$	$a_{15} = 4\cdots 49444$	$a_{20} = 8\cdots 89$
$a_4 = 2\cdots 225$	$a_{10} = 4\cdots 484$	$a_{16} = 5\cdots 56$	$a_{21} = 9\cdots 929$
$a_5 = 4\cdots 41444$	$a_{11} = 4\cdots 4544$	$a_{17} = 6\cdots 656$	$a_{22} = 9\cdots 969$
$a_6 = 4\cdots 4144$	$a_{12} = 4\cdots 4644$		

One will show that, among these numbers with $k \geq 5$ digits, only 44944 is a square. The exclusion of the other numbers can be carried out fairly easily in certain cases, as we show in §2. In the other cases we will solve equations of the type

$$x^2 - dy^2 = k \tag{1}$$

(where $d, k \in \mathbb{Z}^*$, $d > 0$ and $\sqrt{d} \notin \mathbb{Z}$) in integers. The literature concerning equation (1) is rather extensive. In this connection, we mention [1, 2, 3, 4].

We now recall the solving method (in accordance with [2]). We denote by (r, s) the minimal positive solution to the equation

$$x^2 - dy^2 = 1 \tag{2}$$

and by $\varepsilon = r + s\sqrt{d}$. We determine the "small" solutions to equation (1) (if any). They generate all the solutions.

Theorem 1.2 *We denote by $\mu_i = a_i + b_i\sqrt{d}, i = \overline{1, m}$ all the numbers with the property that (a_i, b_i) is a solution in nonnegative integers to equation (1) with $a_i \leq \sqrt{|k|\varepsilon}$ and $b_i \leq \sqrt{\varepsilon|k|/d}$ (if any). If x and y are solutions to (1) then there exist $i, n \in \mathbb{Z}$ such that $1 \leq i \leq m$ and $x + y\sqrt{d} = \pm\mu_i\varepsilon^n$ or $x + y\sqrt{d} = \pm\bar{\mu}_i\varepsilon^n$.*

We will use this theorem in §3.

2 Excluding the simple cases

We assume in this section that $k \geq 5$ and a_n is a square, hence $9a_n$ is a square as well. Make use of simple reasonings, we shall show that this fact is impossible. To this end, we use the symbol of Legendre in some cases.

The 16 cases which have to be excluded will be exposed in a concise form, inasmuch as some of them are quite similar:

$$a_3, a_4, a_{20};\ a_2, a_{15}, a_{18};\ a_{11}, a_{17}.$$

We mention that each of the cases below is concluded by a contradictory assertion, thus proving the impossibility of the corresponding case.

1. We have $9a_2 = 10^k + 449 \equiv (-1)^k + 9 \pmod{11}$. But $\left(\frac{8}{11}\right) = \left(\frac{10}{11}\right) = -1$.

2. It follows by $9a_3 = 2(10^k + 8) = (4x)^2$ that $2^{k-3}5^k = (x-1)(x+1)$. Since $(x-1, x+1) = 2$, we have $5^k \mid x + \varepsilon$ with $\varepsilon \in \{-1, 1\}$, whence $x + 1 \geq 5^k$. Consequently $2^{k-3} \cdot 5^k = x^2 - 1 \geq 5^k(5^k - 2)$, hence $2^{k-3} \geq 5^k - 2$.

3. By $9a_4 = 2 \cdot 10^k + 25 = (5x)^2$, we have $2^{k+1} \cdot 5^{k-2} = (x-1)(x+1)$, whence $2^{k+1} \cdot 5^{k-2} \geq 5^{k-2}(5^{k-2} - 2)$.

4. We have $9a_5 = 4 \cdot 10^k - 27004 = (2x)^2$, whence $10^k - 6751 = x^2$. When k is an odd number, we have $10^k - 6751 \equiv 2 \pmod{11}$, but $\left(\frac{2}{11}\right) = -1$. If $k = 2h$ with $h \geq 3$, then $(10^h - x)(10^h + x) = 6751$, whence $10^h - x = a$, $10^h + x = b$, where we have either $(a, b) = (1, 6751)$ or $(a, b) = (43, 157)$. Since $2 \cdot 10^h = a + b$, it follows that either $2 \cdot 10^h = 6752$ or $2 \cdot 10^h = 200$, although $h \geq 3$.

5. By $9a_6 = 4 \cdot 10^k - 2704 = (4x)^2$ it follows that $5^2 \cdot 10^{k-2} - 169 = x^2$. Since $k \geq 5$, we get that $x^2 = 5^2 \cdot 10^{k-2} - 169 \overset{4}{\equiv} -169 \overset{4}{\equiv} 3$.

6. We have $a_9 = 4 \cdot 11 \cdots 16$, but $11 \cdots 16$ does not occur among the numbers a_i.

7. We have $a_{10} = 4a_1$, and we shall get the contradiction after we study a_1 for $k \geq 5$.

8. We have $9a_{11} = 4 \cdot 10^k + 896 = (8x)^2$, hence $2^{k-4}5^k + 14 = x^2$. It follows that $x \vdots 2$. Therefore $x^2 \vdots 4$, and $k = 5$. In this case we get $x^2 = 6264$.

9. We have $a_{12} = 4a_2$, but $a_2 \neq x^2$.

10. By $9a_{15} = 4 \cdot 10^k + 44996 = (2x)^2$ it follows that $10^k + 11249 = x^2$. Then $10^k + 11249 \equiv (-1)^k + 7 \pmod{11}$, but $\left(\frac{6}{11}\right) = \left(\frac{8}{11}\right) = -1$.

11. Since $4a_{16} = \underbrace{22 \cdots 2}_{k}4$ and $a_3 \neq x^2$, it follows that $a_{16} \neq y^2$.

12. We have $9a_{17} = 6 \cdot 10^k - 96 = (4x)^2$, hence $3 \cdot 2^{k-3}5^k - 6 = x^2$. But $x^2 \vdots 4$ and $3 \cdot 2^{k-3}5^k \vdots 4$ (because $k \geq 5$).

13. We have $9a_{18} = 7 \cdot 10^k - 16 \equiv 7(-1)^k - 5 \pmod{11}$. But $\left(\frac{2}{11}\right) = \left(\frac{10}{11}\right) = -1$.

14. By $9a_{20} = 8 \cdot 10^k + 1 = x^2$ it follows that $(x-1)(x+1) = 2^{k+3}5^k$ and then $2^{k+3}5^k \geq 5^k(5^k - 2)$.

15. We have $a_{21} \equiv 2 \pmod{9}$.

16. We have $a_{22} \equiv 6 \pmod{9}$.

3 The six difficult cases

Just as in the previous cases, the numbers under consideration have $k \geq 5$ digits, and $k - 1$ of these digits are equal.

1. For $a_1 = 11 \cdots 121 = x^2$ it follows that $(10^k - 1)/9 + 10 = x^2$. We denote $y = 3x$ and, since $k \geq 5$, we have $y > 316$ and

$$10^k - y^2 = -89. \tag{3}$$

For $k = 2m$ we have $(10^m - y)(10^m + y) = -89$, whence we get both $10^m - y = -1$ and $10^m + y = 89$, which is a contradiction.

For $k = 2m + 1$, we denote $z = 10^m$ and then

$$y^2 - 10z^2 = 89.$$

The primitive solution of the Pell equation

$$x^2 - 10y^2 = 1$$

is $(r, s) = (19, 6)$. Making use of Theorem 1.2 in the Introduction, we get $b_i \leq \sqrt{\frac{89}{10}\left(19 + 6\sqrt{10}\right)}$, hence $b_i \leq 18$. We find $b_1 = 8, a_1 = 27$, and $b_2 = 10, a_2 = 33$.

It follows that either

$$y + z\sqrt{10} = \left(\pm 27 \pm 8\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t$$

or

$$y + z\sqrt{10} = \left(\pm 33 \pm 10\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t,$$

with $t \in \mathbb{Z}$. Since $y > 0$, $z > 0$, we have only the solutions

$$y + z\sqrt{10} = \left(27 \pm 8\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t$$

and

$$y + z\sqrt{10} = \left(33 \pm 10\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t,$$

with $t \in \mathbb{Z}$. Since $\frac{27+8\sqrt{10}}{19+6\sqrt{10}} < 2$ and $\frac{33+10\sqrt{10}}{19+6\sqrt{10}} < 2$, it follows that $t \in \mathbb{N}$. Let $\left(19 + 6\sqrt{10}\right)^t = a_t + b_t\sqrt{10}$, $a_t, b_t \in \mathbb{N}$, $a_0 = 1, b_0 = 0$. For $t \geq 1$, we have the following equalities:

$$a_t = 19^t + C_t^2 19^{t-2} \cdot 6^2 \cdot 10 + \cdots \tag{4}$$

and

$$b_t = C_t^1 19^{t-1} \cdot 6 + C_t^3 19^{t-3} \cdot 6^3 \cdot 10 + \cdots. \tag{5}$$

We have $a_t \equiv 1 \pmod 3$ and $b_t \equiv 0 \pmod 3$. Since $z = 10^m \equiv 1 \pmod 3$, we only have one of the situations

$$y + z\sqrt{10} = \left(27 - 8\sqrt{10}\right)\left(19 + 6\sqrt{10}\right), \quad t \in \mathbb{N}, \tag{6}$$

and

$$y + z\sqrt{10} = \left(33 + 10\sqrt{10}\right)\left(19 + 6\sqrt{10}\right), \quad t \in \mathbb{N}. \tag{7}$$

a) In the case of the relation (6), we have the identity

$$z = 10^m = 27b_t - 8a_t. \tag{8}$$

Since $k \geq 5$, it follows that $m \geq 2$.

For $m = 2$, the equation (3) takes the form $y^2 - 10^5 = 89$, and has no integer solutions.

For $m \geq 3$, it follows that $8 \mid b_t$. By (5) we have $b_t \equiv 6t \cdot 19^{t-1} \pmod 8$, whence $t = 4h$. It then follows by (4) and (5) that

$$a_t \equiv 6^{4h} \cdot 10^{2h} \pmod{19} \text{ and } 19 \mid b_t.$$

By (8) we get $10^m \equiv -8 \cdot 6^{4h} \cdot 10^{2h} \pmod{19}$, whence

$$\left(\frac{10^m}{19}\right) = \left(\frac{-8 \cdot 6^{4t} \cdot 10^{2h}}{19}\right) = \left(\frac{-2}{19}\right) = 1.$$

Since $\left(\frac{10}{19}\right) = \left(\frac{-9}{19}\right) = (-1) \cdot \left(\frac{3^2}{19}\right) = -1$, it follows that $m = 2f$.

The equation (3) takes the form

$$10^{4f+1} + 89 = y^2. \tag{9}$$

We have $10^4 \equiv 1 \pmod{101}$, hence $10^{4f+1} \equiv 10 \pmod{101}$. In view of (9), it follows that

$$y^2 \equiv 99 \pmod{101}.$$

But $\left(\frac{99}{101}\right) = \left(\frac{-2}{101}\right) = \left(\frac{2}{101}\right) = -1$, and thus a contradiction.

b) It follows by (7) that

$$z = 10^m = 33b_t + 10a_t. \tag{10}$$

Since $m \geq 3$, it follows that $b_t + 2a_t \equiv 0 \pmod 8$. By (4) and (5) we have $a_t \equiv 19^t \pmod 8$ and $b_t \equiv 6t \cdot 19^{t-1} \pmod 8$. Therefore $6t \cdot 19^{t-1} + 2 \cdot 19^t \equiv 0 \pmod 8$, which in turn implies $3t + 19 \equiv 0 \pmod 4$ and $t = 4h+3$. Now (4), (5) and (10) imply that $10^m \equiv 33 \cdot 6^{4h+3} \cdot 10^{2h+1} \pmod{19}$, whence

$$\begin{aligned}\left(\frac{10^m}{19}\right) &= \left(\frac{33 \cdot 6 \cdot 10}{19}\right) = \left(\frac{3^2 \cdot 2^2 \cdot 55}{19}\right) \\ &= \left(\frac{55}{19}\right) = \left(\frac{-2}{19}\right) = -\left(\frac{2}{19}\right) = -(-1)^{(19^2-1)/8} \\ &= 1.\end{aligned}$$

Consequently $m = 2f$, and we get (9) again, which is a contradiction.

2. For $a_7 = 44 \cdots 41 = x^2$ it follows that $4 \cdot \frac{10^k - 1}{9} - 3 = x^2$, hence

$$4 \cdot 10^k - y^2 = 31, \tag{11}$$

where $y = 3x$.

If $k = 2m$, then $(2 \cdot 10^m - y)(2 \cdot 10^m + y) = 31$. Hence

$$2 \cdot 10^m - y = 1 \text{ and } 2 \cdot 10^m + y = 31,$$

which is a contradiction. If $k = 2m + 1$ then, denoting $z = 2 \cdot 10^m$, we get the equation

$$y^2 - 10z^2 = -31.$$

Just as in the previous case, we make use of Theorem 1.2 and, for $y > 0$, $z > 0$, we get that either

$$y + z\sqrt{10} = \left(3 + 2\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t \tag{12}$$

or

$$y + z\sqrt{10} = \left(-3 + 2\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t, \tag{13}$$

with $t \in \mathbb{N}$.

a) By (12) we obtain the equation:

$$2 \cdot 10^m = 3b_t + 2a_t. \tag{14}$$

Since $m \geq 2$, it follows by (5), (6) and (14) that $3 \cdot 6t \cdot 19^{t-1} + 2 \cdot 19^t \equiv 0 \pmod 8$, that is, $9t + 19 \equiv 0 \pmod 4$, whence $t = 4h + 1$. By (4) and (5) we get that $z \equiv 3 \cdot 6^{4h+1} \cdot 10^{2h} \pmod{19}$, that is, $10^m \equiv 3^2 \cdot 6^{4h} \cdot 10^{2h} \pmod{19}$, whence $\left(\frac{10^m}{19}\right) = 1$. Consequently m is an even number.

On the other hand, it follows by (14) that $3b_t + 2a_t \equiv 0 \pmod 5$, that is, $a_t \equiv b_t \pmod 5$. By (4) and (5) it follows that $a_t \equiv (-1)^t \pmod 5$ and $b_t \equiv (-1)^{t-1}t \pmod 5$, whence $t \equiv 4 \pmod 5$. We have $(3 + \sqrt{10})^5 = 4443 + 1405\sqrt{10} \equiv -53 \pmod{281}$, hence $(19 + 6\sqrt{10})^5 = \left((3 + \sqrt{10})^5\right)^2 \equiv 53^2 \equiv -1 \pmod{281}$. Consequently

$$\begin{aligned} y + 2 \cdot 10^m\sqrt{10} &= \left(3 + 2\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^{t+1}\left(19 - 6\sqrt{10}\right) \\ &= \left(-63 + 20\sqrt{10}\right)\left[\left(19 + 6\sqrt{10}\right)^5\right]^{(t+1)/5} \\ &\equiv -63 + 20\sqrt{10} \pmod{281}. \end{aligned}$$

We have taken into account that $t \equiv 4 \pmod 5$ and t is an odd number.

We have $2 \cdot 10^m \equiv 20 \pmod{281}$, that is, $10^{m-1} \equiv 1 \pmod{281}$. Since $10^7 \equiv 53 \pmod{281}$, it follows that $10^{14} \equiv -1 \pmod{281}$ and $10^{28} \equiv 1 \pmod{281}$. Thus we have $\operatorname{ord} \overline{10} = 28$ in $\mathbb{Z}_{281}$, whence $28 \mid m - 1$, which is a contradiction since m is even.

b) It follows by (13) that

$$2 \cdot 10^m = -3b_t + 2a_t. \tag{15}$$

Since $m \geq 2$, it follows that $-3b_t + 2a_t \equiv 0 \pmod 8$. We get by (4) and (5) that $t = 4h + 3$ and then $2 \cdot 10^m \equiv -3 \cdot 6^{4h+3} \cdot 10^{2h+1} \pmod{19}$. Therefore $\left(\frac{10^m}{19}\right) = 1$, and m is even. Also by (15) we get $a_t + b_t \equiv 0 \pmod 5$, and in view of (4) and (5) we have $t \equiv 1 \pmod 5$. The relation (13) can be written as:

$$\begin{aligned} y + 2 \cdot 10^m\sqrt{10} &= \left(-3 + 2\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^{t-1}\left(19 + 6\sqrt{10}\right) \\ &= \left(63 + 20\sqrt{10}\right)\left(\left(19 + 6\sqrt{10}\right)^5\right)^{(t-1)/5} \\ &\equiv 63 + 20\sqrt{10} \pmod{281}. \end{aligned}$$

Just as in the case *a)*, it follows that $2 \cdot 10^m \equiv 20 \pmod{281}$. The relation $10^{m-1} \equiv 1 \pmod{281}$ contradicts the fact that m is even.

3. For $a_8 = 44 \cdots 49 = x^2$ we have $4 \cdot \frac{10^k - 1}{9} + 5 = x^2$. We denote $3x = y$ and then

$$4 \cdot 10^k + 41 = y^2.$$

For $k = 2m$, we get the equalities $y - 2 \cdot 10^m = 1$ and $y + 2 \cdot 10^m = 41$, which is a contradiction because $m \geq 2$.

For $k = 2m+1$ we set $z = 2 \cdot 10^m$, and then $y^2 - 10z^2 = 41$. Whence for $y > 0$ and $z > 0$ we get either

$$y + 2 \cdot 10^m \sqrt{10} = \left(9 - 2\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t, \tag{16}$$

or

$$y + 2 \cdot 10^m \sqrt{10} = \left(9 + 2\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t, \tag{17}$$

where t is a natural number.

a) It follows by (16) that

$$2 \cdot 10^m = 9b_t - 2a_t.$$

Since $2 \cdot 10^m \equiv 2 \pmod 3$, and on the other hand we have by (4) that $a_t \equiv 1 \pmod 3$, we get the contradiction $2 \equiv -2 \pmod 3$.

b) It follows by (17) that

$$2 \cdot 10^m = 2a_t + 9b_t. \tag{18}$$

Then $b_t \equiv 2a_t \pmod 5$, whence $t \equiv 3 \pmod 5$. We also have $b_t + 2a_t \equiv 0 \pmod 4$, hence t is odd.

The equality (17) takes the form

$$\begin{aligned} y + 2 \cdot 10^m \sqrt{10} &= \left(9 + 2\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^{t+2}\left(19 - 6\sqrt{10}\right)^2 \\ &= \left(1929 - 610\sqrt{10}\right)\left[\left(19 + 6\sqrt{10}\right)^5\right]^{(t+2)/5} \\ &\equiv 38 + 48\sqrt{10} \pmod{281}, \end{aligned}$$

since $(t+2)/5$ is odd and $\left(19 + 6\sqrt{10}\right)^5 \equiv -1 \pmod{281}$.

Thus $2 \cdot 10^m \equiv 48 \pmod{281}$, hence $\left(\frac{2 \cdot 10^m}{281}\right) = \left(\frac{48}{281}\right)$. Therefore,

$$(-1)^{\frac{281^2-1}{8}(m+1)} \left(\frac{5^m}{281}\right) = \left(\frac{3}{281}\right).$$

We have $\left(\frac{5}{281}\right) = \left(\frac{281}{5}\right) = \left(\frac{1}{5}\right) = 1$ and $\left(\frac{3}{281}\right) = \left(\frac{281}{3}\right) = \left(\frac{2}{3}\right) = -1$, hence a contradiction.

4. For $a_{13} = 44 \cdots 4944 = x^2$, we have $4 \cdot \frac{10^k - 1}{9} + 500 = x^2$, that is, $y^2 - 25 \cdot 10^{k-2} = 281$, where $y = \frac{3}{4}x$.

If $k = 2m + 2$, then $(y - 5 \cdot 10^m)(y + 5 \cdot 10^m) = 281$, whence $y - 5 \cdot 10^m = 1$ and $y + 5 \cdot 10^m = 281$. One gets the contradiction $10^{m+1} = 280$.

If $k = 2m + 3$, we denote $z = 5 \cdot 10^m$. We have $m \geq 1$ and

$$y^2 - 10z^2 = 281 \tag{19}$$

whence either

$$y+z\sqrt{10}=\left(21+4\sqrt{10}\right)\left(19+6\sqrt{10}\right)^t,\quad t\in\mathbb{N}, \tag{20}$$

or

$$y+z\sqrt{10}=\left(21-4\sqrt{10}\right)\left(19+6\sqrt{10}\right)^t,\quad t\in\mathbb{N}^*. \tag{21}$$

a) By (20) it follows that

$$5\cdot 10^m=21b_t+4a_t,$$

hence $5\cdot 10^m\equiv 4a_t \pmod 3$. Since $a_t\equiv 1 \pmod 3$, we get the contradiction $5\equiv 4 \pmod 3$.

b) By (21), if $t=1$ then $y=159$ and $z=50$, whence $m=1$ and then $k=5$. One thus get the number

$$44944=212^2.$$

For $t\geq 2$, it follows that $y+\sqrt{10}\cdot 5\cdot 10^m=\left(159+50\sqrt{10}\right)\left(19+6\sqrt{10}\right)^s$, $s\geq 1$, hence

$$5\cdot 10^m=159b_s+50a_s. \tag{22}$$

For $m=0$ and $m=2$, equation (19) has no integer solutions, hence we may consider $m\geq 3$. We have by (22) that $b_s\equiv 2a_s \pmod 8$. Hence it follows by (4) and (5) that $6s\cdot 19^{s-1}\equiv 2\cdot 19^s \pmod 8$. Therefore $3s\equiv 19 \pmod 4$ and $s=4h+1$. Also by (22) we have $5\cdot 10^m\equiv 159\cdot 6^s\cdot 10^{2h} \pmod{19}$, hence $\left(\frac{5\cdot 10^m}{19}\right)=\left(\frac{159}{19}\right)\left(\frac{6^s}{19}\right)=\left(\frac{7}{19}\right)$, because $\left(\frac{6}{19}\right)=1$. Since

$$\left(\frac{5}{19}\right)=\left(\frac{-14}{19}\right)=(-1)^{\frac{19-1}{2}}(-1)^{\frac{19^2-1}{8}}\left(\frac{7}{19}\right)=\left(\frac{7}{19}\right),$$

we have $\left(\frac{10^m}{19}\right)=1$, that is, $\left(\frac{10}{19}\right)^m=1$, whence $(-1)^m=1$. Thus $m=2n$.

Equality (19) takes the form

$$y^2=25\cdot 10^{2m+1}+281=25\cdot 10^{4n+1}+281.$$

Since $10^4\equiv 1 \pmod{101}$, it follows that $y^2\equiv 250+281 \pmod{101}$. Hence $y^2\equiv 26 \pmod{101}$, whence $\left(\frac{26}{101}\right)=1$. But $\left(\frac{26}{101}\right)=\left(\frac{-75}{101}\right)=\left(\frac{3}{101}\right)=\left(\frac{101}{3}\right)=\left(\frac{2}{3}\right)=-1$, which is a contradiction.

5. For $a_{14}=44\cdots 45444=x^2$ and $y=3x/2$ we have the equation:

$$y^2-10^k=2249.$$

If $k=2m$, $m\geq 3$, we have either

$$y-10^m=1 \text{ and } y+10^m=2249$$

or

$$y-10^m=13 \text{ and } y+10^m=173,$$

and none of these systems has solutions.

If $k=2m+1$, then $m\geq 2$. With $z=10^m$ we have the equation:

$$y^2-10z^2=2249. \tag{23}$$

The initial solutions (a, b) of the equation are $(57, 10)$, $(147, 44)$, $(153, 46)$, hence the solutions with $y > 0$, $z > 0$ are given by the identities:

$$y + 10^m\sqrt{10} = \left(57 - 10\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t, \tag{24}$$

$$y + 10^m\sqrt{10} = \left(57 + 10\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t, \tag{25}$$

$$y + 10^m\sqrt{10} = \left(147 - 44\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t, \tag{26}$$

$$y + 10^m\sqrt{10} = \left(147 + 44\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t, \tag{27}$$

$$y + 10^m\sqrt{10} = \left(153 - 46\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t, \tag{28}$$

$$y + 10^m\sqrt{10} = \left(153 + 46\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^t, \tag{29}$$

where $t \in \mathbb{N}$. We get by (24) that $10^m = 57b_t - 10a_t$, hence $10^m \equiv -a_t \pmod 3$, which yields the contradiction $1 \equiv -1 \pmod 3$.

If (27) was true, then $10^m = 147b_t + 44a_t$. Since $a_t \equiv 1 \pmod 3$ and $10^m \equiv 1 \pmod 3$, the contradiction $1 \equiv 44 \pmod 3$ follows.

In the case when (28) holds, we get $10^m = 153b_t - 46a_t$, whence the contradiction $1 \equiv -46 \pmod 3$.

We still have to study three situations.

a) We have by (25) that

$$10^m = 57b_t + 10a_t. \tag{30}$$

Since $m \geq 2$, it follows that $b_t + 2a_t \equiv 0 \pmod 4$. By (4) and (5) we have $6t(-1)^{t-1} + 2(-1)^t \equiv 0 \pmod 4$. Therefore $3t - 1 \equiv 0 \pmod 2$, and we get that t is odd. It then follows that $19 \mid a_t$, and by (30) we deduce the contradiction $19 \mid 10^m$.

b) It follows by (26) that

$$10^m = 147b_t - 44a_t. \tag{31}$$

Just as in the case a), by considering congruences $\pmod 4$, we get that t is even.

Also by (31) we have $2b_t + a_t \equiv 0 \pmod 5$, whence by (4) and (5) we get $t \equiv 3 \pmod 5$. Then $(t + 2)/5$ is an even natural number. The relation (26) takes the form

$$\begin{aligned} y + 10^m\sqrt{10} &= \left(147 - 44\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^{t+2}\left(19 - 6\sqrt{10}\right)^2 \\ &\equiv \left(147 - 44\sqrt{10}\right)\left(721 - 228\sqrt{10}\right) \pmod{281} \\ &\equiv \left(147 - 44\sqrt{10}\right)\left(-122 + 53\sqrt{10}\right) \pmod{281}. \end{aligned}$$

It then follows that $10^m \equiv 13159 \pmod{281} \equiv -48 \pmod{281}$, hence $\left(\frac{10^m}{281}\right) = \left(\frac{-48}{281}\right)$. One directly gets a contradiction, observing that $\left(\frac{10}{281}\right) = \left(\frac{-1}{281}\right) = \left(\frac{16}{281}\right) = 1$ and $\left(\frac{3}{281}\right) = -1$.

c) By (29) we have the equation:

$$10^m = 153b_t + 46a_t. \tag{32}$$

For $m=2$, we get the number 102249 which is not a square. Hence $m \geq 3$.

For $m \geq 3$, we have $b_t - 2a_t \equiv 0 \pmod 8$, and then $t = 4h+1$. Also by (32) we have $3b_t + a_t \equiv 0 \pmod 5$, whence $t \equiv 2 \pmod 5$. Therefore $(t-2)/5$ is an odd natural number, which in turn implies that $\left(19+6\sqrt{10}\right)^{(t-2)/5} \equiv -1 \pmod{281}$. By (29) we have the following relations:

$$\begin{aligned} y + 10^m\sqrt{10} &= \left(153+46\sqrt{10}\right)\left(19+6\sqrt{10}\right)^{t-2}\left(19+6\sqrt{10}\right)^2 \\ &\equiv -\left(153+46\sqrt{10}\right)\left(-122-53\sqrt{10}\right) \pmod{281}. \end{aligned}$$

Then $10^m \equiv 13721 \equiv -48 \pmod{281}$, that is, the contradiction from *b)*.

6. For $a_{19} = 88\cdots 81 = x^2$, denoting $y = 3x$ we get the equation:

$$y^2 = 8 \cdot 10^k - 71.$$

If $k = 2m$, then $m \geq 3$. We denote $z = 2\cdot 10^m$ and get the identity:

$$y^2 - 2z^2 = -71.$$

It then follows for $y, z > 0$ that either

$$y + z\sqrt{2} = \left(1+6\sqrt{2}\right)\left(3+2\sqrt{2}\right)^t, \tag{33}$$

or

$$y + z\sqrt{2} = \left(-1+6\sqrt{2}\right)\left(3+2\sqrt{2}\right)^t. \tag{34}$$

We set $\left(3+2\sqrt{2}\right)^t = c_t + d_t\sqrt{2}$. For $t \geq 1$ we then have the equalities:

$$c_t = 3^t + C_t^2 \cdot 3^{t-2} \cdot 2^2 \cdot 2 + \cdots, \tag{35}$$

$$d_t = 2t \cdot 3^{t-1} + C_t^3 \cdot 3^{t-3} \cdot 2^4 + \cdots \tag{36}$$

a) We have by (33) that $d_t + 6c_t \equiv 0 \pmod 8$. Then $2t+18 \equiv 0 \pmod 8$, hence $t = 4h+3$, whence $d_t \equiv 2^{t+(t-1)/2} \pmod 3$. We have $z = d_t + 6c_t$. It follows that $2\cdot 10^m \equiv d_t \pmod 3$, hence $2\cdot 10^m \equiv 2^{6h+4} \pmod 3$, whence $10^m \equiv 2^{6h+3} \pmod 3$, consequently,

$$\left(\frac{10^m}{3}\right) = \left(\frac{2^{6h+3}}{3}\right) = \left(\frac{2^{3k+1}}{3}\right)^2 \cdot \left(\frac{2}{3}\right) = -1.$$

Since $\left(\frac{10^m}{3}\right) = \left(\frac{1}{3}\right)$, a contradiction follows.

b) For $k = 2m$, we consider the equality (34) and we have $2\cdot 10^m = -d_t + 6c_t$, hence $d_t \equiv 6c_t \pmod 8$. It follows that $2t \equiv 18 \pmod 8$, that is, $t = 4h+1$. We then have $2\cdot 10^m \equiv -d_t \equiv -2^{4h+1}\cdot 2^{2h} \pmod 3$. Hence $10^m \equiv -(2^{3h})^2 \pmod 3$ and $\left(\frac{10^m}{3}\right) = \left(\frac{-1}{3}\right) = -1 \neq 1 = \left(\frac{10^m}{3}\right)$.

For $k = 2m+1$ we have $m \geq 2$. Denoting $z = 4\cdot 10^m$, we get the equation:

$$y^2 - 5z^2 = -71.$$

For $y, z > 0$ we have either

$$y + z\sqrt{5} = \left(3 + 4\sqrt{5}\right)\left(9 + 4\sqrt{5}\right)^t, \tag{37}$$

or

$$y + z\sqrt{5} = \left(-3 + 4\sqrt{5}\right)\left(9 + 4\sqrt{5}\right)^t, \tag{38}$$

where t is a natural number.

We put $\left(9 + 4\sqrt{5}\right)^t = e_t + f_t\sqrt{5}$ and then

$$e_t = 9^t + C_t^2 \cdot 9^{t-2} \cdot 4^2 \cdot 5 + \cdots, \tag{39}$$

$$f_t = 4t \cdot 9^{t-1} + C_t^3 \cdot 9^{t-3} \cdot 4^3 \cdot 5 + \cdots \tag{40}$$

It follows by (37) and (38) that $z = 4 \cdot 10^m = 4e_t \pm 3f_t$, hence $4e_t \pm 3f_t \equiv 0 \pmod{8}$. By (39) and (40) we get $4 \pm 4t \equiv 0 \pmod{8}$, whence $t = 2h + 1$.

By $4 \cdot 10^m = 4e_t \pm 3f_t$ we infer $4 \cdot 10^m \equiv e_t \pmod{3}$. Since t is odd, we have $e_t \equiv 0 \pmod{3}$, and the contradiction $4 \cdot 10^m \equiv 0 \pmod{3}$.

Remarks. It would be interesting to solve the similar problem involving numbers written with respect to some basis $b \geq 2$.

It might be more difficult to consider the same problem imposing the condition all-but-one-equal-digits to higher powers, instead of squares.

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