Journal of Integer Sequences, Vol. 3 (2000),
Article 00.2.4

On Generalizations of the Stirling Number Triangles[1]

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Abstract

Sequences of generalized Stirling numbers of both kinds are introduced. These sequences of triangles (i.e. infinite-dimensional lower triangular matrices) of numbers will be denoted by $S2(k;n,m)$ and $S1(k;n,m)$ with $k \in \mathbf{Z}$. The original Stirling number triangles of the second and first kind arise when $k=1$. $S2(2;n,m)$ is identical with the unsigned $S1(2;n,m)$ triangle, called $S1p(2;n,m)$, which also represents the triangle of signless Lah numbers. Certain associated number triangles, denoted by $s2(k;n,m)$ and $s1(k;n,m)$, are also defined. Both $s2(2;n,m)$ and $s1(2;n+1,m+1)$ form Pascal's triangle, and $s2(-1,n,m)$ turns out to be Catalan's triangle.

Generating functions are given for the columns of these triangles. Each $\mathbf{S2}(k)$ and $\mathbf{S1}(k)$ matrix is an example of a Jabotinsky matrix. The generating functions for the rows of these triangular arrays therefore constitute exponential convolution polynomials. The sequences of the row sums of these triangles are also considered.

These triangles are related to the problem of obtaining finite transformations from infinitesimal ones generated by $x^k \frac{d}{dx}$, for $k \in \mathbf{Z}$.

AMS MSC numbers: 11B37, 11B68, 11B83, 11C08, 15A36

1 Overview

Stirling's numbers of the second kind (also called *subset numbers*), and denoted by $S2(n,m)$ (or $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$ in the notation of [3], or $\mathcal{S}_n^{(m)}$ in [1], or sequence A008277 in the database [10]) can be defined by

$$E_x{}^n \equiv (x\, d_x)^n \;=\; \sum_{m=1}^{n} S2(n,m)\, x^m\, d_x{}^m \quad , \quad n \in \mathbf{N}, \tag{1}$$

[1]In memory of my mother Else Gertrud Lang.

where the derivative operator $d_x \equiv \frac{d}{dx}$, and E_x is the *Euler* operator satisfying $E_x\, x^k \;=\; k\, x^k$. A recursion relation can be derived from eq. 1 by considering $x\, d_x (x\, d_x)^{n-1}$, using the convention $S2(n,m) = 0$ if $n < m$ to interpret $S2(n,m)$ as a lower triangular, infinite-dimensional matrix $\mathbf{S2}$:

$$S2(n,m) \;=\; m\, S2(n-1,m) \;+\; S2(n-1,m-1), \tag{2}$$

with initial values $S2(n,0) \equiv 0$ and $S2(1,1) = 1$. Because of eq. 1 these numbers arise when one asks how finite scale transformations (dilations) look, given infinitesimal ones. This is a special case of the exponentiation operation for Lie groups. The generator of the abelian Lie group of scale transformations $x' = \lambda\, x$, $\lambda \in \mathbf{R}_+$, is E_x. In order to exhibit these numbers within this framework consider first

$$\begin{aligned} e^{c\,x\,d_x} \;&=\; \sum_{n=0}^{\infty} \frac{c^n}{n!}\, {E_x}^n \;=\; 1 + \sum_{n=1}^{\infty} \frac{c^n}{n!} \sum_{m=1}^{n} S2(n,m)\, x^m\, {d_x}^m \\ &=\; 1 + \sum_{m=1}^{\infty} \left(\sum_{n=m}^{\infty} \frac{c^n}{n!}\, S2(n,m) \right) x^m\, {d_x}^m \;=\; 1 + \sum_{m=1}^{\infty} G2_m(c)\, x^m\, {d_x}^m \;. \end{aligned} \tag{3}$$

In the third step an interchange of summation has been performed (we ignore questions of convergence here), and in the last step an exponential generating function (e.g.f.) has been introduced for the $m-$th column of the number triangle, or lower triangular matrix, $\mathbf{S2}$. The recursion relation implies $G2_m(c) = \frac{1}{m!}\,(G2(c))^m$, with $G2(c) = exp(c) - 1$; therefore we obtain

$$e^{c\,x\,d_x} \;=\; \sum_{m=0}^{\infty} \frac{1}{m!}\, (G2(c)\, x)^m\, {d_x}^m \;=: e^{(exp(c)-1) x\, d_x} : \;, \tag{4}$$

where we have used the linear normal order symbol $: A :$ from quantum physics. ($: A :$ means expand A in powers of x and d_x, and move all operators d_x to the right-hand side, ignoring the usual commutation rule $[d_x, x] \equiv d_x\, x - x\, d_x = 1$. For example, $: (x\, d_x)^m : = \; x^m\, d_x^m$.) This normal order prescription is applied to each term of the expanded exponential in eq. 4. From Taylor's theorem, we see that for suitable functions f we have

$$e^{c\,x\,d_x}\, f(x) \;=: e^{(exp(c)-1) x\, d_x} :\, f(x) \;=\; f(x + (e^c - 1) x) \;=\; f(x') \;\;, \tag{5}$$

with $x' = e^c\, x$. Therefore the parameter λ for finite scale transformations is $\lambda = e^c$ if c is the parameter for infinitesimal transformations. In the context of Lie groups this fact is found by integrating the ordinary differential equation (see for example [2] and the references given there):

$$\frac{dx(\alpha)}{d\alpha} \;=\; c\, x(\alpha), \tag{6}$$

for curves starting at a fixed $x := x(\alpha = 0)$. The finite transformation maps x to $x' := x(\alpha = 1)$. x' should not be confused with a derivative. In this work we will generalize this to the case $E_{k;x} \equiv x^k\, d_x$ with $k \in \mathbf{Z}$. It is clear from the solution of the differential equation

$$\frac{dx(\alpha)}{d\alpha} \;=\; c\, x^k(\alpha), \;\; \text{with initial condition} \;\; x(\alpha = 0) \;=: x \;\; \text{and its transform} \;\; x' := x(\alpha = 1), \tag{7}$$

that the scale transformation case $k = 1$ which has been treated above is special. For $k \neq 1$, after separation of variables, we obtain the equation $(x')^{1-k} \;-\; x^{1-k} \;=\; (1-k)\, c$. Setting

$$x' \;=\; (1 + g(k;c;x))\, x \tag{8}$$

we have

$$1 + g(k;c;x) \;=\; \left(1 - (k-1)\, c\, x^{k-1}\right)^{-\frac{1}{k-1}} \;. \tag{9}$$

Therefore

$$e^{c\, x^k\, d_x}\, f(x) \;=\; f(x') \;=\; f\left((1 - (k-1)\, c\, x^{k-1})^{-\frac{1}{k-1}}\, x \right) \tag{10}$$

for $k \in \mathbf{Z} \setminus \{1\}$. The case $k = 1$ has been dealt with in eq. 5. It can be recovered from eq. 10 by taking the limit $k - 1 \to 0$.

k−Stirling numbers of the second kind, which we will denote by $S2(k;n,m)$, with $S2(1;n,m) = S2(n,m)$ the ordinary Stirling subset numbers, emerge in a proof, independent of the one implied by eq. 10, of the following operator identity, valid for $k \in \mathbf{Z}$,

$$e^{c\,x^k\,d_x} \;=\; :e^{g(k;c;x)\,x\,d_x}: \;, \tag{11}$$

where $g(k;c;x)$ is defined by eq. 9 for $k \neq 1$ and $g(1;c;x) = G2(c)$ (see eq. 4). By analogy with eq. 1 the $S2(k;n,m)$ number triangle is defined by

$$E_{k;x}{}^{n} \equiv (x^k\,d_x)^n \;=\; \sum_{m=1}^{n} S2(k;n,m)\,x^{m+(k-1)\,n}\,d_x{}^{m} \quad , \quad n \in \mathbf{N}, \quad k \in \mathbf{Z} \;, \tag{12}$$

with the further convention that $S2(k;n,m) = 0$ for $n < m$ and $S2(k;n,0) = 0$. These numbers will be shown to satisfy the recursion relation

$$S2(k;n,m) \;=\; ((k-1)(n-1)+m)\,S2(k;n-1,m) \;+\; S2(k;n-1,m-1), \tag{13}$$

with $S2(k;1,1) = 1$ from eq. 12. The e.g.f. for the $m-$th column of the $\mathbf{S2}(k)$ triangle,

$$G2(k;m;x) \;:=\; \sum_{n=m}^{\infty} S2(k;n,m)\,\frac{x^n}{n!} \;, \tag{14}$$

satisfies

$$G2(k;m;x) \;=\; \frac{1}{m!}\,(G2(k;x))^m \tag{15}$$

for $k \neq 1$, with

$$G2(k;x) \;=\; (k-1)\,g2(k;\frac{x}{k-1}) \;, \tag{16}$$

where

$$g2(k;y) \;:=\; \sum_{n=1}^{\infty} s2(k;n,1)\,y^n \tag{17}$$

is the ordinary generating function (o.g.f.) for the first column of the triangle of numbers $s2(k;n,m)$ which is associated to triangle $S2(k;n,m)$ by[2]

$$s2(k;n,m) \;:=\; (k-1)^{n-m}\,\frac{m!}{n!}\,S2(k;n,m) \;. \tag{18}$$

These number triangles, or lower triangular infinite-dimensional matrices, $\mathbf{s2}(k)$, which are here only defined for $k \in \mathbf{Z} \setminus \{1\}$, obey the recursion relation

$$s2(k;n,m) \;=\; \frac{k-1}{n}\,[(k-1)(n-1)+m]\;s2(k;n-1,m) \;+\; \frac{m}{n}\,s2(k;n-1,m-1) \quad , \tag{19}$$

with

$$s2(k;n,m) \;=\; 0, \;\; n < m \quad , \quad s2(k;n,0) \;=\; 0, \quad s2(k;1,1) = 1 \;. \tag{20}$$

It follows that these numbers are nonnegative. At this stage it is not obvious that they are integers for every $k \in \mathbf{Z} \setminus \{1\}$.
The o.g.f. of the m-th column of the $\mathbf{s2}(k)$ matrix is

$$g2(k;m;y) \;:=\; \sum_{n=m}^{\infty} s2(k;n,m)\,y^n \;=\; (g2(k;y))^m \;, \tag{21}$$

[2]These *associated Stirling numbers of the second kind* are not the ones of [5], p. 76, Table 2.

with

$$g2(k;y) \;=\; y\,c2(1-k;y), \tag{22}$$

and, for $l \in \mathbf{Z}\setminus\{0\}$,

$$c2(l;y) \;=\; \frac{1-(1-l^2\,y)^{\frac{1}{l}}}{l\,y}\;. \tag{23}$$

It is clear from eq. 21 that $\mathbf{s2}(k)$ is a convolution triangle generated from its first column (cf. [4],[6],[9]). Such ordinary convolution triangles will be called *Bell* matrices [9] (see Note 7). For $k \in \mathbf{Z}\setminus\{1\}$, eq. 16 now yields

$$G2(k;x) \;=\; -1 \;+\; (1+(1-k)\,x)^{\frac{1}{1-k}}\;, \tag{24}$$

and letting $k-1 \to 0$ we obtain $G2(1;x) \;=\; e^x - 1 \;=\; G2(x)$. The infinite-dimensional lower triangular matrices $\mathbf{S2}(k)$ with integer entries are examples of *Jabotinsky* matrices (*cf.* [4], which also contains earlier references). Therefore the o.g.f. of the rows of the triangle $\mathbf{S2}(k)$ are exponential (or binomial) convolution polynomials. In other words, the polynomials

$$S2_n(k;x) \;:=\; \sum_{m=1}^{n} S2(k;n,m)\,x^m \quad,\quad S2_0(k;x) := 1\;, \tag{25}$$

satisfy

$$S2_n(k;x+y) \;=\; \sum_{p=0}^{n}\binom{n}{p} S2_p(k;x)\,S2_{n-p}(k;y) \;=\; \sum_{p=0}^{n}\binom{n}{p} S2_p(k;y)\,S2_{n-p}(k;x) \tag{26}$$

for $k \in \mathbf{Z}$. In the notation of the umbral calculus (*cf.* [7]) the polynomials $S2_n(k;x)$ are a special type of *Sheffer* polynomials called *associated polynomial sequences*. An equivalent notation used there for the general case is "Sheffer for $(1, f(t))$". In our case $f(t) \;=\; \overline{G2(k;t)})$, where $\overline{G2(k;t)} \;=\; (-1+(1+t)^{1-k})/(1-k)$ if $k \neq 1$. This is the compositional inverse of $G2(k;t)$ from eq. 24. Also $\overline{G2(1;t)} = ln(1+t)$ can be obtained in the limit as $1-k \to 0$.
For negative k the $\mathbf{S2}(k)$ matrices also contain negative entries. The recursion of eq. 13 shows that it is possible to define nonnegative matrices by

$$S2p(-k;n,m) \;:=\; (-1)^{n-m}\,S2(-k;n,m) \quad,\quad k \in \mathbf{N}_0. \tag{27}$$

The e.g.f. for column m of the triangle $\mathbf{S2}p(-|k|)$ is

$$G2p(-|k|;m;x) \;=\; \frac{1}{m!}\,(G2p(-|k|;x))^m \tag{28}$$

with

$$G2p(-|k|;x) \;=\; -\,G2(-|k|;-x) \;=\; 1 \;-\; (1-(|k|+1)\,x)^{\frac{1}{|k|+1}}. \tag{29}$$

Eqs. 19 and 20 will be seen to imply that the $\mathbf{s2}(-|k|)$ matrices have always nonnegative entries. In Tables 1 and 2 we have listed for some of these $\mathbf{s2}(k)$ and $\mathbf{S2}(k)$ triangles the A-numbers under which they can be viewed in the on-line data-base [11] (see also [10]). This data-base will henceforth be quoted as *EIS* (Encyclopedia of Integer Sequences). These tables also give the A-numbers of the sequences formed by the first columns of the lower triangular matrices, and of the sequences of the row sums of these matrices.
For $l = 2$ the function $c2(l;y)$ defined in eq. 23 generates the well-known *Catalan* numbers. For $l \in \mathbf{Z}\setminus\{0\}$ it defines what we call $l-$*Catalan* numbers. For positive l these sequences were introduced by O. Gerard in *EIS*, who called them *Patalan* numbers. It will be proved later (see Note 11) that $c2(l;y)$ does indeed generate integers. Their explicit form can be found in eq. 77.

The *Stirling numbers of the first kind*, $S1(n,m)$ ($S_n^{(m)}$ in [1], *EIS*: A008275 can be defined from the inversion of Eq. 1 by

$$x^n\,d_x^{\;n} \;=\; \sum_{m=1}^{n} S1(n,m)\,(x\,d_x)^m \quad . \tag{30}$$

We also set $S1(n,0) \equiv 0$ and $S1(n,m) := 0$ for $n < m$. In (infinite-dimensional) matrix notation we can write [4]

$$\mathbf{S1} \cdot \mathbf{S2} \;:=\; \mathbf{1} \;=\; \mathbf{S2} \cdot \mathbf{S1} \quad . \tag{31}$$

The *signless Stirling numbers of the first kind*, also known as *cycle numbers*, $S1p(n,m)$ (or $\left[\begin{array}{c} n \\ m \end{array} \right]$ in the notation of [3]), are

$$S1p(n,m) \;:=\; (-1)^{n-m}\, S1(n,m) \quad . \tag{32}$$

Their recurrence formula is

$$S1p(n+1,m) \;=\; n\, S1p(n,m) \;+\; S1p(n,m-1) \quad , \tag{33}$$

with $S1p(1,1) = 1$, $S1p(n,0) = 0$ and $S1p(n,m) = 0$ for $n < m$.

The *generalized* $k-$*Stirling numbers of the first kind* $S1(k;n,m)$ are defined analogously by inverting eq. 12, *i.e.*

$$x^{kn}\, d_x^{\;n} \;=\; \sum_{m=1}^{n} S1(k;n,m)\, x^{(k-1)(n-m)}\, (x^k\, d_x)^m \quad \text{for} \quad k \in \mathbf{Z}\ ,\ n \in \mathbf{N}\ . \tag{34}$$

In matrix notation:

$$\mathbf{S1}(k) \cdot \mathbf{S2}(k) \;=\; \mathbf{1} \;=\; \mathbf{S2}(k) \cdot \mathbf{S1}(k) \quad \text{for} \quad k \in \mathbf{Z}. \tag{35}$$

For $k \in \mathbf{N}$ we define the *nonnegative* $k-$*Stirling numbers of the first kind* by

$$S1p(k;n,m) \;:=\; (-1)^{n-m}\, S1(k;n,m)\ . \tag{36}$$

For $-k \in \mathbf{N}_0$ the numbers $S1(k;n,m)$ are nonnegative.
The recurrence relation for $k-$Stirling numbers of the first kind is

$$S1(k;n,m) \;=\; -[(k-1)\,m + n - 1]\, S1(k;n-1,m) \;+\; S1(k;n-1,m-1) \quad , \tag{37}$$

with $S1(k;1,1) = 1$, $S1(k;n,0) = 0$ and $S1(k;n,m) = 0$ for $n < m$. For $k \neq 1$ we also introduce the *associated* $k-$*Stirling numbers of the first kind* [3]

$$s1(k;n,m) \;:=\; (1-k)^{n-m}\, \frac{m!}{n!}\, S1(k;n,m)\ , \tag{38}$$

which turn out to be always nonnegative. They satisfy the recursion

$$s1(k;n,m) \;=\; \frac{k-1}{n}\big((k-1)\,m + n - 1\big)\, s1(k;n-1,m) + \frac{m}{n}\, s1(k;n-1,m-1) \tag{39}$$

with $s1(k;n,m) = 0$ for $n < m$, $s1(k;1,1) = 1$ and $s1(k;n,0) := 0$. At this stage it is not obvious that the $s1(k;n,m)$ are in fact integers.
The o.g.f. for the $m-$th column of the number triangle $s1(k;n,m)$ will be shown to be

$$g1(k;m;y) \;=\; \sum_{n=m}^{\infty} s1(k;n,m)\, y^n \;=\; (g1(k;y))^m\ , \tag{40}$$

for $k \in \mathbf{Z} \setminus \{1\}$, with

$$g1(k;y) \;=\; y\, c1(k-1;y), \tag{41}$$

and, for $l \in \mathbf{Z} \setminus \{0\}$,

$$c1(l;y) \;=\; \frac{-1 \;+\; (1 - l\, y)^{-l}}{l^2\, y}\ . \tag{42}$$

Hence $\mathbf{s1}(k)$ is, like $\mathbf{s2}(k)$, a convolution triangle generated from its $m = 1$ column, *i.e.* both are Bell matrices.

[3] These associated Stirling numbers of the first kind are not the ones appearing in [5], p. 75, table 2.

For $l \in \mathbf{N}$ the function $c1(l;y)$ generates the numbers

$$c1(l;y) \;=\; \sum_{n=0}^{\infty} c1_n^{(l)}\, y^n \;,\quad c1_n^{(l)} \;=\; \binom{n+l}{l-1}\, l^{n-1} \;. \tag{43}$$

For $l=2$ this is the *EIS* sequence A001792 $\{1,3,8,20,48,...\}$. For negative l, $c1(l;y)$ becomes a polynomial in y; *e.g.* $c1(-2;y)=1+y$, or $g1(-1;y)=y+y^2$. The coefficients of these polynomials define a triangle of numbers found under the *EIS* number A049323. Their explicit form is, for $l \in \mathbf{N}$,

$$c1_n^{(-l)} \;=\; \binom{l}{n+1}\, l^{n-1} \;\text{ for } n=0,1,...,l-1, \text{ and } 0 \text{ otherwise.} \tag{44}$$

Eq. 43 now implies, using eqs. 41 and 40, that $s1(k;n,m)$ is indeed an integer for every $k \in \mathbf{Z} \setminus \{1\}$. An explicit form for the entries in the first column is

$$s1(k;n,1) \;=\; \begin{cases} \binom{k-2-n}{k-2}\,(k-1)^{n-2} & \text{for } k=2,3,..., \text{and } n \in \mathbf{N} \\ \binom{|k|+1}{n}\,(|k|+1)^{n-2} & \text{for } -k \in \mathbf{N}_0\,,\; n=1,2,...|k|+1 \;. \end{cases} \tag{45}$$

The e.g.f.s for the $m-$th column of the signless $k-$Stirling numbers of the first kind are then, for $k=2,3,...,$

$$G1p(k;m;x) \quad := \quad \sum_{n=m}^{\infty} S1p(k;n,m)\,\frac{x^n}{n!} \;, \tag{46}$$

$$\quad = \quad \frac{1}{m!}\left[(k-1)\,g1(k;\frac{x}{k-1})\right]^m \;. \tag{47}$$

The case $k=1$ corresponds to the ordinary unsigned Stirling numbers $S1p(n,m)$ with e.g.f. for column m given by $G1p(1;m;x) \;=\; \frac{1}{m!}\left(-ln(1-x)\right)^m$. From eqs. 47, 41 and 42,

$$G1p(k;1;x) \;=\; (k-1)\,g1(k;\frac{x}{1-x}) \;=\; \frac{1}{k-1}\,(-1 \;+\; \frac{1}{(1-x)^{k-1}}) \;, \tag{48}$$

and we recover the result for $k=1$ from l'Hôpital's rule in the limit $k-1 \to 0$. Note that $G1(k;1;x) \;=\; -G1p(k;1;-x) \;=\; \overline{G2(k;1;x)}$ for $k \in \mathbf{Z}$.
For $-k \in \mathbf{N}_0$ the e.g.f. for the $m-$th column of the nonnegative triangular matrix $\mathbf{S1}(-|k|)$ is, from eqs. 36 and 46, $G1(-|k|;m;x) \;=\; (-1)^m\, G1p(-|k|;m;-x)$, hence

$$G1(k;1;x) \;=\; \frac{1}{1+|k|}\,(-1 \;+\; (1+x)^{1+|k|}) \quad \text{for} \quad -k \in \mathbf{N}_0 \;. \tag{49}$$

For the signed matrix $\mathbf{S1}(-|k|)$ with elements defined by $S1s(-|k|;n,m) := (-1)^{n-m}\, S1(-|k|,n,m)$ the e.g.f. of the $m-$th column is $G1s(-|k|;m;x) \;=\; (-1)^m\, G1(-|k|;m;-x)$, *i.e.* $G1s(-|k|;x) \equiv G1s(-|k|;1;x) \;=\; (1-(1-x)^{1+|k|})/(1+|k|)$ for $k \in \mathbf{N}_0$.

Tables 3 and 4 give the *EIS* A-numbers of some of the number triangles $\mathbf{s1}(k)$, $\mathbf{S1}(k)$ and $\mathbf{S1p}(k)$. The A-numbers of the $m=1$ column and of the sequence of row sums for each triangle are also given there.

The o.g.f. of the row sequences of triangle $\mathbf{S1}(k)$ are also exponential (or binomial) convolution polynomials. In other words the polynomials

$$S1_n(k;x) \;:=\; \sum_{m=1}^{n} S1(k;n,m)\,x^m \quad,\quad S1_0(k;x) := 1 \;,\quad k \in \mathbf{Z} \;, \tag{50}$$

satisfy eq. 26 with $S2$ replaced by $S1$. In the notation of the umbral calculus (*cf.* [7]) the polynomials $S1_n(k;x)$ are a special type of Sheffer polynomials called associated polynomial sequences or "Sheffer for $(1,f(t))$." Here $f(t) \;=\; \overline{G1(k;t)}$), where $\overline{G1(k;t)} \;=\; G2(k;t)$ is given, for $k \neq 1$, in eq. 24. Also $\overline{G1(1;t)} = G2(1;t) = exp(t) - 1$ is obtained in the limit as $1-k \to 0$.

Each sequence of row sums of a triangle of the type considered in this work is generated by a function which depends on the generating function of the triangle's first ($m = 1$) column. For the $\mathbf{s2}(k)$ and $\mathbf{s1}(k)$ triangles, which can be considered as Bell matrices, these o.g.f.s are, for $k \in \mathbf{Z} \setminus \{1\}$,

$$r2(k;x) \quad = \quad \frac{g2(k;x)}{1 - g2(k;x)} \;=\; \frac{-1 \;+\; [1-(1-k)^2\,x]^{\frac{1}{1-k}}}{k \;-\; [1-(1-k)^2\,x]^{\frac{1}{1-k}}}, \tag{51}$$

$$r1(k;x) \quad = \quad \frac{g1(k;x)}{1 - g1(k;x)} \;=\; \frac{1 \;-\; [1-(k-1)\,x]^{k-1}}{(1+(1-k)^2)\,[1-(k-1)\,x]^{k-1} \;-\; 1} \tag{52}$$

For the $\mathbf{S2}(k)$ ($k \in \mathbf{N}_0$) and $\mathbf{S2p}(k)$ ($-k \in \mathbf{N}_0$) triangles, which can be interpreted as Jabotinsky matrices, the e.g.f.s for the sequences of row sums are

$$R2(k;x) \quad = \quad e^{G2(k;x)} \;-\; 1 \;=\; exp\,[-1 + (1-(k-1)\,x)^{\frac{1}{1-k}}] \;-\; 1 \quad , \tag{53}$$

$$R2p(-|k|;x) \quad = \quad e^{G2p(-|k|;x)} \;-\; 1 \;=\; exp\,[1-(1-(1+|k|)\,x)^{\frac{1}{1+|k|}}] \;-\; 1 \quad . \tag{54}$$

For the $\mathbf{S1p}(k)$ ($k \in \mathbf{N}_0$) and $\mathbf{S1}(k)$ ($-k \in \mathbf{N}_0$) triangles, which can also be interpreted as Jabotinsky matrices, the e.g.f.s for the sequences of row sums are

$$R1p(k;x) \quad = \quad e^{G1p(k;x)} \;-\; 1 \;=\; exp\Big(\frac{1}{k-1}\,[-1+(1-x)^{\frac{1}{k-1}}]\Big) \;-\; 1 \quad , \tag{55}$$

$$R1(-|k|;x) \quad = \quad e^{G1(-|k|;x)} \;-\; 1 \;=\; exp\Big(\frac{1}{1+|k|}[-1+(1+x)^{1+|k|}]\Big) \;-\; 1 \quad . \tag{56}$$

The special case $k = 1$ can be obtained for $R2(k;x)$ and $R1p(k;x)$ by taking the limit as $k - 1 \to 0$.

In Sections 2 and 3 we will give proofs of the results stated above.

2 k−Stirling numbers of the second kind

Definition 1: $S2(k;n,m)$. The *k-Stirling numbers of the second kind*, $S2(k;n,m)$, are defined for $k \in \mathbf{Z}$ by eq. 12.

Lemma 1: The numbers $S2(k;n,m)$ satisfy the recursion relation eq. 13.

Proof: Consider $(x^k\,d_x)^n \;=\; x^k\,d_x\,(x^k\,d_x)^{n-1}$ and use eq. 12 with $n \to n-1$ together with the lower triangular matrix conditions given after this eq. Then compare coefficients of $\{x^m\,{d_x}^m\}_1^n$. □

Note 1: It follows from eq. 13 and the initial conditions that the $S2(k;n,m)$ are always integers.

Definition 2: $s2(k;n,m)$. The *associated k-Stirling numbers of the second kind*, $s2(k;n,m)$, are defined for $k \in \mathbf{Z} \setminus \{1\}$ by eq. 18.

Lemma 2: The numbers $s2(k;n,m)$ satisfy the recursion relation given by eqs. 19 and 20.

Proof: Rewrite eq. 13 for $s2(k;n,m)$. □

Note 2: That the $s2(k;n,m)$ are indeed integers will be proved much later in Lemma 19.

Note 3: For $k = 1$ eqs. 19 and 20 give the (infinite-dimensional) unit matrix $\mathbf{s2}(1) = \mathbf{1}$. This will be used as the definition of $\mathbf{s2}(1)$.

Lemma 3: Nonnegativity of $\mathbf{s2}(k)$. The entries of the lower triangular matrix $\mathbf{s2}(k)$ are nonnegative for each $k \in \mathbf{Z}$.

Proof: If $k - 1 \geq 0$ this follows from eq. 19. For $1 - k \in \mathbf{N}$ the first term in eq. 19 becomes negative if and only if $(1-k)\,(n-1) < m$ and $n - 1 \geq m$ (otherwise $s2(k;n-1,m)$ vanishes). But the first condition contradicts the second. $\square$

Lemma 4: The o.g.f. $g2(k;m,y)$ defined in the first of eqs. 21 for the $m-$th column sequence of the lower triangular matrix $\mathbf{s2}(k)$ with $k \in \mathbf{Z}\setminus\{1\}$ satisfies the first order linear differential-difference equation

$$[1-(k-1)^2\,y]\,g2'(k;m,y)\;-\;m\,(k-1)\,g2(k;m,y)\;-\;m\,g2(k;m-1,y)\;=\;0\;\;, \tag{57}$$

$$g2(k;m,0)=0\;,\;m\in\mathbf{N};\;\;g2'(k;m,y)|_{y=0}=0\;,\;m\in\{2,3,...\}\;;\;g2'(k;1,y)|_{y=0}=s2(k;1,1)=1. \tag{58}$$

The prime denotes differentiation with respect to the variable y.

Proof: Compute $y\,\frac{d}{dy}\sum_{n=m}^{\infty} n\,s2(k;n,m)\,y^n$ with the help of the recurrence relation in eqs. 19 and 20 for $y\neq 0$. For $y=0$ the conditions given in eq. 58 follow from the definition of $g2(k;m,y)$. $\square$

Lemma 5: Using $g2(k;m,y) \;=\; (g2(k;1,y))^m$, $g2(k;y) := g2(k;1,y)$ satisfies the first order differential equation

$$[1-(k-1)^2\,y]\,g2'(k;y)\;-\;(k-1)\,g2(k;y)\;-\;1\;=\;0\;. \tag{59}$$

for $k\in\mathbf{Z}\setminus\{1\}$.

Proof: Immediate from Lemma 4. $\square$

Lemma 6: The solution to the differential eq. 59 with initial condition $g2(k;0)=0$ is, for $k\in\mathbf{Z}\setminus\{1\}$,

$$g2(k;y)\;=\;\frac{1}{[1-(k-1)^2\,y]^{\frac{1}{k-1}}}\,\frac{1-[1-(k-1)^2\,y]^{\frac{1}{k-1}}}{k-1}\;=:\;\frac{y}{[1-(k-1)^2\,y]^{\frac{1}{k-1}}}\,c2(k-1;y)\;. \tag{60}$$

Proof: Standard integration of a first order inhomogeneous differential equation of the form $g'(y)+f(y)\,g(y)=k(y)$. $\square$

Note 4: *Generalized Catalan numbers.* The l-Catalan numbers (for $l\in\mathbf{Z}\setminus\{0\}$) have

$$c2(l;x)\;:=\;\frac{1-[1-l^2\,x]^{\frac{1}{l}}}{l\,x} \tag{61}$$

as o.g.f. The case $l=2$ corresponds to the ordinary Catalan numbers (*EIS* A000108. For positive l these numbers have been called Patalan numbers by Gerard in *EIS* (*cf.* A025748–A025755 for $l=3..10$).

That $c2(l;y)$ generates integers will follow later from the fact that $s2(k;n,m)$ is always an integer (see Notes 2 and 11). Because $c2(-l;x)\;=\;c2(l;x)/(1-l^2\,x)^{\frac{1}{l}}$, one can write $g2(k;y)\;=\;y\,c2(1-k;y)$, as stated in eq. 22.

Consider the expansion $1/(1-l^2\,x)^{1/l}\;=\;\sum_{n=0}^{\infty} b_n^{(l)}\,x^n$, where $b_n^{(l)}\;=\;l^n\,(\prod_{j=1}^{n}(j\,l+1-l))/n!$ and $b_0^{(l)}\;=\;1$, $n\geq 1$. Therefore the sequence $\{c2_n^{(-l)}\}_{n=0}^{\infty}$ generated by $c2(-l;x)$ for $l\in\mathbf{N}$ is the (ordinary) convolution of the sequence $\{b_n^{(l)}\}_{n=0}^{\infty}$ with the sequence $\{c2_n^{(l)}\}_{n=0}^{\infty}$. See *e.g.* *EIS* A035323 for $l=-10$.

Since we have put $\mathbf{s2}(1)\;=\;\mathbf{1}$ we take $g2(1;y)\;=\;y$.

Lemma 7: The e.g.f. for the m-th column sequence of the $k-$Stirling triangle of the second kind $\mathbf{S2}(k)$, defined in eq. 14, is $G2(k;m;x)\;=\;\frac{1}{m!}\,(G2(k;1;x))^m$, $m\in\mathbf{N}$, with $G2(k;1;x)\;\equiv\;G2(k;x)\;=\;(k-1)\,g2(k;\frac{x}{k-1})$ for $k\neq 1$ and $G2(1;1;x)\;\equiv\;G2(1;x)\;=\;exp(x)-1$.

Proof: For $k\neq 1$ substitute $S2(k;n,m)$ from eq. 18 into the definition of $G2(k;m;x)$, and then use eq. 21. For the ordinary Stirling numbers, *i.e.* for $k=1$, the stated result is well-known [1]. $\square$

Lemma 8: For $k\in\mathbf{Z}\setminus\{1\}$,

$$e^{c\,x^k\,d_x}\;=\;\sum_{0}^{\infty}\frac{1}{m!}\left[g2(k;\frac{c}{k-1}\,x^{k-1})\,(k-1)\,x\right]^m\,d_x^{\;m}\;=:e^{g(k;c;x)\,x\,d_x}:\,, \tag{62}$$

where $g(k;c;x) := g2(k; \frac{c}{k-1}\, x^{k-1})\,(k-1)$, and the normal order $:A:$ notation has been explained in the paragraph following eq. 4.

Proof: Similar to that for ordinary Stirling numbers of the second kind, as explained in Section 1, eqs. 3 and 4. Expand the exponential and insert the definition of $S2(k;n,m)$ from eq. 12 using the triangle convention stated there. Then exchange the row summation with the column summation (ignoring questions of convergence). After replacing $S2(k;n,m)$ by $s2(k;n,m)$, using eq. 18 (and remembering that $k \neq 1$) we find the o.g.f. $g2(k;m;\frac{c}{k-1}\, x^{k-1})$ inside the column summation. The convolution property eq. 21 (Lemmas 4,5 and 6) then yields the first eq. of the lemma. The second follows from the definition of normal order, which is applied to each term in the expanded exponential. □

Note 5: For $k \neq 1$, if we insert the o.g.f. $g2(k;y)$ given in Lemma 6, or eqs. 22 and 23, we obtain the formula for $g(k;c;x)$ given in eq. 9. For $k=1$ we obtain $g(1;c;x) = exp(c) - 1$ from eq. 5.

Corollary 1: The operator identity in eq. 11, proved in Lemma 8, implies the shift property shown in eq. 10.

Proof: An applicaton of Taylor's theorem. □

Note 6: A third proof of the shift property in eq. 10 can be given by using the well-known multiple commutator formula for $exp\,(\mathbf{B})\, x^l\, exp(-\mathbf{B})$ for $l \in \mathbf{N}_0$, setting the operator $\mathbf{B} = c\, E_{k;x} = c\, x^k\, d_x$ for $k \in \mathbf{Z}$ and the commutator $[E_{k;x}, x^l] = l\, x^{l+k-1}$. For $k=1$ we find $exp(c\, x\, d_x)\, x^l\, 1 = (exp(c)\, x)^l\, exp(c\, x\, d_x)\, 1 = (exp(c)\, x)^l$. For $k \neq 1$ we first obtain $exp(c\, E_{k;x})\, x^l\, exp(-c\, E_{k;x}) = \sum_{n=0}^{\infty} \frac{1}{n!} (l/(k-1))_n\, (c\,(k-1)\, x^{k-1})^n$ using the rising factorial (or *Pochhammer*) symbol $(\nu)_n := \nu\,(\nu+1) \cdots (\nu+n-1)$. This implies $exp(c\, x^k\, d_x)\, x^l\, 1 = [(1+g(k;c;x))\, x]^l\, 1$ with $1+g(k;c;x)$ given in eq. 9. The 1 on the right-hand side stands for any $x-$independent operator or function. Hence the shift property eq. 10 holds for polynomials and (formally) for power series $f(x)$.

Lemma 9: For $k \in \mathbf{Z}$ the e.g.f. of the row polynomials $S2_n(k;x)$ defined in eq. 25, $\mathcal{G}2(k;z,x) := \sum_{n=0}^{\infty} S2_n(k;x)\, z^n/n!$, is given by

$$\mathcal{G}2(k;z,x) \;=\; e^{x\, G2(k;z)}\ , \tag{63}$$

where $G2(k;z)$ is the e.g.f. for the first ($m=1$) column sequence of the triangular matrix $\mathbf{S2}(k)$ given in eq. 24.

Proof: Separate the $n=0$ term in the definition of $\mathcal{G}2(k;z,x)$ and insert in the remaining expression the definition of the row polynomials eq. 25. Then interchange the row and column summation indices and use the definition of the e.g.f. $G2(k;m;z)$ given in eq. 14. The convolution property Lemma 7, or eq. 15, then leads to the desired result. □

Note 7: Another way to state Lemma 9 is to write

$$S2(k;n,m) \;=\; \left[\frac{z^n}{n!}\right]\, [x^m]\, e^{x\, G2(k;z)}\ , \tag{64}$$

where $[y^k]\, f(y)$ denotes the coefficient of y^k in the expansion of $f(y)$. For each $k \in \mathbf{Z}$ a matrix constructed in this way from the entries of its first ($m=1$) column (collected in the e.g.f. $G2(k;z)$) is called a Jabotinsky matrix. (See [4] for references to the original works. Note that we use Knuth's $n!\, F_n(x)$ as row (or Jabotinsky) polynomials. Knuth's $f(z)$ corresponds to our e.g.f. for the $m=1$ column sequence.)

Another notation is used in the umbral calculus (*cf.* [7]). The row polynomials $E_n(x) = \sum_{n=1}^{m} J(n,m)\, x^n$ built from a lower triangular Jabotinsky matrix $J(n,m)$ are there called associated polynomial sequences. Their defining function is the compositional inverse of the e.g.f. $f(t)$ used by Knuth and in the present work (*cf.* [7], p. 53). $\{E_n(x)\}$ are special Sheffer polynomials for $(1, \bar{f}(t))$ in the umbral notation (*cf.* [7], p. 107).

Yet another description of such convolution polynomials can be found in [9], where Jabotinsky matrices appear as a special case of so-called *Riordan* matrices (if one uses exponential generating functions). The corresponding matrix product furnishes a so-called *Bell* subgroup of the Riordan group (*cf.* [9], p. 238). In the sequel we shall reserve the names Riordan and Bell matrices for the case of ordinary convolutions.

Lemma 10: The exponential (or binomial) convolution property given in eq. 26 for polynomials $S2_n(k;x), n \in \mathbf{N}_0$ and fixed k, is equivalent to the functional equation

$$\mathcal{G}2(k;z,x+y) \; = \; \mathcal{G}2(k;z,x)\,\mathcal{G}2(k;z,y) \; , \tag{65}$$

which follows from eq. 63 for the e.g.f. $\mathcal{G}2(k;z,x)$ defined in Lemma 9.

Proof: Fix k and compare the coefficients of $z^n/n!$ on both sides of this equation. □

Proposition 1: Exponential convolution property of the $S2_n(k;x)$ polynomials. The row polynomials $S2_n(k;x)$ defined in eq. 25 for $n \in \mathbf{N}_0$ satisfy for each $k \in \mathbf{Z}$ the exponential convolution property shown in eq. 26.

Proof: Lemma 10 with Lemma 9. □

Lemma 11: Row sums of ordinary convolution matrices [6]. The o.g.f. $r(x) := \sum_{n=1}^{\infty} r_n\, x^n$ of the row sums $r_n := \sum_{m=1}^{n} s(n,m)$ of a lower triangular ordinary convolution matrix $\{s(n,m)\}_{n\geq m\geq 1}$ is given by

$$r(x) \; := \; \frac{g(x)}{1-g(x)} \; , \tag{66}$$

where $g(x)$ is the o.g.f. of the first $(m=1)$ column of the matrix $s(n,m)$.

Proof: Consider a lower triangular convolution matrix. By definition, the o.g.f. $g(m;x)$ for its $m-$th column sequence is given by $g(m;x) \; = \; (g(1;x))^m = g(x)^m$ for $m \in \mathbf{N}$. The result follows by inserting into $r(x)$ the definition of the row sums r_n, interchanging row and column summation indices and using the definition and convolution property of $g(m;x)$. □

Lemma 12: Row sums of exponential convolution matrices. The e.g.f. $R(x) := \sum_{n=1}^{\infty} R_n\, x^n/n!$ of the row sums $R_n := \sum_{m=1}^{n} S(n,m)$ of a lower triangular exponential convolution matrix $\{S(n,m)\}_{n\geq m\geq 1}$ is given by

$$R(x) \; := \; e^{G(x)} \; - \; 1 \; , \tag{67}$$

where $G(x)$ is the e.g.f. of the first $(m=1)$ column sequence of the matrix $\{S(n,m)\}_{n\geq m\geq 1}$.

Proof: Analogous to the proof of Lemma 11. □

Proposition 2: O.g.f. for row sums of the $\mathbf{s2}(k)$ triangles. For $k \in \mathbf{Z}\setminus\{1\}$ the o.g.f. of the sequence of row sums of the lower triangular matrix $\mathbf{s2}(k)$ is given by eq. 51.

Proof: Lemma 11 and the $g2(k;x)$ result from Lemma 6, eq. 60. □

Proposition 3: E.g.f. for the sequence of row sums of the $\mathbf{S2}(k)$ and $\mathbf{S2p}(k)$ triangles. For $k \in \mathbf{N_0}$ the e.g.f. of the sequence of row sums of the nonnegative lower triangular matrix $\mathbf{S2}(k)$, resp. $\mathbf{S2p}(k)$, defined from eq. 13, resp. eq. 27, is given by eq. 53, resp. eq. 54.

Proof: Lemma 12 and $G2(k;x)$, resp. $G2p(-k;x)$, from eq. 24, resp. eq. 29. □

3 k−Stirling numbers of the first kind

$k-$Stirling numbers of the first kind can be defined as the elements of the (infinite-dimensional, lower triangular) inverse matrix $\mathbf{S1}(k)$ to the matrix $\mathbf{S2}(k)$ formed from the $k-$Stirling numbers of the second kind.

Definition 3: *$k-$Stirling numbers of the first kind*, $S1(k;n,m)$, are defined by

$$x^n\, {d_x}^n \; = \; \sum_{m=1}^{n} S1(k;n,m)\, x^{-m(k-1)}\, (x^k\, d_x)^m \quad , \text{ for } k \in \mathbf{Z},\; n \in \mathbf{N} \; . \tag{68}$$

Note that this equation is obtained from eq. 34 by multiplication by $x^{-n(k-1)}$ on the left. Therefore the equations are equivalent for every k provided $x \neq 0$. We set $S1(k;n,m) = 0$ if $n < m$, *i.e.* $\mathbf{S1}(k)$ is a lower triangular matrix.

Lemma 13: $\mathbf{S2}(k) \cdot \mathbf{S1}(k) = \mathbf{1}$, or

$$\sum_{m=p}^{n} S2(k;n,m)\, S1(k;m,p) = \delta_{n,p} \tag{69}$$

for fixed $k \in \mathbf{Z}$, $n \in \mathbf{N}$ and $p \in \mathbf{N}$, where $\delta_{n,p}$ is the Kronecker symbol.

Proof: Insert eq. 68 with $n \to m$ and $m \to p$ into the defining eq. 12 for the $S2(k;n,m)$ numbers, and then extend the $p-$sum from m to n, using lower triangularity of each matrix $\mathbf{S1}(k)$. After interchanging the summations over m and p we find, for all $k \in \mathbf{Z}$, $n \in \mathbf{N}$ and $x \neq 0$,

$$\mathcal{O}_x(k;n) := x^{-(k-1)n}\,(x^k\, d_x)^n = \sum_{p=1}^{n} \delta(k;n,p)\, \mathcal{O}_x(k;p) \ , \tag{70}$$

with $\delta(k;n,p) := \sum_{m=p}^{n} S2(k;n,m)\, S1(k;m,p)$. Since the operators $\{\mathcal{O}_x(k;p)\}_{p=1}^{n}$ acting on functions $f \in C^n$ are a linearly independent[4], eq. 70 implies $\delta(k;n,p) = \delta_{n,p}$ for each k. □

Similarly, one finds

$$\mathbf{S1}(k) \cdot \mathbf{S2}(k) = \mathbf{1} \tag{71}$$

after inserting eq. 12 with $n \to m$, $m \to p$ into eq. 68. Now we compare coefficients of the operators $\{x^p\, {d_x}^p\}_1^n$.

Lemma 14: The $k-$Stirling numbers of the first kind satisfy the recurrence given in eq. 37.

Proof: Use $x^n\, {d_x}^n = (x\, d_x - (n-1))\, x^{n-1}\, {d_x}^{n-1}$ and insert eq. 68 in both sides of this identity. After differentiation, remembering the triangularity of $\mathbf{S1}(k)$, we compare coefficients of the linearly independent operators $\{\mathcal{O}_x(k;m)\}_{m=1}^{n}$ defined in eq. 70. □

Note 8: It is obvious from the recurrence 37 together with the initial conditions that all $S1(k;n,m)$ are integers for $k \in \mathbf{Z}$.

Definition 4: $s1(k;n,m)$. The *associated $k-$Stirling numbers of the first kind*, $s1(k;n,m)$, are defined for $k \in \mathbf{Z} \setminus \{1\}$ by eq. 38.

Lemma 15: The numbers $s1(k;n,m)$ satisfy the recurrence given in eq. 39.

Proof: Rewrite the recurrence relation eq. 37 for $S1(k;n,m)$ with $k \neq 1$. The lower triangularity of the matrix $\mathbf{s1}(k)$ is inherited from $\mathbf{S1}(k)$. □

Note 9: For $k = 1$ eq. 39 gives the unit matrix $\mathbf{s1}(1) = \mathbf{1}$. This will be used as the definition of $\mathbf{s1}(1)$.
Lemma 16: Nonnegativity of $\mathbf{s1}(k)$. The entries of the lower triangular matrix $\mathbf{s1}(k)$ are nonnegative for each k.

Proof: If $k-1 \geq 0$ this follows from eq. 39. For $1-k \in \mathbf{N}$ this follows from the fact that $s1(k;n-1,m) = 0$ if $n-1 > (1-k)\,m$, *i.e.* if the coefficient of the first term in the recurrence eq. 39 is negative. This will be shown by induction on m. For $m = 1$ the assertion is true because only the first term in the recurrence is present, and since $s1(k,2-k,1) = 0$, due to the vanishing coefficient of the first term in its recursion, the recurrence shows that $s1(k;n-1,1)$ vanishes for $n-1 = 2-k, 3-k, \ldots$ (if $n-1 = 2-k$ the multiplier in the first recursion term vanishes). Assuming the assertion holds for given $m \geq 1$, *i.e.* $s1(k;n-1,m) = 0$ for $n-1 > (1-k)\,m$, leads to a vanishing second term in the $s1(k;n-1,m+1)$ recurrence for all

[4] This linear independence can be proved by applying the differentiation operators $\frac{1}{p!}\,\mathcal{O}_x(k;p)$ for fixed $k \in \mathbf{Z}$ and $p = 1, \ldots, n$ to the monomials x^q, for $q = 1, .., n$. The linear independence is then inferred from the non-singularity of the $n \times n$ matrix $A_{q,p}(k) = \frac{1}{p!} \prod_{j=0}^{p-1}(q + j(k-1))$. In fact, $Det\, \mathbf{A}(k) = +1$ for each $k \in \mathbf{Z}$ and $n \in \mathbf{N}$.

$n-1 > (1-k)\,m + 1$. Therefore, $s1(k;(1-k)\,(m+1) + 1, m+1)$ will be zero because the coefficient of the first term of this recurrence vanishes and the second term is absent since $(1-k)\,(m+1) > (1-k)\,m$. Then $s1(k;n-1,m+1)$ vanishes recursively for all $n-1 \geq (1-k)\,(m+1) + 1$. □

Lemma 17: The o.g.f. for the m-th column of $\mathbf{s1}(k)$ (see eqs. 40, 41 and 42). $\mathbf{s1}(k)$ is a Bell matrix (see Note 7 for this name), *i.e.* the o.g.f. for the sequence $\{s1(k;n,m)\}_{n=1}^{\infty}$ is given by $g1(k;m;y) = (g1(k;1;y))^m$ and

$$g1(k;y) \;:=\; g1(k;1;y) \;=\; \frac{-1 + (1-(k-1)\,y)^{-(k-1)}}{(k-1)^2} \quad \text{for} \quad k \in \mathbf{Z} \setminus \{1\}\ . \tag{72}$$

Since we have set $\mathbf{s1}(1) = \mathbf{1}$ we take $g1(1;y) = y$.

Proof: From the recurrence relation eq. 39 we find, for $k \in \mathbf{Z}$, the first-order linear differential-difference equation

$$[1-(k-1)\,y]\,g1'(k;m;y) \;-\; m\,(k-1)^2\,g1(k;m;y) \;-\; m\,g1(k;m-1;y) \;=\; 0 \ , \tag{73}$$

$$g1(k;m;0) = 0\ ,\ m \in \mathbf{N};\quad g1'(k;m;y)|_{y=0} = s1(k;1,1)\,\delta_{m,1} = \delta_{m,1}. \tag{74}$$

The prime denotes differentiation with respect to y. The $y = 0$ conditions follow from the definition of $g1(k;m;y)$ in eq. 40. Eq. 73 is solved using $g1(k;m;y) = (g1(k;1;y))^m$, which results in a standard linear inhomogeneous differential equation for $g1(k;y) := g1(k;1;y)$, namely

$$[1-(k-1)\,y]\,g1'(k;y) \;-\; (k-1)^2\,g1(k;y) \;-\; 1 \;=\; 0\,, \tag{75}$$

with the initial condition $g1(k;0) = 0$. The solution is given by equation eq. 72 (cf. eq. 41, 42). □

Note 10: Generalized *EIS* A001792 sequences. Analogous to the generalized Catalan numbers generated by $c2(l;y)$ of eq. 23 (see Note 4), we can use $c1(l;y)$ defined in eq. 42 as the o.g.f. for sequences $\{c1_n^{(l)}\}_{n=0}^{\infty}$. We find that $c1(1;y) = 1/(1-y)$ generates *EIS* A000012 (powers of 1), $c1(2;y)$ is the o.g.f. for the sequence A001792(n). The *EIS* A-numbers for the sequences for $l = k-1$ are found in the second column of Table 3 for $l = 1, ..., 5$ and $l = -1, ..., -6$. See also *EIS* A053113.

In order to have $g1(1;y) = y$ we set $c1(0;y) \equiv 1$ (see eq. 41). An explicit expression for $c1_n^{(l)}$ with $l \in \mathbf{N}$ is given in eq. 43. Also $c1_n^{(0)} = \delta_{n,0}$, and $c1(-l;x)$ is a polynomial in x for $l \in \mathbf{N}$. For example, $c1(-3;x) = 1 + 3\,x + 3\,x^2$. The triangle of coefficients in these polynomials can be found as *EIS* A049323 (increasing powers of x), or A033842 (decreasing powers of x). The explicit form for these coefficients is given in eq. 44.

Lemma 18: The entries of the matrix $\mathbf{s1}(k)$ are integers for all $k \in \mathbf{Z}$.

Proof: The first column of $\mathbf{s1}(k)$ consists of integers since $c1(k-1;y)$ generates the integers $c1_n^{(k-1)}$ given explicitly in eqs. 43 and 44, and $g1(k;y)$ is given by eq. 41 (see Lemma 17). The case $k = 1$ is trivial. Since $\mathbf{s1}(k)$ is an ordinary convolution triangle (or Bell matrix) it is sufficient to prove that the first column consists of integers. □

Lemma 19: The entries of the matrix $\mathbf{s2}(k)$ are integers for all $k \in \mathbf{Z}$.

Proof: Once this has been established, all entries of $s2(k;n,m)$ are nonnegative integers by Lemma 3. For the proof we first substitute eqs. 18 and 38 into eq. 69. Define, for $k \in \mathbf{Z}$, the signed matrix $\mathbf{s2s}(k)$ by $s2s(k;n,m) := (-1)^{n-m}\,s2(k;n,m)$. Then eq. 69 implies

$$\mathbf{s2s}(k)\,\cdot\,\mathbf{s1}(\mathbf{k}) \;=\; \mathbf{1}\ . \tag{76}$$

Using the fact that the $s1(k;n,m)$ are integers from the previous lemma (from Lemma 16 they are even known to be nonnegative) this equation allows us to carry out the proof recursively. We omit the details. □

Note 11: Using Lemmas 16 and 19, eqs. 21 and 22 show that $c2(l;y) = \sum_{n=0}^{\infty} c2_n^{(l)}\,y^n$ defined in eq. 23 generates positive integers for all $l \in \mathbf{Z} \setminus \{0\}$. Their explicit form is given by

$$c2_n^{(l)} \;=\; l^n\,\prod_{j=1}^{n}(j\,l-1)/(n+1)!\ . \tag{77}$$

By definition $c2(0;y) := 1$.

Lemma 20: The e.g.f. for the $m-th$ column sequence of the unsigned $k-$Stirling triangle of the first kind, $\mathbf{S1p}(k)$, defined in eq. 36 for $k \in \mathbf{N}$, is $G1p(k;m;x) = \frac{1}{m!}\,(G1p(k;1;x))^m$, $m \in \mathbf{N}$, with $G1p(k;1;x) \equiv G1p(k;x) = (k-1)\,g1(k;\frac{x}{k-1})$ for $k = 2,3,...$ and $G1p(1;1;x) \equiv G1p(1;x) = -\,ln(1-x)$.

Proof: For $k \geq 2$ substitute $S1p(k;n,m)$ from eqs. 36 and 38 into the definition of $G1p(k;m;x)$ given in eq. 46. In this way the o.g.f. $g1(k;m;y)$ appears in the desired form. The result for the ordinary unsigned Stirling numbers ($k=1$) is well-known [1]. □

Note 12: Explicit form for $G1p(k;m;x)$, $k > 1$: eq. 48 and Lemma 20. Equation 48 follows from the o.g.f. $g1(k;m;y)$ in eqs. 40 and 72. This shows that $G1p(k;1;x) = -\overline{G2(k;-x)}$, the negative compositional inverse of $G2(k;-x)$ of eq. 24. Inverse Jabotinsky matrices like $\mathbf{S2}$ and $\mathbf{S1}$ (*cf.* eqs. 69 and 71) have first column e.g.f.'s which are inverse to each other in the compositional sense [4].

Lemma 21: Row polynomials for $\mathbf{S1}(k)$. For $k \in \mathbf{Z}$ the e.g.f. of the row polynomials $S1_n(k;x) := \sum_{m=1}^{n} S1(k;n,m)\,x^m$, $n \in \mathbf{N}$, and $S1_0(k;x) := 1$ is

$$\mathcal{G}1(k;z,x) := \sum_{n=0}^{\infty} S1_n(k;x)\,z^n/n! \;=\; e^{x\,G1(k;z)}\ , \tag{78}$$

where $G1(k;z) = (-1 + (1+z)^{1-k})/(1-k)$ for $k \neq 1$, and $G1(1;z) = ln(1+z)$ are the e.g.f.s for the first ($m=1$) column sequences of the triangular matrices $\mathbf{S1}(k)$.

Proof: Analogous to that of Lemma 9.

Note 13: $S1(k;n,m) = \left[\frac{z^n}{n!}\right]\,[x^m]\,e^{x\,G1(k;z)}$ (cf. Note 7).

Proposition 5: Exponential convolution property of the $S1_n(k;x)$ polynomials. The row polynomials $S1_n(k;x)$ defined in Lemma 21 for $n \in \mathbf{N}_0$, satisfy for each $k \in \mathbf{Z}$ the exponential (or binomial) convolution property shown in eq. 26 with $S2$ replaced everywhere by $S1$.

Proof: For fixed k, compare the coefficients of $z^n/n!$ on both sides of the identity $\mathcal{G}1(k;z,x+y) = \mathcal{G}1(k;z,x)\,\mathcal{G}1(k;z,y)$. □

Note 14: In the notation of the umbral calculus (*cf.* [7]) the polynomials $S1_n(k;x)$ are called associated polynomial (or Sheffer) sequences for $(1, \overline{G1(k;t)} = G2(k;t))$. For $k \neq 1$ $G2(k;t)$ is given in eq. 24. Also $\overline{G1(1;t)} = G2(1;t) = exp(t) - 1$.

Proposition 6: O.g.f. for row sums of $\mathbf{s1}(k)$ triangles. For $k \in \mathbf{Z} \setminus \{1\}$ the o.g.f. of the sequence of row sums of the lower triangular matrix $\mathbf{s1}(k)$ is given by eq. 52.

Proof: Lemma 11 and the $g1(k;x)$ result in Lemma 17. □

Proposition 7: E.g.f. of the sequence of row sums of $\mathbf{S1p}(k)$ and $\mathbf{S1}(-|k|)$ triangles. For $k \in \mathbf{N_0}$ the e.g.f. of the sequence of row sums of the nonnegative lower triangular matrix $\mathbf{S1p}(k)$, resp. $\mathbf{S1}(-|k|)$, defined in eq. 36, resp. eq. 37, is given by eq. 55, resp. eq. 56.

Proof: Lemma 12 and $G1p(k;x)$, resp. $G1(-|k|;x)$, from Lemma 20, *i.e.* eq. 48, resp. eq. 49. □

Note 15: Row-sums of signed $\mathbf{S1}(k)$, $k \in \mathbf{N}$, resp. $\mathbf{S1s}(-|k|)$ triangles. Here Lemma 12 applies with the e.g.f.s $G1(k;x)$, resp. $G1s(-|k|;x)$, given in the first line after eq. 78, resp. in the paragraph after eq. 49.

Acknowledgements

The author would like to thank Stefan Theisen for a conversation at a very early stage of this work (Note 6, case $k = 1$). Thanks go also to Norbert Dragon who pointed out his web-pages (ref. [2]). This work has its origin in an exercise in the author's 1998/1999 lectures on conformal field theory (*Konforme Feldtheorie*, Blatt 1, Aufgabe 2, available as a ps.gz file under http://www-itp.physik.uni-karlsruhe.de/˜wl/Uebungen.html).

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Table 1: Associated k-Stirling number triangles of the second kind

$\mathbf{s2(k)}$, $\mathbf{k \neq 1}$ $\mathbf{s2(1) := 1}$

k	A-number of triangle	A-number of sequence of first column	A-number of sequence of row sums
⋮			
-5	A049224	A025751 (Gerard)	A025759 (Gerard)
-4	A049223	A025750 (Gerard)	A025758 (Gerard)
-3	A049213	A025749 (Gerard)	A025757 (Gerard)
-2	A048966	A025748 (Gerard)	A025756 (Gerard)
-1	A033184 (Catalan)	A000108$(n-1)$	A000108 (Catalan)
0	A023531 (**1** matrix)	A000007$(n-1)$	A000012 (powers of 1)
2	A007318$(n-1, m-1)$ (Pascal)	A000012	A000079 (powers of 2)
3	A035324	A001700$(n-1)$	A049027
4	A035529	A034171$(n-1)$	A049028
5	A048882	A034255$(n-1)$	A048965
6	A049375	A034687	A039746
⋮			

Table 2: k-Stirling number triangles of the second kind

$\mathbf{S2(k), k = 0, 1, 2, ...,} \qquad \mathbf{S2p(k), k = 0, -1, -2, ...}$

k	A-number of triangle	A-number of sequence of first column	A-number of sequence of row sums
⋮			
-5	A013988	A008543$(n-1)$ (Keane)	A028844
-4	A011801	A008546$(n-1)$ (Keane)	A028575
-3	A000369	A008545$(n-1)$ (Keane)	A016036
-2	A004747	A008544$(n-1)$ (Keane)	A015735
-1	A001497$(n-1, m-1)$ (Bessel)	A001147$(n-1)$ (double factorials)	A001515 (Riordan)
0	A023531 (**1** matrix)	A000007$(n-1)$	A000012 (powers of 1)
1	A008277 (Stirling 2nd kind)	A000012 (powers of 1)	A000110 (Bell)
2	A008297 (unsigned Lah)	A000142 (factorials)	A000262 (Riordan)
3	A035342	A001147 (2-factorials)	A049118
4	A035469	A007559 (3-factorials)	A049119
5	A049029	A007696 (4-factorials)	A049120
6	A049385	A008548 (5-factorials)	A049412
⋮			

Table 3: Associated k-Stirling number triangles of the first kind

$\mathbf{s1(k),\ k \neq 1 \quad s1(1) := 1}$

k	A-number of triangle	A-number of sequence of first column	A-number of sequence of row sums
⋮			
-5	A049327	A049323(5,m)	A049351
-4	A049326	A049323(4,m)	A049350
-3	A049325	A049323(3,m)	A049349
-2	A049324	A049323(2,m)	A049348
-1	A030528	A019590=A049323$(1,m)$	A000045$(n+1)$ (Fibonacci)
0	A023531 (**1** matrix)	A000007$(n-1)$=A049323(0,m)	A000012 (powers of 1)
2	A007318$(n-1,m-1)$ (Pascal)	A000012 (powers of 1)	A000079 (powers of 2)
3	A030523	A001792	A039717
4	A030524	A036068	A043553
5	A030526	A036070	A045624
6	A030527	A036083	A046088
⋮			

Table 4: k-Stirling number triangles of the first kind

$\mathbf{S1p(k), k = 0, 1, 2, ...,} \qquad \mathbf{S1(k), k = 0, -1, -2, ...}$

k	A-number of triangle	A-number of sequence of first column	A-number of sequence of row sums
⋮			
-5	A049411	A008279(5,$n-1$) (numbperm)	A049431
-4	A049424	A008279(4,$n-1$) (numbperm)	A049427
-3	A049410	A008279(3,$n-1$) (numbperm)	A049426
-2	A049404	A008279(2,$n-1$) (numbperm)	A049425
-1	A049403	A008279(1,$n-1$) (numbperm)	A000085
0	A023531 (**1** matrix)	A000007($n-1$)	A000012 (powers of 1)
1	A008275 (unsigned Stirling 1st kind)	A000142($n-1$)	A000142 (factorials)
2	A008297 (unsigned Lah)	A000142 (factorials)	A000262 (Riordan)
3	A046089	A001710($n+1$) (Mitrinovic^2)	A049376
4	A035469	A001715($n+2$) (Mitrinovic^2)	A049377
5	A049353	A001720($n+3$) (Mitrinovic^2)	A049378
6	A049374	A001725($n+4$) (Mitrinovic^2)	A049402
⋮			

(Concerned with sequences A000007, A000012, A000045, A000079, A000085, A000108, A000110, A000142, A000262, A000369, A001147, A001497, A001515, A001700, A001710, A001715, A001720, A001725, A001792, A004747, A007318, A007559, A007696, A008275, A008277, A008279, A008297, A008543, A008544, A008545, A008546, A008548, A011801, A013988, A015735, A016036, A019590, A023531, A025748, A025748-A025755, A025749, A025750, A025751, A025756, A025757, A025758, A025759, A028575, A028844, A030523, A030524, A030526, A030527, A030528, A033184, A033842, A034171, A034255, A034687, A035323, A035324, A035342, A035469, A035529, A036068, A036070, A036083, A039717, A039746, A043553, A045624, A046088, A046089, A048882, A048965, A048966, A049027, A049028, A049029, A049118, A049119, A049120, A049213, A049223, A049224, A049323, A049324, A049325, A049326, A049327, A049348, A049349, A049350, A049351, A049353, A049374, A049375, A049376, A049377, A049378, A049385, A049402, A049403, A049404, A049410, A049411, A049412, A049424, A049425, A049426, A049427, A049431, and A053113.)

Received February 11, 2000; published in Journal of Integer Sequences September 13, 2000; minor editorial changes November 30, 2000; fixed OEIS links August 11 2012.

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