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Hankel Matrices and Lattice Paths

Wen-jin Woan
Department of Mathematics
Howard University
Washington, D.C. 20059, USA
Email address: wwoan@howard.edu

Abstract

Let H be the Hankel matrix formed from a sequence of real numbers $S = \{a_0 = 1, a_1, a_2, a_3, ...\}$, and let L denote the lower triangular matrix obtained from the Gaussian column reduction of H. This paper gives a matrix-theoretic proof that the associated Stieltjes matrix S_L is a tri-diagonal matrix. It is also shown that for any sequence (of nonzero real numbers) $T = \{d_0 = 1, d_1, d_2, d_3, ...\}$ there are infinitely many sequences such that the determinant sequence of the Hankel matrix formed from those sequences is T.

1. Introduction. In this paper we give a matrix-theoretic proof (Theorem 2.1) of one of the main theorems in [1]. In Section 2 we discuss the connection between the decomposition of a Hankel matrix and Stieltjes matrices, and in Section 3 we discuss the connection between certain lattice paths and Hankel matrices. Section 4 presents an explicit formula for the decomposition of a Hankel matrix.

Definition 1.1. Let $S = \{a_0 = 1, a_1, a_2, a_3, ...\}$ be a sequence of real numbers. The Hankel matrix generated by S is the infinite matrix

$$H = \begin{bmatrix} 1 & a_1 & a_2 & a_3 & a_4 & . \\ a_1 & a_2 & a_3 & a_4 & a_5 & . \\ a_2 & a_3 & a_4 & a_5 & a_6 & . \\ a_3 & a_4 & a_5 & a_6 & a_7 & . \\ a_4 & a_5 & a_6 & a_7 & a_8 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Definition 1.2. A lower triangular matrix

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . \\ l_{10} & 1 & 0 & 0 & 0 & . \\ l_{20} & l_{21} & 1 & 0 & 0 & . \\ l_{30} & l_{31} & l_{32} & 1 & 0 & . \\ l_{40} & l_{41} & l_{42} & l_{43} & 1 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

is said to be a Riordan matrix if there exist Taylor series $g(x) = 1 + a_1x + a_2x^2 + ... + a_nx^n + ...$ and $f(x) = x + b_2x^2 + b_3x^3 + ... + b_nx^n + ...$ such that for every $k \geq 0$ the k-th column has ordinary generating function $g(x)(f(x))^k$.

Definition 1.3. The Stieltjes matrix of a lower triangular matrix L is the matrix S_L which satisfies $LS_L = L^r$ where L^r is the matrix obtained from L by deleting the first row of L.

Thus

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . \\ l_{10} & 1 & 0 & 0 & 0 & . \\ l_{20} & l_{21} & 1 & 0 & 0 & . \\ l_{30} & l_{31} & l_{32} & 1 & 0 & . \\ l_{40} & l_{41} & l_{42} & l_{43} & 1 & . \\ . & . & . & . & . & . \end{bmatrix} S_L = \begin{bmatrix} l_{10} & 1 & 0 & 0 & 0 & . \\ l_{20} & l_{21} & 1 & 0 & 0 & . \\ l_{30} & l_{31} & l_{32} & 1 & 0 & . \\ l_{40} & l_{41} & l_{42} & l_{43} & 1 & . \\ . & . & . & . & . & . \end{bmatrix}$$

and so

$$S_L = L^{-1}L^r = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . \\ -l_{10} & 1 & 0 & 0 & 0 & . \\ \times & -l_{21} & 1 & 0 & 0 & . \\ \times & \times & -l_{32} & 1 & 0 & . \\ \times & \times & \times & -l_{43} & 1 & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} l_{10} & 1 & 0 & 0 & 0 & . \\ l_{20} & l_{21} & 1 & 0 & 0 & . \\ l_{30} & l_{31} & l_{32} & 1 & 0 & . \\ l_{40} & l_{41} & l_{42} & l_{43} & 1 & . \\ . & . & . & . & . & . \end{bmatrix}$$

$$= \begin{bmatrix} b_0 & 1 & 0 & 0 & 0 & . \\ c_0 & b_1 & 1 & 0 & 0 & . \\ \times & c_1 & b_2 & 1 & 0 & . \\ \times & \times & c_2 & b_3 & 1 & . \\ \times & \times & \times & c_3 & b_4 & . \\ . & . & . & . & . & . \end{bmatrix}$$

where

$$b_0 = l_{10},\ b_k = l_{k+1,k} - l_{k,k-1},\ k > 0,$$

$$c_0 = l_{2,0} - l_{1,0}^2,\ c_k = (l_{k,k-1}l_{k+1,k} - l_{k+1,k-1}) - l_{k+1,k}^2 + l_{k+2,k},\ k > 0.$$

Definition 1.4. Let L and S_L be as in Definition 1.3. We define

$$D_L = \begin{bmatrix} d_0 & 0 & 0 & 0 & 0 & . \\ 0 & d_1 & 0 & 0 & 0 & . \\ 0 & 0 & d_2 & 0 & 0 & . \\ 0 & 0 & 0 & d_3 & 0 & . \\ 0 & 0 & 0 & 0 & d_4 & . \\ . & . & . & . & . & . \end{bmatrix}$$

to be the diagonal matrix with diagonal entries given by $d_0 = 1$, $d_{k+1} = d_k c_k$ for $k > 0$.

2. Stieltjes and Hankel Matrices.

The following two theorems are proved in [1].

Theorem 2.1. Let L be a lower triangular matrix and let $D = D_L$ be the diagonal matrix with nonzero diagonal entries $\{d_i\}$ as in Definition 1.4. Then LDL^t is a Hankel matrix if and only if S_L is a tri-diagonal matrix, i.e. if and only if

$$S_L = \begin{bmatrix} b_0 & 1 & 0 & 0 & 0 & . \\ c_0 & b_1 & 1 & 0 & 0 & . \\ 0 & c_1 & b_2 & 1 & 0 & . \\ 0 & 0 & c_2 & b_3 & 1 & . \\ 0 & 0 & 0 & c_3 & b_4 & . \\ . & . & . & . & . & . \end{bmatrix}$$

where $b_0 = l_{1,0}$, $c_0 = d_1$, $b_k = l_{k+1,k} - l_{k,k-1}$, $c_k = \frac{d_{k+1}}{d_k}$, $k \geq 1$.

PROOF. Let $H = LDL^t$ be a Hankel matrix. Then

$L = H(DL^t)^{-1}$,

$L^r = (H(DL^t)^{-1})^r = H^r(DL^t)^{-1}$,

$S_L = L^{-1}L^r = L^{-1}(H^r(DL^t)^{-1}) = (L^{-1}H^r)(DL^t)^{-1}$.

Since H is a Hankel matrix, deleting the first row has the same effect as deleting the first column.

$$L^{-1}H = DL^t = \begin{bmatrix} d_0 & d_0l_{10} & d_0l_{20} & d_0l_{3,0} & d_0l_{4,0} & . \\ 0 & d_1 & d_1l_{21} & d_1l_{31} & d_1l_{41} & . \\ 0 & 0 & d_2 & d_2l_{32} & d_2l_{42} & . \\ 0 & 0 & 0 & d_3 & d_3l_{43} & . \\ 0 & 0 & 0 & 0 & d_4 & . \\ . & . & . & . & . & . \end{bmatrix},$$

$$L^{-1}H^r = L^{-1}H^c = (L^{-1}H)^c = \begin{bmatrix} d_0l_{10} & d_0l_{20} & d_0l_{30} & d_0l_{4,0} & . \\ d_1 & d_1l_{21} & d_1l_{31} & d_1l_{41} & . \\ 0 & d_2 & d_2l_{32} & d_2l_{42} & . \\ 0 & 0 & d_3 & d_3l_{43} & . \\ 0 & 0 & 0 & d_4 & . \\ . & . & . & . & . \end{bmatrix},$$

$$S_L = (L^{-1}H)^c(DL^t)^{-1} = \begin{bmatrix} d_0l_{10} & d_0l_{20} & d_0l_{30} & d_0l_{4,0} & . \\ d_1 & d_1l_{21} & d_1l_{31} & d_1l_{41} & . \\ 0 & d_2 & d_2l_{32} & d_2l_{42} & . \\ 0 & 0 & d_3 & d_3l_{43} & . \\ 0 & 0 & 0 & d_4 & . \\ . & . & . & . & . \end{bmatrix} \begin{bmatrix} \frac{1}{d_0} & \times & \times & \times & \times & . \\ 0 & \frac{1}{d_1} & \times & \times & \times & . \\ 0 & 0 & \frac{1}{d_2} & \times & \times & . \\ 0 & 0 & 0 & \frac{1}{d_3} & \times & . \\ 0 & 0 & 0 & 0 & \frac{1}{d_4} & . \\ . & . & . & . & . & . \end{bmatrix}$$

$$= \begin{bmatrix} b_0 & 1 & 0 & 0 & 0 & . \\ c_0 & b_1 & 1 & 0 & 0 & . \\ 0 & c_1 & b_2 & 1 & 0 & . \\ 0 & 0 & c_2 & b_3 & 1 & . \\ 0 & 0 & 0 & c_3 & b_4 & . \\ . & . & . & . & . & . \end{bmatrix}$$

where

$$b_0 = l_{1,0}\ , \quad c_0 = \frac{d_1}{d_0} = d_1\ , \quad b_k = l_{k+1,k} - l_{k,k-1}\ , \quad c_k = \frac{d_{k+1}}{d_k}\ , \quad k \geq 1.$$

Conversely, let S_L be a tri-diagonal matrix and let $H = LDL^t$. Then
$L^{-1}H^r = L^{-1}(LDL^t)^r = L^{-1}(L^rDL^t) = (L^{-1}L^r)DL^t = S_LDL^t$

$$= \begin{bmatrix} b_0 & 1 & 0 & 0 & 0 & . \\ c_0 & b_1 & 1 & 0 & 0 & . \\ 0 & c_1 & b_2 & 1 & 0 & . \\ 0 & 0 & c_2 & b_3 & 1 & . \\ 0 & 0 & 0 & c_3 & b_4 & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} d_0 & d_0l_{10} & d_0l_{20} & d_0l_{3,0} & d_0l_{4,0} & . \\ 0 & d_1 & d_1l_{21} & d_1l_{31} & d_1l_{41} & . \\ 0 & 0 & d_2 & d_2l_{32} & d_2l_{42} & . \\ 0 & 0 & 0 & d_3 & d_3l_{43} & . \\ 0 & 0 & 0 & 0 & d_4 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Therefore
$(L^{-1}H^r)_{n,k} = c_{n-1}d_{n-1}l_{k,n-1} + b_nd_nl_{k,n} + d_{n+1}l_{k,n+1}$
$= \frac{d_n}{d_{n-1}}d_{n-1}l_{k,n-1} + b_nd_nl_{k,n} + c_nd_nl_{k,n+1}$
$= d_n(l_{k,n-1} + b_nl_{k,n} + c_nl_{k,n+1})$
$= d_nl_{k+1,n} = (DL^t)_{n,k+1} = (DL^t)^c_{n,k} = (L^{-1}H)^c_{n,k} = (L^{-1}H^c)_{n,k}$.
We have shown that $L^{-1}H^r = L^{-1}H^c$, and so $H^r = H^c$. Hence H is a Hankel matrix. ■

Theorem 2.2. L is a Riordan matrix (i.e. $b_k = b_1 = b$ and $c_k = c_1 = c$ for $k \geq 1$) if and only if $f = x(1 + bf + cf^2)$ and

$$g = \frac{1}{1 - xb_0 - xc_0f}\ ,$$

where f, g are as in Definition 1.2.

See [1] for the proof.

Corollary 2.3. Let $T = \{d_0 = 1, d_1, d_2\ , d_3, ...\}$ be any sequence of (nonzero) real numbers. Then there exists a sequence $S = \{a_0 = 1, a_1, a_2, a_3, ...\}$ such that T is equal to the sequence of diagonal entries of D in the decomposition $H = LDL^t$ of the Hankel matrix generated by S .

PROOF. As in Theorem 2.1, let $c_0 = d_1$, $c_k = \frac{d_{k+1}}{d_k}$, $k \geq 1$, and form the Stieltjes matrix

$$S_L = \begin{bmatrix} b_0 & 1 & 0 & 0 & 0 & . \\ c_0 & b_1 & 1 & 0 & 0 & . \\ 0 & c_1 & b_2 & 1 & 0 & . \\ 0 & 0 & c_2 & b_3 & 1 & . \\ 0 & 0 & 0 & c_3 & b_4 & . \\ . & . & . & . & . & . \end{bmatrix}$$

where the b_is are arbitrary. By Definition 1.3 there is a lower triangular matrix L such that $LS_L = L^r$. Let S be the sequence formed by the first column of L and let H denote the Hankel matrix generated by S. By Theorem 2.1 the diagonal entries of D in the decomposition $H = LDL^t$ form the sequence T. ■

Example 2.4. Let $T = \{1, 1, 2, 5, 14, 42, 132, ...\}$ be the Catalan sequence (A000108 in [2]) and let

$$S_L = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & . \\ 1 & 0 & 1 & 0 & 0 & . \\ 0 & 2 & 0 & 1 & 0 & . \\ 0 & 0 & \frac{5}{2} & 0 & 1 & . \\ 0 & 0 & 0 & \frac{14}{5} & 0 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Then

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 0 & 1 & 0 & 0 & 0 & . \\ 1 & 0 & 1 & 0 & 0 & . \\ 0 & 3 & 0 & 1 & 0 & . \\ 3 & 0 & \frac{11}{2} & 0 & 1 & . \\ . & . & . & . & . & . \end{bmatrix},$$

$$LDL^t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 0 & 1 & 0 & 0 & 0 & . \\ 1 & 0 & 1 & 0 & 0 & . \\ 0 & 3 & 0 & 1 & 0 & . \\ 3 & 0 & \frac{11}{2} & 0 & 1 & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 0 & 1 & 0 & 0 & 0 & . \\ 0 & 0 & 2 & 0 & 0 & . \\ 0 & 0 & 0 & 5 & 0 & . \\ 0 & 0 & 0 & 0 & 14 & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 3 & . \\ 0 & 1 & 0 & 3 & 0 & . \\ 0 & 0 & 1 & 0 & \frac{11}{2} & . \\ 0 & 0 & 0 & 1 & 0 & . \\ 0 & 0 & 0 & 0 & 1 & . \\ . & . & . & . & . & . \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 3 & . \\ 0 & 1 & 0 & 3 & 0 & . \\ 1 & 0 & 3 & 0 & 14 & . \\ 0 & 3 & 0 & 14 & 0 & . \\ 3 & 0 & 14 & 0 & \frac{167}{2} & . \\ . & . & . & . & . & . \end{bmatrix} = H.$$

3. Lattice Paths and Hankel Matrices

We consider those lattice paths in the Cartesian plane running from $(0,0)$ that use steps from $S = \{u = (1,1),\ h = (1,0),\ d = (1,-1)\}$ with assigned weights 1 for u, w_1 for h and w_2 for d. Let $L(n,k)$ be the set of paths that never go below the x-axis and end at (n,k). The weight of a path is the product of the weights of its steps. Let $l_{n,k}$ be the sum of the weights of all the paths in $L(n,k)$. See also [3], [4].

Theorem 3.1. Let $L = (l_{n,k})_{n,k\geq 0}$. Then L is a lower triangular matrix, the Stieltjes matrix of L is

$$S_L = \begin{bmatrix} w_1 & 1 & 0 & 0 & 0 & . \\ w_2 & w_1 & 1 & 0 & 0 & . \\ 0 & w_2 & w_1 & 1 & 0 & . \\ 0 & 0 & w_2 & w_1 & 1 & . \\ 0 & 0 & 0 & w_2 & w_1 & . \\ . & . & . & . & . & . \end{bmatrix}$$

and $H = LDL^t$ is the Hankel matrix generated by the first column of L and $d_k = w_2^k$ for $k > 0$.

PROOF. From Theorem 2.1. ■

Example 3.2. For $w_1 = 0,\ w_2 = 1$, L is the Catalan matrix. For $w_1 = t,\ w_2 = 1$, L is the t-Motzkin matrix. In both cases D is the identity matrix. For example, when $t = 1$,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 1 & 1 & 0 & 0 & 0 & . \\ 2 & 2 & 1 & 0 & 0 & . \\ 4 & 5 & 3 & 1 & 0 & . \\ 9 & 12 & 9 & 4 & 1 & . \\ . & . & . & . & . & . \end{bmatrix},$$

$$LDL^t = \begin{bmatrix} 1 & 1 & 2 & 4 & 9 & . \\ 1 & 2 & 4 & 9 & 21 & . \\ 2 & 4 & 9 & 21 & 51 & . \\ 4 & 9 & 21 & 51 & 127 & . \\ 9 & 21 & 51 & 127 & 323 & . \\ . & . & . & . & . & . \end{bmatrix} = H$$

where $S = \{1, 1, 2, 4, 9, 21, 51, ...\}$ is the Motzkin sequence A001006.

Theorem 3.3. If w_1, w_2 depend on the height k, i.e. $w_1(k) = b_k$ and $w_2(k+1) = c_k$, then

$$S_L = \begin{bmatrix} b_0 & 1 & 0 & 0 & 0 & . \\ c_0 & b_1 & 1 & 0 & 0 & . \\ 0 & c_1 & b_2 & 1 & 0 & . \\ 0 & 0 & c_2 & b_3 & 1 & . \\ 0 & 0 & 0 & c_3 & b_4 & . \\ . & . & . & . & . & . \end{bmatrix}$$

and $H = LDL^t$ is the Hankel matrix generated by the first column of L and $d_k = \Pi_{i\leq k} c_i$.

PROOF. From Theorem 2.1. ■

See Example 2.4 for an illustration.

4. Gaussian Column Reduction

Let $S = \{a_0 = 1, a_1, a_2, a_3, ...\}$ be a sequence of real numbers and let H denote the Hankel matrix generated by S. All the results in this section are well-known in matrix theory. We shall express the entries of L in term of S. We assume that H is positive definite.

Lemma 4.1. The decomposition of a positive definite Hankel matrix $H = LDU$ is unique and $U = L^t$, where L is a lower triangular matrix with diagonal entries 1, D is a diagonal matrix and U is an upper triangular matrix with diagonal entries 1.

Proof. Let $LDU = H = L_1D_1U_1$. Then $DUU_1^{-1} = L^{-1}L_1D_1$ is both an upper and lower triangular matrix, hence $UU_1^{-1} = L^{-1}L_1 = I$ is the infinite identity matrix. ■

Let H_n be the truncated submatrix of H with $n \geq 0$. For example,

$$H_3 = \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 1 & a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_6 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix}.$$

Let $H_n(k)$ be the matrix obtained from H_n by replacing the last column of H_n by $a_k, a_{k+1}, a_{k+2}, ..., a_{k+n}$. For example,

$$H_3(1) = \begin{bmatrix} 1 & a_1 & a_2 & a_1 \\ a_1 & a_2 & a_3 & a_2 \\ a_2 & a_3 & a_4 & a_3 \\ a_3 & a_4 & a_5 & a_4 \end{bmatrix}, \quad H_3(5) = \begin{bmatrix} 1 & a_1 & a_2 & a_5 \\ a_1 & a_2 & a_3 & a_6 \\ a_2 & a_3 & a_4 & a_7 \\ a_3 & a_4 & a_5 & a_8 \end{bmatrix}.$$

.

Let $h_i = \det H_i$ and define an infinite upper triangular matrix $R = (r_{n,k})$ in term of (n, k)-cofactor of H_k by $r_{n,k} = 0$ for $k < n$, and

$$r_{n,k} = \frac{1}{h_{k-1}}(-1)^{n+k+2} \det \begin{bmatrix} 1 & a_1 & a_2 & . & a_{k-1} \\ a_1 & a_2 & a_3 & .. & a_k \\ a_2 & a_3 & a_4 & . & a_{k+1} \\ . & . & . & . & .. \\ a_{n-1} & a_n & a_{n+1} & . & a_{k+n-2} \\ a_{n+1} & a_{n+2} & a_{n+3} & . & a_{k+n} \\ . & . & . & . & . \\ a_k & a_{k+1} & a_{k+2} & . & a_{k+k} \end{bmatrix}$$

for $k \geq n$. For example,

$$r_{2,4} = \frac{1}{h_3}(-1)^{(2+4)+2} \det \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{bmatrix}.$$

Remark 4.2. $HR = LD$, where $L = (l_{n,k})$ is the Gaussian column reduction of the Hankel matrix H and D is the diagonal matrix with diagonal entries $\{d_i\}$, $R^{-1} = L^t$ with $d_i = \frac{h_i}{h_{i-1}}$ and $l_{n,k} = \frac{1}{h_{k-1}} \det H_k(n)$.

Remark 4.3. If L is a Riordan matrix, then for $i \geq 1$, $c = c_i = \frac{d_{i+1}}{d_i} = \frac{h_{i+1}h_{i-1}}{h_i h_i}$ and $b = b_i = l_{i+1,i} - l_{i,i-1} = \frac{1}{h_{i-1}} \det H_i(i+1) - \frac{1}{h_{i-2}} \det H_{i-1}(i)$ is a recurrence relation for the sequence S.

Example 4.4. Let $S = \{1, 3, 13, 63, 321, 1683, 8989, 48639, 265729, ...\}$ be the central Delannoy numbers A001850 and let H be the Hankel matrix generated by S. Then

$$H = \begin{bmatrix} 1 & 3 & 13 & 63 & . \\ 3 & 13 & 63 & 321 & . \\ 13 & 63 & 321 & 1683 & . \\ 63 & 321 & 1683 & 8989 & . \\ . & . & . & . & . \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & -3 & 5 & -9 & . \\ 0 & 1 & -6 & 21 & . \\ 0 & 0 & 1 & -9 & . \\ 0 & 0 & 0 & 1 & . \\ . & . & . & . & . \end{bmatrix},$$

$$LD = HR = \begin{bmatrix} 1 & 0 & 0 & 0 & . \\ 3 & 4 & 0 & 0 & . \\ 13 & 24 & 8 & 0 & . \\ 63 & 132 & 72 & 16 & . \\ . & . & . & . & . \end{bmatrix},$$

$$R^t HR = D = \begin{bmatrix} 1 & 0 & 0 & 0 & . \\ 0 & 4 & 0 & 0 & . \\ 0 & 0 & 8 & 0 & . \\ 0 & 0 & 0 & 16 & . \\ . & . & . & . & . \end{bmatrix},$$

$$L = HRD^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & . \\ 3 & 1 & 0 & 0 & . \\ 13 & 6 & 1 & 0 & . \\ 63 & 33 & 9 & 1 & . \\ . & . & . & . & . \end{bmatrix},$$

$$S_L = L^{-1}L^r = R^t L^r = \begin{bmatrix} 1 & 0 & 0 & 0 & . \\ -3 & 1 & 0 & 0 & . \\ 5 & -6 & 1 & 0 & . \\ -9 & 21 & -9 & 1 & . \\ . & . & . & . & . \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & . \\ 13 & 6 & 1 & 0 & 0 & . \\ 63 & 33 & 9 & 1 & 0 & . \\ 321 & 180 & 62 & 12 & 1 & . \\ . & . & . & . & . & . \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 0 & 0 & . \\ 4 & 3 & 1 & 0 & . \\ 0 & 2 & 3 & 1 & . \\ 0 & 0 & 2 & 3 & . \\ . & . & . & . & . \end{bmatrix},$$

$$LDL^t = \begin{bmatrix} 1 & 0 & 0 & 0 & . \\ 3 & 1 & 0 & 0 & . \\ 13 & 6 & 1 & 0 & . \\ 63 & 33 & 9 & 1 & . \\ . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & . \\ 0 & 4 & 0 & 0 & . \\ 0 & 0 & 8 & 0 & . \\ 0 & 0 & 0 & 16 & . \\ . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 & 3 & 13 & 63 & . \\ 0 & 1 & 6 & 33 & . \\ 0 & 0 & 1 & 9 & . \\ 0 & 0 & 0 & 1 & . \\ . & . & . & . & . \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 13 & 63 & . \\ 3 & 13 & 63 & 321 & . \\ 13 & 63 & 321 & 1683 & . \\ 63 & 321 & 1683 & 8989 & . \\ . & . & . & . & . \end{bmatrix} = H.$$

Remark 4.5. If H is the Hankel matrix corresponding to a sequence S, then by Theorem 3.1 and Theorem 3.3 we may use lattice paths to find L, the Gaussian column reduction of H.

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