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Integral Representations of Catalan and Related Numbers

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Abstract

We derive integral representations for the Catalan numbers $C(n)$, shifted Catalan numbers $C(n+p)$, and the numbers $n! \cdot C(n)$ and $C(n) \cdot B(n)$, where $B(n)$ are the Bell numbers, for $n = 0, 1 \ldots$. Our method is to use inverse Mellin transform. All these numbers are power moments of positive functions, and their representations turn out to be unique.

The Catalan numbers $C(n)$, $n = 0, 1, 2, \ldots$, defined by

$$C(n) = \frac{\binom{2n}{n}}{n+1} \quad , \tag{1}$$

are among the most ubiquitous sequences in enumerative combinatorics. Stanley [13] cites no less than 66 different combinatorial settings where these numbers appear. The first few Catalan numbers are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862$$

for $n = 0 \ldots 9$. A plethora of information about the $C(n)$'s can be found in [11], under sequence no. A000108.

In this note we derive an integral representation of $C(n)$ as the n-th power moment of a certain non-negative function $W_C(x)$ on the positive half-axis. We also study the ramifications of this representation for other integer sequences involving $C(n)$.

To this end we seek a function $W_C(x)$ such that

$$\int_0^\infty x^n W_C(x)dx \quad = \quad C(n) \tag{2}$$

$$= \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n+2)} \quad , \quad n = 0, 1, \ldots \quad . \tag{3}$$

Replacing n by a complex variable $s-1$, we rewrite Eq.(3) as

$$\int_0^\infty x^{s-1} W_C(x) dx = \frac{4^{s-1}\Gamma(s-1/2)}{\sqrt{\pi}\Gamma(s+1)} \quad , \quad \text{Re } s > 1 \quad , \tag{4}$$

which implies that

$$W_C(x) = \mathcal{M}^{-1}\left[\frac{4^{s-1}\Gamma(s-1/2)}{\sqrt{\pi}\Gamma(s+1)}; x\right], \tag{5}$$

where $\mathcal{M}^{-1}[f^*(s); x] = f(x)$ is the inverse Mellin transform [12], with $f^*(s) = \mathcal{M}[f(x); s] = \int_0^\infty x^{s-1} f(x) dx$ the Mellin transform of $f(x)$. We note the following property of $\mathcal{M}$ [12] :

$$\mathcal{M}\left[x^b f(ax^h); s\right] = \frac{1}{h} a^{-\frac{s+b}{h}} f^*\left(\frac{s+b}{h}\right), \qquad b \in R, \quad h > 0, \tag{6}$$

which, when specialized to $a = \frac{1}{4}$, $b = -\frac{1}{2}$ and $h = 1$, implies that

$$\mathcal{M}\left[x^{-\frac{1}{2}} f\left(\frac{x}{4}\right); s\right] = 4^s f^*(s-1/2)/2 \quad . \tag{7}$$

Adopting the standard notation $(y)_+^\alpha = y^\alpha$ if $y > 0$, $(y)_+^\alpha = 0$ otherwise, and using the formula 2.2(1), p.151 of [5] :

$$\mathcal{M}\left[(1-x)_+^{\alpha-1}; s\right] = \Gamma(\alpha)\frac{\Gamma(s)}{\Gamma(\alpha+s)} \quad , \qquad \alpha > 0, \quad s > 0, \tag{8}$$

we can apply Eq.(7) with $f(x) = (1-x)_+^{\alpha-1}$ and $\alpha = \frac{3}{2}$. This yields

$$W_C(x) = \frac{x^{-\frac{1}{2}}}{\pi}\left(1 - \frac{x}{4}\right)_+^{\frac{1}{2}}. \tag{9}$$

The function $W_C(x)$ is displayed on Fig.(1). The desired integral representation of $C(n)$ is then

$$C(n) = \int_0^4 x^n \left(\frac{\sqrt{\frac{4-x}{x}}}{2\pi}\right) dx \quad . \tag{10}$$

This is the solution of the Hausdorff moment problem on $[0, 4]$, which is always unique [1], and so the representation of Eq.(10) is also unique.

By the same token we can find the solution of

$$\int_0^\infty x^n W_{C,p}(x) dx = C(n+p), \quad n = 0, 1, 2 \ldots, \quad p = 1, 2, \ldots, \tag{11}$$

i.e. the unique representation of the shifted Catalan numbers $C(n+p)$, as the Hausdorff moments of

$$W_{C,p}(x) = \frac{x^{p-\frac{1}{2}}}{\pi}\left(1 - \frac{x}{4}\right)_+^{\frac{1}{2}}. \tag{12}$$

The Mellin convolution property for products of Mellin transforms, in its simplest incarnation, states ([12], [5]) that if $\mathcal{M}[W_{1,2}(x); s] = \rho_{1,2}(s)$ then

$$\mathcal{M}^{-1}[\rho_1(s)\rho_2(s); x] = W_{12}(x) \equiv \int_0^\infty \frac{1}{t} W_1\left(\frac{x}{t}\right) W_2(t) dt \quad . \tag{13}$$

Observe that $W_{1,2}(x) > 0$ implies $W_{12}(x) > 0$.

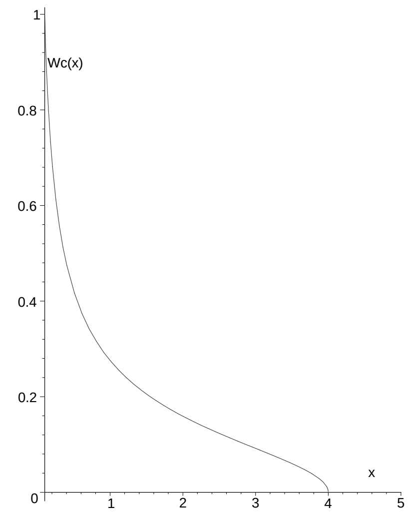

Figure 1: : The function $W_C(x)$, s. Eq.(9). This function diverges at $x = 0$.

As an application of Eq.(13) we look for an integral representation of the sequence $n! \cdot C(n)$ whose initial terms are $1, 1, 4, 30, 336, 5040, 95040, 2162160, 57657600, 1764322560$, for $n = 0 \ldots 9$; compare [11], no. A001761. Using Eq.(9) and performing the Mellin convolution in Eq.(13) with $W_1(x) = e^{-x}$ and $W_2(x) = W_C(x)$, one ends up with the following Stieltjes moment problem :

$$\int_0^\infty x^n W_{1C}(x)dx = n! \cdot C(n) = \frac{(2n)!}{(n+1)!} \quad , \quad n = 0, 1, \ldots, \tag{14}$$

with the solution

$$W_{1C}(x) = \frac{1}{2\pi\sqrt{x}} \int_{\frac{x}{4}}^\infty e^{-t} \frac{\sqrt{4t-x}}{t} dt \tag{15}$$

$$= -\frac{1}{2} + \frac{1}{\sqrt{\pi x}} e^{-\frac{x}{4}} + \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{x}}{2}\right) \quad , \tag{16}$$

where erf(y) is the error function. The function $W_{1C}(x)$ is shown in Fig.(2). As $W_{1C}(x) > 0$, the (sufficient) Carleman condition $(\sum_{n=1}^\infty (\frac{(2n)!}{(n+1)!})^{-\frac{1}{2n}} = \infty)$ (cf. Ref.[1]) indicates that the solution $W_{1C}(x)$ of Eq.(16) is also unique. Similar results are obtained by using $W_{C,p}(x)$ instead of $W_C(x)$ in Eq.(13).

Another use of Eq.(13) is illustrated by considering the sequence $C(n) \cdot B(n)$, where $B(n)$ are the Bell numbers (see [11], no. A000110, and [2]). The initial terms of this sequence are 1, 1, 4, 25, 210, 2184, 26796, 376233, 5920200, 102816714, for $n = 0 \ldots 9$. For this last sequence see [11], no. A064299. The weight function whose n-th moment is equal to $B(n)$ is

$$W_B(x) = \frac{1}{e} \sum_{k=1}^\infty \frac{\delta(x-k)}{k!} \quad , \tag{17}$$

which is a consequence of Dobiński formula, $B(n) = \frac{1}{e}\sum_{k=1}^\infty \frac{k^n}{k!}$, see [2]. In Eq.(17), $\delta(y)$ is Dirac's delta function. By Mellin convolution of $W_B(x)$ with $W_C(x)$ one obtains

$$W_{BC}(x) = \frac{1}{2\pi e} \sum_{k=1}^\infty \frac{1}{k k!} \sqrt{\frac{4k-x}{x}} H\left(4 - \frac{x}{k}\right) \quad , \tag{18}$$

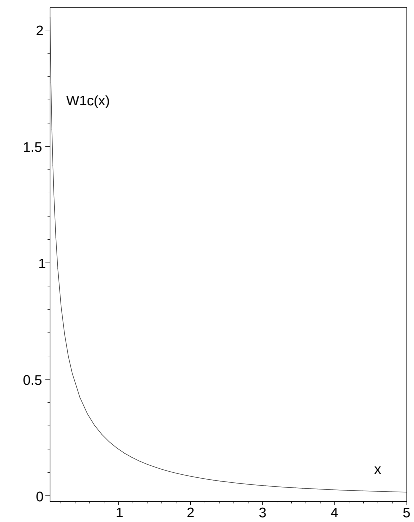

Figure 2: : The function $W_{1C}(x)$, s. Eq.(16). This function diverges at $x = 0$.

which, via Carleman's criterion, is the only positive function such that its n-th moment is equal to $C(n) \cdot B(n)$. In Eq.(18) $H(y)$ is the Heaviside function. The function $W_{BC}(x)$ is displayed on Fig.(3).

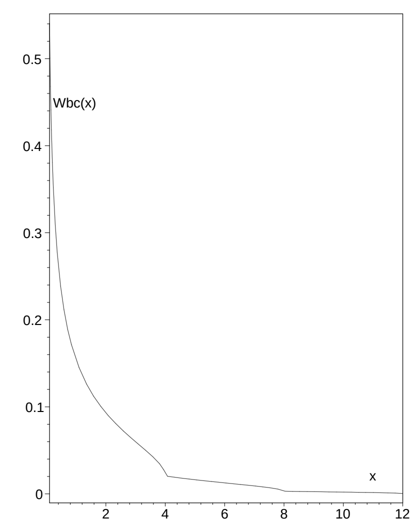

Figure 3: : The function $W_{BC}(x)$, s. Eq.(18). This function diverges at $x = 0$.

The last sequence that will concern us here is $(n!)^2 C(n) = \frac{(2n)!}{n+1}$. Its initial terms are

$$1, 1, 8, 180, 8064, 604800, 68428800, 10897286400, 2324754432000$$

for $n = 0 \ldots 9$; compare [11], no. A060593. Proceeding as in Eqs.(3) and (4), we are looking for $W_3(x)$ satisfying

$$\int_0^\infty x^{s-1} W_3(x) dx = \frac{4^{s-1}\Gamma(s-1/2)\Gamma^2(s)}{\sqrt{\pi}\Gamma(s+1)} \quad , \quad \text{Re } s > 1 \quad . \tag{19}$$

It appears that when studying Eq.(19) it is possible to avoid using $W_C(x)$. As the first step we observe from

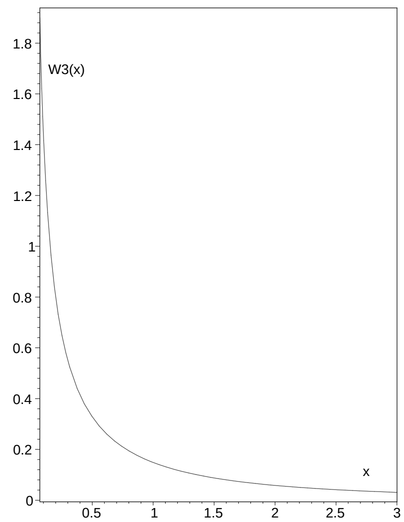

Figure 4: : The function $W_3(x)$, s. Eq.(24). This function diverges at $x = 0$.

Eq.(6) that

$$\mathcal{M}^{-1}\left[\Gamma\left(s-\frac{1}{2}\right);x\right] = \frac{e^{-x}}{\sqrt{x}} \qquad . \tag{20}$$

In addition, the following relation holds :

$$\mathcal{M}^{-1}\left[\frac{\Gamma^2(s)}{\Gamma(s+1)};x\right] = -Ei(-x) \qquad , \tag{21}$$

which is the formula 8.1(1), p.182 of [5]. In Eq.(21) $Ei(y)$ is the exponential integral function. Combining Eqs.(20) and (21) in the Mellin convolution we obtain

$$\mathcal{M}^{-1}\left[\frac{\Gamma(s-1/2)\Gamma^2(s)}{\Gamma(s+1)};x\right] = -\frac{1}{\sqrt{x}}\int_0^\infty t^{-\frac{1}{2}}t^{-\frac{x}{t}}Ei(-t)dt \tag{22}$$

$$= 2\sqrt{\frac{\pi}{x}}e^{-2\sqrt{x}} + 4\sqrt{\pi}Ei(-2\sqrt{x}) \qquad , \quad x > 0 \quad . \tag{23}$$

In writing Eq.(23) we have used the formula 2.5.4.2, p.72 of [6]. Finally, we use Eq.(6) again (with $a = \frac{1}{4}$, $b = 0$ and $h = 1$) and from Eq.(19) we get the solution

$$W_3(x) = \frac{1}{\sqrt{x}}e^{-\sqrt{x}} + Ei(-\sqrt{x}) \qquad , \tag{24}$$

which is plotted in Fig.(4). As $W_3(x) > 0$, by Carleman's criterion the solution is again unique.

Remark: E. P.Wigner [14] has demonstrated that Eq.(10), under a suitable parametrization, describes the distribution function of eigenvalues of an ensemble of random, symmetric, real matrices.

Integral representations of other combinatorial numbers can be found in [3]. For further applications of Mellin convolution formula Eq.(13), one may consult [5], [10], [7], [8] , [4] and [9].

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(Concerned with sequences A000108, A000110, A001761, A060593, A064299.)

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