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A Common Generating Function for Catalan Numbers and Other Integer Sequences

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Abstract

Catalan numbers and other integer sequences (such as the triangular numbers) are shown to be particular cases of the same sequence array $g(n,m) = \frac{(2n+m)!}{m!n!(n+1)!}$. Some features of the sequence array are pointed out and a unique generating function is proposed.

1 Introduction

Catalan numbers can be found in many different combinatorial problems, as shown by Stanley [1], and exhaustive information about this sequence can be found in [2]. In this note I show that the Catalan numbers (A000108) and other known sequences (triangular numbers A000217, A034827, A001700, A002457, A002802, A002803, A007004, A024489) can be derived by the same generating function and are related to the same polynomial set.

2 The polynomials $j_m(y)$

Consider the following recurrence relation defining the polynomials $j_m(y)$:

$$j_0(y) = 1;$$
$$j_{m+1}(y) = y j_m(y) + \sum_{s=0}^{m} j_s(y)\, j_{m-s}(y)\,. \tag{1}$$

It may immediately be noticed that for $y = 0$ this formula coincides with the recursive definition of the Catalan numbers:

$$C_{s+1} = \sum_{m=0}^{s} C_m C_{s-m} \tag{2}$$

where

$$C_n = \frac{2n!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n}. \tag{3}$$

This means that C_n is the zero-order coefficient of the nth-order polynomial $j_n(y)$, i.e.,

$$j_m(y) = \sum_{q=0}^{m} e(m,q)\ y^q \tag{4}$$

and $e(m,0) = C_m$. The first few values of $e(m,q)$ are shown in Table 1.

$m \diagdown q$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	1							
1	1	1						
2	2	3	1					
3	5	10	6	1				
4	14	35	30	10	1			
5	42	126	140	70	15	1		
6	132	462	630	420	140	21	1	
7	429	1716	2772	2310	1050	252	28	1

Table 1: Some of the coefficients $e(m,q)$

Equation (4) can be introduced into (1) to obtain

$$\sum_{q=0}^{m+1} e(m+1,q)\ y^q = \sum_{q=0}^{m} e(m,q)\ y^{q+1} + \sum_{s=0}^{m}\sum_{p=0}^{m-s} e(m-s,p)\ \sum_{r=0}^{s} e(s,r)\ y^{p+r}$$

and transformed as follows:

$$\begin{aligned}
\textstyle\sum_{q=0}^{m+1} e(m+1,q)\ y^q &= \textstyle\sum_{q=1}^{m+1} e(m,q-1)\ y^q + \sum_{s=0}^{m}\sum_{p=0}^{m-s} e(m-s,p)\ \sum_{r=0}^{s} e(s,r)\ y^{p+r} = \\
&= \textstyle\sum_{q=1}^{m+1} e(m,q-1)\ y^q + \sum_{p=0}^{m}\ \sum_{r=0}^{m-p}\ y^{p+r}\ \left[\sum_{s=r}^{m-p} e(m-s,p)\ e(s,r)\right] = \\
&= \textstyle\sum_{q=1}^{m+1} e(m,q-1)\ y^q + \sum_{p=0}^{m}\ \sum_{q=p}^{m}\ y^q\ \left[\sum_{s=q-p}^{m-p} e(m-s,p)\ e(s,q-p)\right] = \\
&= \textstyle\sum_{q=1}^{m+1} e(m,q-1)\ y^q + \sum_{q=0}^{m}\ \sum_{p=0}^{q}\ \left[\sum_{s=q-p}^{m-p} e(m-s,p)\ e(s,q-p)\right] y^q = \\
&= \textstyle\sum_{q=1}^{m+1} e(m,q-1)\ y^q + \sum_{q=0}^{m}\ \sum_{p=0}^{q}\ \left[\sum_{l=q}^{m} e(m-l+p,p)\ e(l-p,q-p)\right] y^q.
\end{aligned}$$

The following set of equations can then be obtained for any natural number m:

(1) for $q = 0$:

$$e(m+1,0) = \sum_{s=0}^{m} e(m-s,0)\ e(s,0),$$

whose solution is

$$e(m,0) = C_m = \frac{1}{m+1}\binom{2m}{m}. \tag{5}$$

(2) for $0 < q \le m$:

$$e(m+1,q) = e(m,q-1) + \sum_{p=0}^{q} \sum_{l=q}^{m} e(m-l+p,p)\; e(l-p,q-p). \tag{6}$$

(3) for $q = m+1$:

$$e(m+1,m+1) = e(m,m) = \cdots = 1. \tag{7}$$

It is useful to introduce the modified matrix $g(n,k)$ defined as follows:

$$\begin{aligned} g(n,k) &= e(n+k,k) \\ e(n,k) &= g(n-k,k) \end{aligned} \tag{8}$$

Table 2 reports the first few values:

$m \diagdown q$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	1	1	1	1	1	1	1	1
1	1	3	6	10	15	21	28	36
2	2	10	30	70	140	252	420	660
3	5	35	140	420	1050	2310	4620	8580
4	14	126	630	2310	6930	18018	42042	90090
5	42	462	2772	12012	42042	126126	336336	816816
6	132	1716	12012	60060	240240	816816	2450448	6651216
7	429	6435	51480	291720	1312740	4988412	16628040	49884120

Table 2: Some values of the coefficients $g(m,q)$

Equations (5), (6), (7) then become:

(1) for $q = 0$:

$$g(n,0) = C_n. \tag{9}$$

(2) for $0 < q \le n+1$:

$$g(n+1,q) = g(n+1,q-1) + \sum_{p=0}^{q} \sum_{l=0}^{n} g(n-l,p)\; q(l,q-p), \tag{10}$$

where $m-q$ was replaced by $n = m-q$. Moreover, from (7) we get $g(0,n) = 1$.

The solution of (10) can be written as follows:

$$g(n,q) = \frac{(2n+q)!}{q!n!(n+1)!} = C_n \binom{2n+q}{q}. \tag{11}$$

In fact, (9) is satisfied, and substituting (11) into (10), we get

$$\begin{aligned} C_{n+1}\binom{2(n+1)+q}{q} &= C_{n+1}\binom{2(n+1)+q-1}{q-1} + \\ &+ \sum_{l=0}^{n} C_{n-l}C_l \left[\sum_{p=0}^{q} \binom{2(n-l)+p}{p} \binom{2l+q-p}{q-p} \right] \end{aligned} \tag{12}$$

It is possible to show that (see appendix)

$$\sum_{p=0}^{q} \binom{2(n-l)+p}{p}\binom{2l+q-p}{q-p} = \binom{2n+q+1}{q},$$

and

$$\binom{2(n+1)+q}{q} - \binom{2(n+1)+q-1}{q-1} = \binom{2(n+1)+q-1}{q-1}\left[\frac{2(n+1)+q}{q} - 1\right]$$

$$= \left\{\frac{[2(n+1)+q-1]!}{[q-1]!\,[2(n+1)]!}\right\}\left[\frac{2(n+1)}{q}\right] = \left\{\frac{(2n+q+1)!}{q!\,(2n+1)!}\right\} = \binom{2n+q+1}{q}.$$

By using the recursive definition (2) of Catalan numbers, (12) becomes an indentity.

3 Some features of the array $g(n, q)$

The sequence

$$g(n, q) = \frac{2n+q!}{q!n!(n+1)!} = C_n\binom{2n+q}{q}$$

can be seen as a generalization of the Catalan sequence, as it reduces to the Catalan sequence for $q = 0$. There are also some other interesting features. In Table 3 I report the known names of the integer sequences, referenced in *The On-line Encyclopedia of Integer Sequences* [2], that can be extracted from the matrix $g(n, q)$.

		A000108	A001700	A002457	A002802	A002803	none
	$m \diagdown q$	**0**	**1**	**2**	**3**	**4**	**5**
-	**0**	1	1	1	1	1	1
A000217	**1**	1	3	6	10	15	21
A034827	**2**	2	10	30	70	140	252
none	**3**	5	35	140	420	1050	2310
none	**4**	14	126	630	2310	6930	18018
-	**5**	42	462	2772	12012	42042	126126
-	**6**	132	1716	12012	60060	240240	816816

Table 3: $g(m, q)$ numbers and names of known sequences

The first five columns correspond to the known sequences: A000108 (Catalan), A001700, A002457, A002802, A002803. The first two rows correspond to the sequences A000217 ($g(1, q)$, triangular numbers) and A034827. For the other rows and columns no reference was found by the author. Also the sequence on the main diagonal $g(k, k)$ (1,3,30,420,6930,126126,...) is known as A007004 and the sequence on the diagonal $g(k, k+1)$ (1,6,70,1050,18018, ...) is known as A024489.

4 Generating function

Consider the algebraic equation in J:

$$-xJ^2 + (1 - yx)J - 1 = 0, \tag{13}$$

and its solutions

$$J(x, y) = \frac{(1 - yx) \pm \sqrt{(1 - yx)^2 - 4x}}{2x}. \tag{14}$$

Let now suppose that $J(x, y)$ admits a Taylor expansion in x (which excludes the solution $J(x, y) = \frac{(1-yx)+\sqrt{(1-yx)^2-4x}}{2x}$ unlimited in $x = 0$)

$$J(x, y) = \sum_{m=0}^{\infty} j_m\,(y)\,x^m. \tag{15}$$

Substituting (15) into equation (13) we get

$$\begin{array}{c} 0 = -x \sum_{m=0}^{\infty} j_m\,(y)\,x^m \sum_{m=0}^{\infty} j_n\,(y)\,x^n + \sum_{m=0}^{\infty} j_m\,(y)\,x^m - yx \sum_{m=0}^{\infty} j_m\,(y)\,x^m - 1 = \\ = -x \sum_{s=0}^{\infty} j_s\,(y) \sum_{m=s}^{\infty} j_{s-m}\,(y)\,x^s + \sum_{m=0}^{\infty} j_m\,(y)\,x^m - y \sum_{m=0}^{\infty} j_m\,(y)\,x^{m+1} - 1 = \\ = -\sum_{s=0}^{\infty} \sum_{m=0}^{s} j_s\,(y)\,j_{s-m}\,(y)\,x^{s+1} + \sum_{s=0}^{\infty} j_s\,(y)\,x^s - y \sum_{s=0}^{\infty} j_s\,(y)\,x^{s+1} - 1 = \\ = j_0(y) - 1 + \sum_{s=0}^{\infty} \left[-\sum_{m=0}^{s} j_s\,(y)\,j_{s-m}\,(y) + j_{s+1}\,(y) - y j_{s-1}\,(y)\right] x^{s+1}. \end{array}$$

Then

$$\begin{array}{c} j_0(y) = 1 \\ j_{s+1}\,(y) = y\ j_s\,(y) + \sum_{m=0}^{s} j_s\,(y)\,j_{s-m}\,(y)\,, \end{array} \tag{16}$$

which is the recursive definition given by (1). This means that

$$j_m\,(y) = \lim_{x \to 0} \frac{1}{m!} \frac{d^m J(x, y)}{dx^m},$$

and for $y = 0$ the function $J(x, 0)$ is the generating function of the Catalan sequence

$$\begin{array}{c} j_m(0) = C_m \\ J(x, 0) = \frac{1-\sqrt{1-4x}}{2x} = C_a(x). \end{array}$$

Now, using Equations (4) and (15), we get

$$\begin{array}{c} J(x, y) = \sum_{m=0}^{\infty} \sum_{q=0}^{m} e(m, q)\ y^q x^m = \sum_{q=0}^{\infty} \sum_{m=q}^{\infty} e(m, q)\ y^q x^m = \\ = \sum_{q=0}^{\infty} y^q \sum_{m=0}^{\infty} e(m + q, q)\ x^{m+q} = \sum_{q=0}^{\infty} (yx)^q \sum_{m=0}^{\infty} g(m, q)\ x^m. \end{array} \tag{17}$$

Then, the function

$$L(x, z) = \frac{(1 - z) - \sqrt{(1 - z)^2 - 4x}}{2x} = J(x, z/x)$$

can be expanded to get (see equation (17))

$$L(x,z) = \sum_{q=0}^{\infty}\sum_{m=0}^{\infty} g(m,q)\ x^m z^q, \tag{18}$$

and this can be seen to be the generating function of $g(m,q)$:

$$g(m,q) = \lim_{x,z\to 0} \frac{1}{m!q!}\frac{\partial^{m+q}L(x,z)}{\partial x^m \partial z^q}.$$

It is interesting to observe that

$$\begin{aligned} L(x,z) &= \sum_{q=0}^{\infty}\sum_{m=0}^{\infty} g(m,q)\ x^m z^q = \sum_{m=0}^{\infty} C_m \sum_{q=0}^{\infty} \binom{2m+q}{q} z^q x^m = \\ &= \sum_{m=0}^{\infty} C_m T_m(z) x^m \end{aligned}$$

with

$$T_m(z) = \sum_{q=0}^{\infty} \binom{2m+q}{q} z^q.$$

It can be proven that

$$T_m(z) = \frac{1}{(1-z)^{2m+1}}$$

as

$$\frac{1}{q!}\frac{d^q T_m(z)}{dz^q} = \frac{(2m+1)\cdots(2m+q)}{q!\ (1-z)^{2m+1+q}}$$

and

$$\lim_{z\to 0}\frac{1}{q!}\frac{d^q T_m(z)}{dz^q} = \frac{(2m+q)!}{q!\ 2m!} = \binom{2m+q}{q}.$$

The generating function $L(x,z)$ can then be written also in the form

$$L(x,z) = \frac{1}{(1-z)}\sum_{n=0}^{\infty} C_n \left[\frac{x}{(1-z)^2}\right]^n = \frac{1}{(1-z)} C_a \left[\frac{x}{(1-z)^2}\right]$$

that better shows the strong link existing between the sequence array $g(n,q)$ and the Catalan numbers.

5 Appendix

The following binomial identity:

$$\sum_{p=0}^{q} \binom{m+p}{m}\binom{n+q-p}{n} = \binom{m+n+q+1}{q} \tag{19}$$

holds for any non-negative integers m, n, q.

Proof. We define

$$M(m,n,q) = \sum_{p=0}^{q} \binom{m+p}{m} \binom{n+q-p}{n}.$$

Then the proposition (19) is equivalent to

$$M(m,n,q) = \binom{m+n+q+1}{q}$$

The proof is based on the use of the binomial identity

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1} \tag{20}$$

that can also be written as

$$\sum_{k=0}^{m} \binom{r+k}{r} = \binom{m+r+1}{r+1}.$$

The identity (19) holds for $n = 0$ and any m, q. In fact,

$$M(m,0,q) = \sum_{p=0}^{q} \binom{m+p}{m} \binom{q-p}{0} = \sum_{p=0}^{q} \binom{m+p}{m} = \binom{m+q+1}{m+1} = \binom{m+q+1}{q},$$

where the binomial identity (20) was used.

For any $n \neq 0$, using (20) again, and with q, m natural numbers,

$$M(m,n,q) = \sum_{p=0}^{q} \binom{m+p}{m} \binom{n+q-p}{n} = \sum_{p=0}^{q} \binom{m+p}{m} \binom{(n-1)+q-p+1}{(n-1)+1} =$$
$$= \sum_{p=0}^{q} \binom{m+p}{m} \sum_{k=0}^{q-p} \binom{n-1+k}{n-1} = \sum_{k=0}^{q} \sum_{p=0}^{q-k} \binom{m+p}{m} \binom{n-1+k}{n-1} =$$
$$= \sum_{k=0}^{q} \binom{q-k+m+1}{m+1} \binom{n-1+k}{n-1} = M(m+1, n-1, q).$$

By repeatedly applying the rule $M(m,n,q) = M(m+1, n-1, q)$, it is easy to obtain

$$M(m,n,q) = M(m+n, 0, q) = \binom{m+n+q+1}{q}.$$

■

References

[1] R. P. Stanley. *Enumerative Combinatorics*, Vol. 2. Cambridge University Press, 1999.

[2] N. J. A. Sloane. *On-Line Encyclopedia of Integer Sequences.* published electronically at http://www.research.att.com/∼njas/sequences.

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(Concerned with sequences A000108 A000217 A034827 A001700 A002802 A002803 A007004 A024489 A002457.)

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