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Domino Tilings and Products of Fibonacci and Pell Numbers

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Abstract

In this brief note, we prove a result which was "accidentally" found thanks to Neil Sloane's Online Encyclopedia of Integer Sequences. Namely, we prove via elementary techniques that the number of domino tilings of the graph $W_4 \times P_{n-1}$ *equals* $f_n p_n$*, the product of the* n^{th} *Fibonacci number and the* n^{th} *Pell number.*

1 Introduction

Recently I received an electronic mail message [1] in which I was notified that a pair of duplicate sequences existed in Neil Sloane's Online Encyclopedia of Integer Sequences [3]. This involved sequence A001582 and the former sequence A003763. One of these sequences was described as the product of Fibonacci and Pell numbers, while the other was the number of domino tilings of the graph $W_4 \times P_{n-1}$.

I soon notified Neil Sloane of this fact and he promptly combined the two sequence entries into one entry, A001582. Upon combining these two entries, he also noted that it was not officially a theorem (that the product of the Fibonacci and Pell numbers was always equal to the number of domino tilings of $W_4 \times P_{n-1}$), but that is was "certainly true."

The primary goal of this short note is to prove this result so that its status is indeed a theorem. We close the paper by noting two other results relating domino tiling sequences in [2] with Fibonacci numbers.

Before proving the main result, we supply some definitions and notation for the sake of completeness. First, we note that the graph W_4 is $K_4 - e$. Moreover, the graph P_n is the path graph on n vertices. The Fibonacci numbers A000045 are defined as the sequence of integers

$$f_1 = 1, f_2 = 1, \text{ and } f_{n+2} = f_{n+1} + f_n \text{ for all } n \geq 0. \tag{1}$$

The Pell numbers A000129 are defined as the sequence of integers

$$p_1 = 1, p_2 = 2, \text{ and } p_{n+2} = 2p_{n+1} + p_n \text{ for all } n \geq 0. \tag{2}$$

2 The Results

Theorem 2.1. *The number of domino tilings of $W_4 \times P_{n-1}$ equals $f_n p_n$ for all $n \geq 2$.*

Proof. Thanks to an article of Faase [2, page 146], we know that the number of domino tilings of $W_4 \times P_{n-1}$, which we will denote by c_n, satisfies $c_2 = 2$, $c_3 = 10$, $c_4 = 36$, $c_5 = 145$ and

$$c_{n+4} - 2c_{n+3} - 7c_{n+2} - 2c_{n+1} + c_n = 0 \tag{3}$$

for all $n \geq 2$. (This is slightly different than the notation used in Faase, where he defines a function $C(n)$ wherein $c_n = C(n-1)$ and then does all work in terms of $C(n)$ rather than c_n.)

It is easy to check that $f_n p_n$ satisfies the initial conditions above. Hence, we focus on checking that $f_n p_n$ satisfies (3).

Repeated applications of (1) and (2) yield

$$f_{n+2} = f_n + f_{n+1}, \; f_{n+3} = f_n + 2f_{n+1}, \text{ and } f_{n+4} = 2f_n + 3f_{n+1} \tag{4}$$

while

$$p_{n+2} = p_n + 2p_{n+1}, \; p_{n+3} = 2p_n + 5p_{n+1}, \text{ and } p_{n+4} = 5p_n + 12p_{n+1}. \tag{5}$$

We now substitute the findings in (4) and (5) into the left-hand side of (3) and simplify:

$$\begin{aligned}
& f_{n+4}p_{n+4} - 2f_{n+3}p_{n+3} - 7f_{n+2}p_{n+2} - 2f_{n+1}p_{n+1} + f_n p_n \\
= \; & (2f_n + 3f_{n+1})(5p_n + 12p_{n+1}) - 2(f_n + 2f_{n+1})(2p_n + 5p_{n+1}) - 7(f_n + f_{n+1})(p_n + 2p_{n+1}) \\
& -2f_{n+1}p_{n+1} + f_n p_n \\
= \; & f_n p_n(10 - 4 - 7 + 1) + f_{n+1}p_n(15 - 8 - 7) + f_n p_{n+1}(24 - 10 - 14) + f_{n+1}p_{n+1}(36 - 20 - 14 - 2) \\
= \; & 0
\end{aligned}$$

Thus, $c_n = f_n p_n$ for all $n \geq 2$ and the proof is complete. □

We note that at least two other domino tiling sequences that appear in [2] are also closely related to Fibonacci numbers.

We first consider sequence A003775 which appears in [2, p. 147] in relationship to the number of domino tilings of $P_5 \times P_{2n}$. Based on this sequence, we define $d_1 = 1, d_2 = 8, d_3 = 95, d_4 = 1183$ and

$$d_{n+4} - 15d_{n+3} + 32d_{n+2} - 15d_{n+1} + d_n = 0 \tag{6}$$

for all $n \geq 1$. Computational experimentation indicates that $f_{2n-1} \mid d_n$ for several values of n. This leads to the following theorem:

Theorem 2.2. *For all $n \geq 1$, $d_n = g_n h_n$ where g_n is the n^{th} term in the sequence A004253 and $h_n = f_{2n-1}$ for all $n \geq 1$.*

Remark: Note that A004253 is related to the number of domino tilings in $K_3 \times P_{2n}$ and $S_4 \times P_{2n}$ as noted by Faase in [2, pp. 146-147]. Moreover, A004253 is quite similar to A028475.

Proof. The proof here follows a similar line of argument to that of Theorem 2.1. From the information given in A004253, we know that

$$g_1 = 1, g_2 = 4, \text{ and } g_{n+2} = 5g_{n+1} - g_n. \tag{7}$$

Moreover, we have

$$h_1 = 1, h_2 = 2, \text{ and } h_{n+2} = 3h_{n+1} - h_n. \tag{8}$$

It is easy to check that $d_n = g_n h_n$ for $1 \leq n \leq 4$, so we focus our attention on the recurrence (6). We know from (7) that

$$g_{n+2} = 5g_{n+1} - g_n, \; g_{n+3} = 24g_{n+1} - 5g_n, \text{ and } g_{n+4} = 115g_{n+1} - 24g_n \tag{9}$$

while (8) yields

$$h_{n+2} = 3h_{n+1} - h_n, \; h_{n+3} = 8h_{n+1} - 3h_n, \text{ and } h_{n+4} = 21h_{n+1} - 8h_n. \tag{10}$$

We now substitute the information from (9) and (10) into the left-hand side of (6) and obtain the following:

$$\begin{aligned}
& g_{n+4}h_{n+4} - 15g_{n+3}h_{n+3} + 32g_{n+2}h_{n+2} - 15g_{n+1}h_{n+1} + g_n h_n \\
= \; & (115g_{n+1} - 24g_n)(21h_{n+1} - 8h_n) - 15(24g_{n+1} - 5g_n)(8h_{n+1} - 3h_n) + 32(5g_{n+1} - g_n)(3h_{n+1} - h_n) \\
& -15g_{n+1}h_{n+1} + g_n h_n \\
= \; & g_n h_n(192 - 225 + 32 + 1) + g_{n+1}h_n(-920 + 1080 - 160) \\
& +g_n h_{n+1}(-504 + 600 - 96) + g_{n+1}h_{n+1}(2415 - 2880 + 480 - 15) \\
= \; & 0
\end{aligned}$$

This completes the proof.

□

Finally, we consider the last sequence of values in [2], which is related to the number of domino tilings of $P_8 \times P_n$ and appears as A028470 in [3]. The first 32 values of the sequence are as follows:

n	C_n
1	1
2	34
3	153
4	2245
5	14824
6	167089
7	1292697
8	12988816
9	108435745
10	1031151241
11	8940739824
12	82741005829
13	731164253833
14	6675498237130
15	59554200469113
16	540061286536921
17	4841110033666048
18	43752732573098281
19	393139145126822985
20	3547073578562247994
21	31910388243436817641
22	287665106926232833093
23	2589464895903294456096
24	23333526083922816720025
25	210103825878043857266833
26	1892830605678515060701072
27	17046328120997609883612969
28	153554399246902845860302369
29	1382974514097522648618420280
30	12457255314954679645007780869
31	112199448394764215277422176953
32	1010618564986361239515088848178

The recurrence relation satisfied by the values of C_n is given by

$$\begin{aligned} C_{n+32} \quad = \quad & 153C_{n+30} - 7480C_{n+28} + 151623C_{n+26} - 1552087C_{n+24} + 8933976C_{n+22} - 30536233C_{n+20} \\ & +63544113C_{n+18} - 81114784C_{n+16} + 63544113C_{n+14} - 30536233C_{n+12} + 8933976C_{n+10} \\ & -1552087C_{n+8} + 151623C_{n+6} - 7480C_{n+4} + 153C_{n+2} - C_n \end{aligned}$$

for all $n \geq 1$.

We note that $f_{n+1} \,|\, C_n$ for several values of n.

n	C_n/f_{n+1}
1	1
2	17
3	51
4	449
5	1853
6	12853
7	61557
8	382024
9	1971559
10	11585969
11	62088471
12	355111613
13	1939427729
14	10943439733
15	60338602299
16	338172377293
17	1873494595072
18	10464657396101
19	58113694771149
20	324052035315389
21	1801727076022631
22	10038214290617749
23	55845947547948897
24	311010011115265801
25	1730773816266538081
26	9636747170211055304
27	53636683818362516979
28	298610928685279993661
29	1662149072277201394907
30	9253169548548380483401
31	51507590702129135617317
32	286734628936105610236201
33	1596133084485272225826727
34	8885301696820653301727657
35	49461269578409945493024768
36	275337533349256635378114713
37	1532712030034359905504838377
38	8532165290411528453455489081
39	47495826004154548026644356683
40	264395102236750242015097270393

This leads to the conjecture that $f_{n+1} \,|\, C_n$ for all $n \geq 1$. We leave the proof of this (assuming it is true) to the reader, as the calculations involved herein would be straightforward but tedious.

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References

[1] Personal communication from Frank Ellerman, frank.ellermann@t-online.de, February 9, 2002.

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