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Bounds for Sets of Remainders

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Abstract

Let $s(n)$ be the number of different remainders $n \bmod k$, where $1 \leq k \leq \lfloor n/2 \rfloor$. This rather natural sequence is sequence A283190 in the OEIS, and although some basic facts are known, surprisingly, it has been barely studied. First, we prove the asymptotic formula $s(n) = cn + O(n/(\log n \log\log n))$, where c is an explicit constant. Then we focus on the difference between the consecutive terms $s(n)$ and $s(n+1)$. It turns out that the value can always increase by at most one, but there exist arbitrarily large decreases. We show that the upper bound on the difference is $O(\log\log n)$. Finally, we consider "iterated remainder sets". These are related to a problem arising from Pierce expansions, and we prove bounds for the size of these sets as well.

1 Introduction

Let us fix an integer n. Then a natural question is the following: What are the remainders $r = n \bmod k$? The sum of such remainders with $1 \leq k \leq n$ was already considered by Lucas [5, p. 373] and more recently in [8, 10, 3]. In the present paper, we are simply interested in the number of distinct remainders. Surprisingly, it seems that this question has been barely studied.

If $k > n$, then we get $r = n$ for all k.

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If $\lfloor n/2 \rfloor + 1 \leq k \leq n$, then $r = n - k$, so in this range we simply get all the integers between 0 and $\lceil n/2 \rceil - 1$.

Therefore, the interesting cases are where $1 \leq k \leq \lfloor n/2 \rfloor$, and we define the set

$$S(n) := \{n \bmod k \colon k \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}\}.$$

Note that clearly $S(n) \subseteq \{0, 1, \ldots, \lfloor n/2 \rfloor - 1\}$. Now, let us define

$$s(n) := |S(n)|.$$

In other words, $s(n)$ is the number of different values $n \bmod k$ for $1 \leq k \leq \lfloor n/2 \rfloor$. This is precisely sequence A283190 in the OEIS (On-Line Encyclopedia of Integer Sequences) [6]. We present the first few numbers of the sequence in Table 1.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$s(n)$	0	1	1	1	2	1	2	2	2	3	4	2	3	3	3	4	5	3

Table 1: First 18 values of $s(n)$.

Although it seems like a natural sequence, it was only entered into the OEIS in 2017 by T. Kerscher. R. Israel then observed that $s(n)/n$ seems to converge to approximately 0.2296. He asked about the actual value of the constant on the StackExchange website [4]. This was answered by the user Empy2 (and added as a comment to the OEIS entry by M. Peake):

$$\lim_{n\to\infty} \frac{s(n)}{n} = \sum_{p \in \mathbb{P}} \frac{1}{p(p+1)} \cdot \prod_{\substack{q \in \mathbb{P}, \\ q < p}} \left(1 - \frac{1}{q}\right) \approx 0.2296. \tag{1}$$

The answer on the StackExchange website also contains an explanation for why this is true. While the argument is relatively simple, we did not find a rigorous proof in print. In this paper, in Section 3, we give a proof of (1), including a bound for the error term.

Then we investigate the differences between consecutive terms of $s(n)$. Looking at the first few values of the sequence $s(n)$, one sees that the values of $s(n+1)$ compared to $s(n)$ usually either stay the same or increase or decrease by 1. They never seem to increase by more than 1, but sometimes they decrease by 2 or, as it turns out, even more. In Section 4 we show that these differences can get arbitrarily large, while being double logarithmically bounded in terms of n.

In Section 5, we generalize the set $S(n) =: S_1(n)$ to "iterated remainder sets" via $S_{j+1}(n) := \{n \bmod k \colon k \in S_j(n) \setminus \{0\}\}$ for $j \geq 1$. These sets are related to an older problem arising from Pierce expansions. In several papers [9, 2, 1] the following problem was considered: If we fix a positive integer n, choose another integer $1 \leq a \leq n$, and repeatedly set $a := n \bmod a$, what is the largest number of steps performed before reaching $a = 0$? This problem remains open. More on this and the relation to our iterated remainder sets in Section 5. In Section 6 we prove some bounds for these sets.

Finally, in Section 7 we pose some open problems.

However, we first present all the main results in the next section.

2 Main results and some lemmas

Let us define the constant from (1), namely

$$c := \sum_{p} \frac{1}{p(p+1)} \cdot \prod_{p'<p} \left(1 - \frac{1}{p'}\right) \approx 0.2296.$$

In Section 3 we will prove the following asymptotics for $s(n)$.

Theorem 1. *We have*

$$s(n) = cn + O\left(\frac{n}{\log n \log\log n}\right).$$

In particular,

$$\lim_{n\to\infty} \frac{s(n)}{n} = c.$$

We believe that our bound for the error term is not sharp. Numerical experiments suggest that $O(n^{1/3})$ might be closer to the truth; see Section 7. In any case, we know that overall $s(n)$ grows linearly; however, due to the error term, this does not provide much information on the differences $s(n+1) - s(n)$.

As mentioned in the introduction, looking at the first few values of $s(n)$, it seems that the values of $s(n+1)$ compared to $s(n)$ usually either stay the same or increase or decrease by 1. It turns out that they never increase by more than 1, but sometimes they do decrease by 2 or even more. For example, $s(131) = 33$ and $s(132) = 30$. The first time that the value decreases by 4 happens at $n = 17291$, where $s(17291) = 3975$ and $s(17292) = 3971$. We have searched up to $n = 10^7$ and have not found a decrease by more than 4 in this range. However, it turns out that arbitrarily large decreases do exist, and we will provide a construction for such n. On the other hand, we can prove bounds on the decreases in terms of n. In Section 4 we will, in particular, prove the following results.

Theorem 2. *For $n \geq 1$ we have*

$$s(n) - O(\log\log n) \leq s(n+1) \leq s(n) + 1.$$

Moreover,

$$\liminf_{n\to\infty} s(n+1) - s(n) = -\infty.$$

Now, let us define the sets of iterated remainders of n inductively by

$$S_0(n) := \{1, 2, \dots, \lfloor n/2 \rfloor\} \quad \text{and}$$
$$S_j(n) := \{n \bmod k \colon k \in S_{j-1}(n) \setminus \{0\}\} \quad \text{for } j \geq 1.$$

Note that, in particular, $S(n) = S_1(n)$. Moreover, analogously to $s(n)$, we define

$$s_j(n) := |S_j(n)|.$$

In Section 6, we will prove the following bounds.

Theorem 3. *For $j \geq 0$ we have*

$$\frac{1}{(j+2)!} \leq \liminf_{n\to\infty} \frac{s_j(n)}{n} \leq \limsup_{n\to\infty} \frac{s_j(n)}{n} \leq \frac{1}{j+2}.$$

While the bounds are clearly not sharp (for example, set $j = 1$ and compare to Theorem 1), it seems that for $j \geq 2$ there is indeed a gap between the limit inferior and limit superior; see Section 7.

Before moving on, we state two simple lemmas. All results on $s(n)$ will be based on the following equivalence.

Lemma 4. *Let $r \in \{0, 1, \dots, \lfloor n/2 \rfloor - 1\}$. Then $r \in S(n)$ if and only if $n - r$ has a proper divisor $k \geq r + 1$.*

Proof. First, assume $r \in S(n)$. This means that there exist integers $k \in \{1, 2, \dots, \lfloor n/2 \rfloor\}$ and q such that

$$n = kq + r$$

and $r \leq k - 1$. In other words, k is a divisor of $n - r$ with $k \geq r + 1$. Moreover, $k \leq \lfloor n/2 \rfloor$ implies $q \geq 2$, and therefore k is indeed a proper divisor of $n - r$.

Now assume conversely that $n - r$ has a proper divisor $k \geq r + 1$. Then we have

$$n - r = kq$$

for some $q \geq 2$. This means that $r = n \bmod k$. Moreover, $q \geq 2$ implies $k = (n - r)/q \leq \lfloor n/2 \rfloor$, and so by definition $r \in S(n)$. □

Finally, let us give a slightly more precise statement than $S(n) \subseteq \{0, 1, \dots \lfloor n/2 \rfloor - 1\}$.

Lemma 5. *We have $S(n) \subseteq \{0, 1, \dots, \lfloor (n-2)/3 \rfloor\}$.*

Proof. Let $r \in S(n)$. Then by Lemma 4, the number $n - r$ has a proper divisor $k \geq r + 1$. In other words, $n - r = kq$ with $q \geq 2$. This implies $n - r \geq (r + 1) \cdot 2$, and rewriting the inequality yields $n \geq 3r + 2$. □

Note that throughout the paper, p will denote a prime. In particular, if we sum or take the union over an index p, the numbers p are implied to be primes.

3 Asymptotics for $s(n)$ (proof of Theorem 1)

In this section, we want to prove that $s(n)$ asymptotically behaves like cn and compute a bound for the error term. We do this with a straightforward sieving argument. In preparation for this, let us define the set of integers that are divisible by p but not by any smaller prime $p' < p$:

$$D_p := \{m \in \mathbb{Z} \colon p \mid m, \text{ and } p' \nmid m \text{ for every prime } p' < p\}. \tag{2}$$

As usual, let $\pi(n)$ denote the number of primes $p \leq n$.

The next lemma is standard; we provide a proof for completeness. hii

Lemma 6. *Let a, t be positive integers and p a prime. Then*

$$|D_p \cap [a+1, a+t]| = t \cdot \frac{1}{p} \cdot \prod_{p' < p} \left(1 - \frac{1}{p'}\right) + E(a, t, p),$$

and the error term $E(a, t, p)$ is bounded by

$$|E(a, t, p)| \leq 2^{\pi(p-1)}.$$

Proof. This follows from a simple inclusion–exclusion argument. Indeed, we have

$$\begin{aligned} |D_p \cap [a+1, a+t]| = &\left(\left\lfloor \frac{a+t}{p} \right\rfloor - \left\lfloor \frac{a}{p} \right\rfloor\right) - \sum_{p_1 < p} \left(\left\lfloor \frac{a+t}{pp_1} \right\rfloor - \left\lfloor \frac{a}{pp_1} \right\rfloor\right) \\ &+ \sum_{p_1 < p_2 < p} \left(\left\lfloor \frac{a+t}{pp_1p_2} \right\rfloor - \left\lfloor \frac{a}{pp_1p_2} \right\rfloor\right) - + \dots \\ &\pm \left(\left\lfloor \frac{a+t}{p \prod_{p' < p} p'} \right\rfloor - \left\lfloor \frac{a}{p \prod_{p' < p} p'} \right\rfloor\right). \end{aligned}$$

Each difference of the shape $\lfloor (a+t)/q \rfloor - \lfloor a/q \rfloor$ is equal to $t/q + \delta$, with some error term δ with $|\delta| < 1$. There are exactly $2^{\pi(p-1)}$ such differences. Therefore,

$$\begin{aligned} |D_p \cap [a+1, a+t]| = &\frac{t}{p} - \sum_{p_1 < p} \frac{t}{pp_1} + \sum_{p_1 < p_2 < p} \frac{t}{pp_1p_2} - + \dots \pm \frac{t}{p \prod_{p' < p} p'} + E(a, t, p) \\ = &\, t \cdot \frac{1}{p} \cdot \prod_{p' < p} \left(1 - \frac{1}{p'}\right) + E(a, t, p), \end{aligned}$$

with $|E(a, t, p)| < 2^{\pi(p-1)}$, as desired.

□

We will also use the next lemma, which is a classical upper bound for the prime counting function [7, Corollary 1].

Lemma 7. *For every $n \geq 2$ we have*

$$\pi(n) \leq 1.3 \cdot \frac{n}{\log n}.$$

Finally, we will use the following simple estimate.

Lemma 8. *We have*

$$\sum_{p > n} \frac{1}{p^2} = O\left(\frac{1}{n \log n}\right).$$

Proof. By the prime number theorem, the k-th p_k prime is asymptotically of the size $p_k \sim k \log k$. The lemma follows from a density argument. To be rigorous, we can do the following estimates for sufficiently large n, where C_1, C_2, C_3 are some positive constants:

$$\begin{aligned}\sum_{p>n} \frac{1}{p^2} &\le C_1 \cdot \sum_{k \ge n/\log n} \frac{1}{(k \log k)^2} \le C_2 \cdot \int_{n/\log n}^{\infty} \frac{1}{(t \log t)^2}\, dt \\ &\le C_2 \cdot \frac{1}{(\log(n/\log n))^2} \cdot \int_{n/\log n}^{\infty} \frac{1}{t^2}\, dt \\ &\le C_3 \cdot \frac{1}{(\log n)^2} \cdot \frac{1}{n/\log n} = C_3 \cdot \frac{1}{n \log n}.\end{aligned}$$

□

Now we are ready to prove the asymptotics for $s(n)$.

Proof of Theorem 1. We want to use Lemma 4 to count the remainders $r \in S(n) \setminus \{0\}$. Note that $n - r$ having a proper divisor $k \ge r + 1$ is equivalent to $n - r$ having a prime factor $p \le (n-r)/(r+1)$. Moreover, note that the last inequality is equivalent to $r \le (n-p)/(p+1)$. We want to count the remainders r systematically by going through the prime numbers p. Therefore, we define for every n and every prime $p < n$ the set

$$R_p(n) := \left\{ r \colon 1 \le r \le \frac{n-p}{p+1},\ p \mid n - r,\ \text{and } p' \nmid n - r \text{ for every prime } p' < p \right\}.$$

Then we have

$$S(n) \setminus \{0\} = \bigcup_{p<n} R_p(n).$$

Since all sets in the union are disjoint by construction, we have in particular

$$s(n) - 1 = |S(n) \setminus \{0\}| = \sum_{p<n} |R_p(n)|. \tag{3}$$

Instead of counting the numbers $r \in R_p(n)$, we can equivalently count the numbers $m = n-r$. In other words, if we define

$$M_p(n) := \left\{ m \in \left[n - \frac{n-p}{p+1}, n-1 \right] : p \mid m,\ \text{and } p' \nmid m \text{ for every prime } p' < p \right\},$$

then we have $|R_p(n)| = |M_p(n)|$.

Let $X = X(n)$ be some threshold function with $1 < X(n) < n$ that we will fix later. Our strategy is to compute

$$s(n) = 1 + \sum_{p<n} |M_p(n)| = \sum_{p<X} |M_p(n)| + \underbrace{1 + \sum_{X \le p < n} |M_p(n)|}_{=: E_1(n)}. \tag{4}$$

We do this because on the one hand, if p is small compared to n, we can compute $|M_p(n)|$ relatively precisely. On the other hand, if p is sufficiently large, $|M_p(n)|$ is small, and we can estimate the error term $E_1(n)$ trivially. Let us do the latter first. We have

$$|E_1(n)| = 1 + \sum_{X \le p < n} |M_p(n)| \le 1 + \sum_{p \ge X} \frac{n}{(p+1)p} \le 1 + n \cdot \sum_{p \ge X} \frac{1}{p^2} = O\left(\frac{n}{X \log X}\right), \tag{5}$$

where we used Lemma 8 for the last estimate.

In order to compute the main term in (4), for each $p < X$ we apply Lemma 6 with $a = n - \frac{n-p}{p+1} - 1$ and $t = \frac{n-p}{p+1}$. This gives us

$$|M_p(n)| = \frac{n-p}{p+1} \cdot \frac{1}{p} \cdot \prod_{p' < p} \left(1 - \frac{1}{p'}\right) + E_2(n, p),$$

with

$$|E_2(n, p)| \le 2^{\pi(p-1)}.$$

Now we can write

$$\begin{aligned}\sum_{p<X} |M_p(n)| &= \sum_{p<X} \frac{n-p}{p+1} \cdot \frac{1}{p} \cdot \prod_{p'<p} \left(1 - \frac{1}{p'}\right) + \sum_{p<X} E_2(n,p) \\ &= n \cdot \sum_{p<X} \frac{1}{p(p+1)} \cdot \prod_{p'<p} \left(1 - \frac{1}{p'}\right) \underbrace{- \sum_{p<X} \frac{1}{p+1} \cdot \prod_{p'<p} \left(1 - \frac{1}{p'}\right) + \sum_{p<X} E_2(n,p)}_{=:E_2(n)},\end{aligned}$$

with

$$|E_2(n)| \le X + \sum_{p<X} 2^{\pi(p-1)} \le X + 2^{\pi(X)}, \tag{6}$$

where in the error term we estimated the first sum very crudely by X, and for the second sum we used the definition of $\pi(x)$.

Finally, recall that

$$c = \sum_{p} \frac{1}{p(p+1)} \cdot \prod_{p'<p} \left(1 - \frac{1}{p'}\right).$$

Thus, we can write

$$\sum_{p<X} |M_p(n)| = cn \underbrace{- n \cdot \sum_{p \ge X} \frac{1}{p(p+1)} \cdot \prod_{p'<p} \left(1 - \frac{1}{p'}\right)}_{=:-E_3(n)} + E_2(n), \tag{7}$$

with

$$|E_3(n)| \le n \cdot \sum_{p \ge X} \frac{1}{p^2} = O\left(\frac{n}{X \log X}\right), \tag{8}$$

where we used Lemma 8 for the last estimate.

Overall, we have from (4) and (7) that

$$s(n) = cn + E_1(n) + E_2(n) + E_3(n).$$

Using the bounds for the error terms from (5), (6), and (8), and setting $X = X(n) = \log n$, we get

$$\begin{aligned} s(n) &= cn + O\left(\frac{n}{X \log X}\right) + O(X) + O(2^{\pi(X)}) \\ &= cn + O\left(\frac{n}{\log n \log\log n}\right) + O(\log n) + O(2^{\pi(\log n)}). \end{aligned}$$

The theorem follows upon noting that with Lemma 7 for sufficiently large n we get

$$2^{\pi(\log n)} \leq 2^{1.3 \log n / \log\log n} \leq n^{1/\log\log n} \leq n^{1/2} = O\left(\frac{n}{\log n \log\log n}\right).$$

□

Remark 9. At first glance, the last estimate in the proof above appears quite rough. However, increasing $X(n)$ from $\log n$ even to just $(\log n)^{1+\varepsilon}$ does not work using the same arguments. To improve the bound, one would need to find stronger error bounds.

4 Differences between consecutive terms of $s(n)$ (proof of Theorem 2)

In this section, we try to understand better how $s(n)$ changes as n increases by 1.

First, observe the following relation between elements in $S(n+1)$ and $S(n)$.

Lemma 10. *For every $n \geq 1$ we have*

$$S(n+1) \setminus \{0\} \subseteq \{r + 1 \colon r \in S(n)\}.$$

Proof. Let $r \in S(n+1) \setminus \{0\}$. Then by Lemma 4 the integer $n + 1 - r$ has a proper divisor $k \geq r + 1$. Now since $n - (r - 1) = n + 1 - r$ has a proper divisor $k \geq r + 1 \geq r$, we have $r - 1 \in S(n)$. □

From this, we immediately see that the value of $s(n)$ can increase by at most one at a time:

Proposition 11. *For every $n \geq 1$ we have*

$$s(n+1) \leq s(n) + 1.$$

Proof. We have

$$s(n+1) - 1 = |S(n+1)| - 1 \leq |S(n+1) \setminus \{0\}| \leq |S(n)| = s(n),$$

where we applied Lemma 10 for the last inequality. □

Note that $0 = n \bmod 1$ is always in $S(n+1)$ and recall Lemma 10, and its proof. We can interpret it in the following way: The set $S(n+1)$ consists precisely of the element 0 and all the elements $r+1$ that were "transferred" from $S(n)$ to $S(n+1)$ by being increased by 1. Thus, to understand the difference $s(n+1) - s(n)$, we need to understand how many elements r were not transferred. We denote the set of "not transferred elements" by

$$T(n, n+1) := \{r \colon r \in S(n) \text{ and } r+1 \notin S(n+1)\}.$$

Then we have

$$s(n+1) = s(n) + 1 - |T(n, n+1)|. \tag{9}$$

Next, we characterize the set $T(n, n+1)$.

Lemma 12. *We have $r \in T(n, n+1)$ if and only if $r+1$ is the largest proper divisor of $n - r$.*

Proof. This follows directly from Lemma 4: $r \in S(n)$ is equivalent to $n - r$ having a proper divisor $\geq r+1$, and $r+1 \notin S(n+1)$ is equivalent to $(n+1) - (r+1) = n - r$ not having a proper divisor $\geq r+2$. □

To count such occurrences, we will use the following simple lemma.

Lemma 13. *Let n, d be positive integers. Then d is the largest proper divisor of n if and only if $n/d = p$ for some prime p and every prime factor of d is $\geq p$.*

Now, for even n, the situation is rather simple:

Proposition 14. *Let $n \geq 2$ be even. Then*

$$T(n, n+1) = \begin{cases} \{(n-2)/3\}, & \text{if } n \equiv 2 \pmod 3; \\ \emptyset, & \text{otherwise.} \end{cases}$$

In particular,

$$s(n+1) = \begin{cases} s(n), & \text{if } n \equiv 2 \pmod 3; \\ s(n)+1, & \text{otherwise.} \end{cases}$$

Proof. By Lemma 12 we have $r \in T(n, n+1)$ if and only if $r+1$ is the largest proper divisor of $n - r$. By Lemma 13, this is the case if and only if $(n-r)/(r+1) = p$ for some prime p and every prime factor of $r+1$ is $\geq p$. Let us find all values of r for which this is the case. Assume first that r is odd. Then $(n-r)/(r+1) = \text{odd}/\text{even} = p$, which is impossible. Now assume that r is even, and we have $(n-r)/(r+1) = \text{even}/\text{odd} = p$. This means that $p = 2$ and $n = 3r + 2$. So this happens if and only if $n \equiv 2 \pmod 3$ and $r = (n-2)/3$.

The second part of the lemma now follows immediately from formula (9). □

For odd n, we will prove that $|T(n, n+1)|$ is at most double logarithmic in n.

Let us first use Lemmas 12 and 13 to characterize the sets $T(n, n+1)$ more precisely.

Lemma 15. *Assume that $n+1 = p_1^{x_1} \cdots p_\ell^{x_\ell}$ is the prime factorization of $n+1$ with $p_1 < \cdots < p_\ell$. Let $1 \leq r < (n-2)/3$. Then $r \in T(n, n+1)$ if and only if all of the following conditions are satisfied:*

(i) $r + 1 = p_I^{x_I} \cdots p_\ell^{x_\ell}$ for some index $2 \leq I \leq \ell$;

(ii) $(n+1)/(r+1) - 1 = p_1^{x_1} \cdots p_{I-1}^{x_{I-1}} - 1 = p$ for some prime p;

(iii) $p \leq p_I$.

Moreover, $r = 0 \in T(n, n+1)$ if and only if $n = p_1^{x_1} \cdots p_\ell^{x_\ell} - 1$ is prime. Finally, if $n \equiv 2 \pmod 3$, then $r = (n-2)/3 \in T(n, n+1)$.

Proof. By Lemma 12 we have $r \in T(n, n+1)$ if and only if $r+1$ is the largest proper divisor of $n - r$. By Lemma 13 this is equivalent to

$$\frac{n-r}{r+1} = \frac{n+1}{r+1} - 1 = p \tag{10}$$

$$\text{and every prime factor of } r+1 \text{ is at least of size } p. \tag{11}$$

Let $1 \leq r < (n-2)/3$ and assume that $n+1 = p_1^{x_1} \cdots p_\ell^{x_\ell}$. We check that (10) and (11) are equivalent to the three conditions in the statement of the lemma.

Assume first that (10) and (11) hold. Then since $r + 1 \mid n + 1$, we have

$$r + 1 = p_1^{y_1} \cdots p_\ell^{y_\ell}, \quad \text{with } 0 \leq y_i \leq x_i \text{ for } 1 \leq i \leq \ell.$$

Let I be the index such that $p_1 < \cdots < p_{I-1} < p \leq p_I < \cdots < p_\ell$ (where $I = 1$ is allowed and means that $p \leq p_1$). Then (11) implies that $y_1 = \cdots = y_{I-1} = 0$. Moreover, for every $I \leq i \leq \ell$ we can write

$$\frac{n+1}{r+1} = p_i^{x_i - y_i} \cdot A = p + 1,$$

with some integer $A \geq 1$. Then $p_I \geq p$ implies $x_i - y_i = 0$ for all $i = I, \ldots, \ell$, except if $p_i = 3$, $x_i - y_i = 1$, $A = 1$ and $p = 2$. The exceptional situation is $(n+1)/(r+1) = 3$, which is equivalent to $r = (n-2)/3$ and was excluded. In the other situations, we have $y_I = x_I, \ldots, y_\ell = x_\ell$, and the three conditions in the lemma are clearly satisfied (the condition $I \geq 2$ follows from $(n+1)/(r+1) = p + 1 > 1$).

Conversely, it is clear that the three conditions in the lemma imply (10) and (11).

The statement for $r = 0$ holds because always $0 \in S(n)$ and by Lemma 4 we have $1 \in S(n+1)$ if and only if $n = (n+1) - 1$ is composite.

Finally, assume $r = (n-2)/3$ is an integer. Then $n \bmod (n+1)/3 = r$ and so $r \in S(n)$. On the other hand, $r + 1 > (n-2)/3 = \lfloor (n+1-2)/3 \rfloor$ and so by Lemma 5 we have $r + 1 \notin S(n+1)$. ∎

Proposition 16. *For every odd $n \geq 1$ we have*

$$s(n) - s(n+1) = O(\log\log n).$$

Proof. In view of formula (9) it suffices to prove that

$$|T(n, n+1) \setminus \{0, (n-2)/3\}| = O(\log\log n).$$

In other words, we want to show that there are at most $O(\log\log n)$ numbers r satisfying the three conditions in Lemma 15. Fix an odd $n \geq 1$ and let

$$n + 1 = p_1^{x_1} \cdots p_\ell^{x_\ell}$$

be the prime factorization of $n+1$ with $p_1 < \cdots < p_\ell$. Then by Lemma 15, $r+1$ must be of the shape $r + 1 = p_I^{x_I} \cdots p_\ell^{x_\ell}$ for some index $2 \leq I \leq \ell$. Moreover, for these indices I, we have

$$p_I \geq p = p_1^{x_1} \cdots p_{I-1}^{x_{I-1}} - 1 \geq p_2 \cdots p_{I-1}.$$

Assume that there are t such numbers r, corresponding to the indices $I_1 < \cdots < I_t$. Then we can weaken the above inequality to

$$p_{I_j} \geq p_{I_2} p_{I_3} \cdots p_{I_{j-1}}.$$

Using this inequality inductively, we get

$$\begin{aligned} n = p_1^{x_1} \cdots p_\ell^{x_\ell} - 1 &\geq p_{I_2} p_{I_3} \cdots p_{I_t} \\ &\geq (p_{I_2} p_{I_3} \cdots p_{I_{t-1}})^2 \geq (p_{I_2} p_{I_3} \cdots p_{I_{t-2}})^4 \geq \cdots \geq p_{I_2}^{2^{t-2}} \geq 3^{2^{t-2}}. \end{aligned}$$

This implies

$$t - 2 \leq \log_2 \log_3(n),$$

and so $t = O(\log\log n)$, as desired. □

Proposition 17. *We have*

$$\liminf_{n\to\infty} s(n+1) - s(n) = -\infty.$$

Proof. Let us set

$$t(n) := |T(n, n+1)|.$$

In view of formula (9), our strategy is to use Lemma 15 to recursively construct numbers $n^{(1)}, n^{(2)}, \ldots$ with the properties

- $n^{(j)}$ is prime and
- $t(n^{(j)}) \geq j$.

For good intuition, let us set the first three values straight away:

$$n^{(1)} := 3 = 2^2 - 1, \quad n^{(2)} := 11 = 2^2 \cdot 3 - 1, \quad n^{(3)} := 131 = 2^2 \cdot 3 \cdot 11 - 1.$$

This works because of Lemma 15 and the following facts: $2^2 - 1$ is prime and $2^2 - 1 \geq 3$, and $2^2 \cdot 3 - 1$ is prime and $2^2 \cdot 3 - 1 \geq 11$, and also $2^2 \cdot 3 \cdot 11 - 1$ is prime.

Now assume we have already constructed $n^{(j)}$ for some $j \geq 3$. Set

$$P := \prod_{p \leq n^{(j)}} p.$$

For reasons that will become apparent in (12), we want to choose an integer y with the following properties:

(a) $\gcd(xP + y,\, P) = 1$ for all integers x. This is equivalent to $y \not\equiv 0 \pmod{p}$ for all $p \mid P$.

(b) $\gcd(y(n^{(j)} + 1) - 1,\, P) = 1$. This is equivalent to $y(n^{(j)} + 1) \not\equiv 1 \pmod{p}$ for all $p \mid P$ with $p \nmid n^{(j)} + 1$.

Overall, we want y to satisfy

$$y \not\equiv 0, (n^{(j)} + 1)^{-1} \pmod{p} \qquad \text{for all } p \mid P.$$

For each $p \geq 3$, this is clearly possible, since at most two residue classes need to be avoided. For the case $p = 2$, note that since $n^{(j)}$ is a prime, $n^{(j)} + 1$ is even, so $p \mid n^{(j)} + 1$ and we only need to avoid $y \equiv 0 \pmod{2}$.

Therefore, by the Chinese remainder theorem, there exists an integer $1 \leq y \leq P$ with properties (a) and (b).

We fix such an integer y and consider the arithmetic progression

$$a_x := (n^{(j)} + 1)(xP + y) - 1 = xP(n^{(j)} + 1) + y(n^{(j)} + 1) - 1, \quad x \geq 1. \tag{12}$$

Now our property (b) implies that

$$\gcd(P(n^{(j)} + 1),\ y(n^{(j)} + 1) - 1) = 1.$$

Thus, by Dirichlet's prime number theorem, there exists an integer $x \geq 1$ such that a_x is a prime. We set

$$n^{(j+1)} := (n^{(j)} + 1)(xP + y) - 1 = a_x.$$

By construction, $n^{(j+1)}$ is prime, and we finally only need to check that indeed $t(n^{(j+1)}) \geq t(n^{(j)}) + 1 \geq j + 1$.

It is easy to see that by construction $n^{(j+1)} \equiv 2 \pmod{3}$ for all $j \geq 2$ and $n^{(j+1)}$ is prime, so the special cases from Lemma 15, namely $r = 0$ and $r = (n^{(j+1)} - 2)/3$, are always in $T(n^{(j+1)}, n^{(j+1)} + 1)$.

Moreover, let us write $n^{(j)} = p_1^{x_1} \cdots p_\ell^{x_\ell} - 1$ with $p_1 < \cdots < p_\ell$. Then we have $n^{(j+1)} = p_1^{x_1} \cdots p_\ell^{x_\ell} \cdot Q - 1$, where all prime factors in $Q = xP + y$ are larger than p_ℓ by property (a) and the definition of P. Thus, if r satisfied the three conditions in Lemma 15 for $n^{(j)}$, then $r' = (r - 1) \cdot Q + 1$ satisfies the three conditions in Lemma 15 for $n^{(j+1)}$. The increase comes from the fact that now $r' = Q - 1$ satisfies the three conditions as well. □

Proof of Theorem 2. Combine Propositions 11, 14, 16, and 17. □

5 Iterated remainder sets and their relation to the "n mod a problem"

Recall that we defined the sets of iterated remainders of n inductively by

$$S_0(n) := \{1, 2, \ldots, \lfloor n/2 \rfloor\} \quad \text{and}$$
$$S_j(n) := \{n \bmod k \colon k \in S_{j-1}(n) \setminus \{0\}\} \quad \text{for } j \geq 1.$$

These sets are related to an older open problem about the length of the Pierce expansion. Shallit [9] first studied the problem, which we state as follows:

Fix a positive integer n and choose another integer $1 \leq a \leq n$. Set $a_0 := a$ and $a_{j+1} := n \bmod a_j$ for $j \geq 0$ and as long as $a_j > 0$. For example, for $(n, a) = (35, 22)$, we get $a_0 = 22, a_1 = 13, a_2 = 9, a_3 = 8, a_4 = 3, a_5 = 2, a_6 = 1, a_7 = 0$. Now let us define $P(n, a)$ to be the integer t such that $a_t = 0$. So, for example, $P(35, 22) = 7$. Finally, let us set

$$P(n) := \max_{1 \leq a \leq n} P(n, a).$$

The problem is to obtain upper and lower bounds for $P(n)$ in terms of n. In particular, experiments suggest that the upper bound should be sublinear; however, this seems to be hard to prove. The best known bounds are due to Chase and Pandey [1], who slightly improved the bounds by Erdős and Shallit [2]: We have

$$\frac{\log n}{\log \log n} \ll P(n) \ll n^{\frac{1}{3} - \frac{2}{177} + \varepsilon} \tag{13}$$

for sufficiently large n.

This problem is directly related to our sets $S_j(n)$ via the following two simple lemmas.

Lemma 18. *Let $j \geq 1$ and $n \geq 1$. Then $r \in S_j(n)$ if and only if there exists an integer $\lfloor n/2 \rfloor + 1 \leq a \leq n$ such that in the above notation $a_{j+1} = r$.*

Proof. This follows directly from the definitions. The index shift comes from the fact that we are assuming $\lfloor n/2 \rfloor + 1 \leq a \leq n$ and so a_1 can be exactly every element from $S_0(n)$. □

Lemma 19. *The following statements are equivalent:*

1. $P(n) = t$;
2. $S_{t+1}(n) = \{0\}$;
3. $|S_{t+1}(n)| = 1$ *and* $|S_{t+1+j}(n)| = 0$ *for all* $j \geq 1$.

Proof. When computing $P(n) = \max_{1 \le a \le n} P(n,a)$, we may restrict ourselves to $\lfloor n/2 \rfloor + 1 \le a \le n$, since if $a < \lfloor n/2 + 1 \rfloor$, the starting value $n - a$ gives $P(n, n-a) = P(n,a) + 1$.

Now the equivalences follow from Lemma 18 and the fact that $S_j(n) = \emptyset$ if and only if $S_{j-1} = \emptyset$ or $S_{j-1} = \{0\}$. □

Remark 20. The known bounds (13) imply that for sufficiently large n, we have $|S_j(n)| = 0$ for all $j \ge n^{1/3}$. On the other hand, there exists a constant c such that for all sufficiently large n we have $|S_j(n)| \ge 1$ for $j \le c \log n / \log\log n$.

6 Bounds for iterated remainders

Recall that we defined

$$s_j(n) := |S_j(n)|.$$

In this section, we prove upper and lower bounds for $s_j(n)$.

We start with a simple upper bound.

Lemma 21. *For all $j \ge 0$ and $n \ge 1$ we have*

$$s_j(n) - 1 \le \max S_j(n) \le \frac{n}{j+2}.$$

Proof. The first inequality is clear since $\min S_j(n) = 0$. We show the second inequality by induction. For $j = 0$ this is clearly satisfied by definition. Now assume that $\max S_j(n) \le n/(j+2)$ for some $j \ge 0$. Then for every $r \in S_{j+1}(n)$ there exists a $k \in S_j(n)$ with $k > r$ such that $n = qk + r$. By induction hypothesis we have $k \le n/(j+2)$; hence $q \ge j+2$. Now

$$r + 1 \le k = \frac{n-r}{q}$$

implies

$$r \le \frac{n-q}{q+1} \le \frac{n}{j+3},$$

as desired. □

Remark 22. The arguments from [2] for the upper bound on $P(n)$ in fact give stronger upper bounds when j is relatively large compared to n. For example, since $\max S_j(n)$ strictly decreases as j increases, one can show that $\max S_j(n)$ is roughly bounded by $2\sqrt{n} - j$ for $j > \sqrt{n}$. One can do even better (see [2, proof of Theorem 2]), using the fact that the number of divisors of m is $O(m^\varepsilon)$. It turns out that for $j > n^{1/3+\varepsilon}$ we get roughly the upper bound $2n^{2/3+\varepsilon} - j \cdot n^{1/3}$.

Finally, we prove a lower bound. In particular, we show that the sequence $s_j(n)$ grows linearly (even if $\lim_{n\to\infty} s_j(n)/n$ might not exist).

Lemma 23. *For every $j \geq 0$ and $n \geq N(j)$ there exists an integer $x_j = x_j(n)$ such that*

$$S_j(n) \supseteq \{r \colon j \leq r \leq \frac{n-j-1}{j+2},\ r \equiv x_j \pmod{(j+1)!}\}. \tag{14}$$

In particular,

$$\liminf_{n\to\infty} \frac{s_j(n)}{n} \geq \frac{1}{(j+2)!}. \tag{15}$$

Proof. Fix some $J \geq 0$ and let $n \geq N(J)$. We want to prove (14) for $j = 0, 1, \dots, J$ with finite induction. For $j = 0$ the inclusion (14) is clearly true with $x_0 = 0$, since $S_0(n) = \{0, 1, \dots, \lfloor n/2 \rfloor\}$. Now assume that (14) holds for some $0 \leq j \leq J-1$. Our goal is to show

$$S_{j+1}(n) \supseteq \{r \colon j+1 \leq r \leq \frac{n-j-2}{j+3},\ r \equiv n-(j+2)x_j \pmod{(j+2)!}\};$$

i.e., we set $x_{j+1} = n - (j+2)x_j$. Assume r is in the set on the right hand side. Then the condition on the residue class implies that we can write $r = n - (j+2)x_j - q(j+2)!$ with some integer q. This implies

$$n - r = (j+2)(x_j + q(j+1)!) = (j+2)k,$$

where we set $k := x_j + q(j+1)!$. Thus, to prove $r \in S_{j+1}(n)$ it suffices to show that $k \in S_j(n)$ and $k \geq r+1$.

Since $n - r = (j+2)k$, the condition $k \geq r+1$ is equivalent to $n - r \geq (j+2)(r+1)$. This is equivalent to $r \leq (n-j-2)/(j+3)$, which is satisfied by assumption.

Clearly, $k \equiv x_j \pmod{(j+1)!}$, so in order to show $k \in S_j(n)$ we only need to check $j \leq k \leq (n-j-1)/(j+2)$. First, $k \leq (n-j-1)/(j+2)$ is equivalent to $n - r \leq (j+2)(n-j-1)/(j+2) = n-j-1$, so this indeed holds for $r \geq j+1$. Finally, $k \geq j$ is equivalent to $n - r \geq (j+2)j$, which holds since $r \leq (n-j-2)/(j+3) \leq n-(j+2)j$ for sufficiently large n.

The bound (15) follows immediately from (14), since for large n the interval length is $(n-j-1)/(j+2) - j + 1 \sim n/(j+2)$ and every $(j+1)!$-th number is included. □

Proof of Theorem 3. Combine Lemmas 21 and 23. □

7 Numerical experiments and open problems

Recall that in Theorem 1 we proved

$$s(n) = cn + O\left(\frac{n}{\log n \log\log n}\right).$$

The bound for the error term seems very large. We have computed the values for $s(n)$ for $n \leq 10^7$ and determined the points $(n, s(n) - cn)$ where $s(n) - cn$ reaches a new maximum

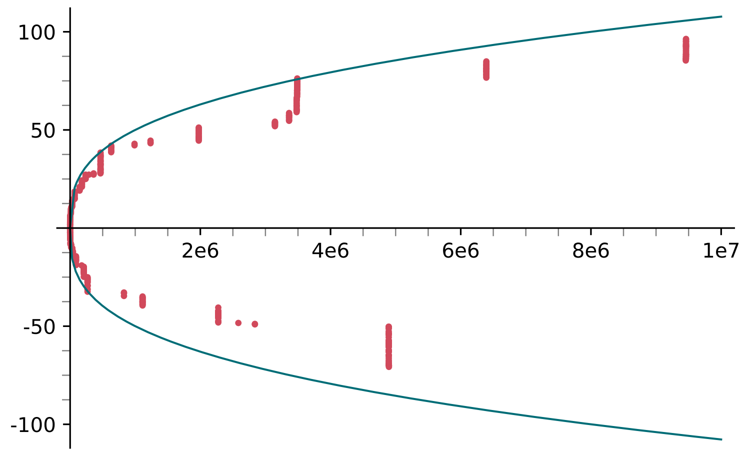


Figure 1: The function $2n^{1/3}$ and the largest deviations of $s(n)$ from cn.

or minimum. These points are plotted in Figure 1, together with the graph of $2n^{1/3}$. This suggests that perhaps the correct bound for the error term might be $O(n^{1/3})$. In any case, we propose the following problem.

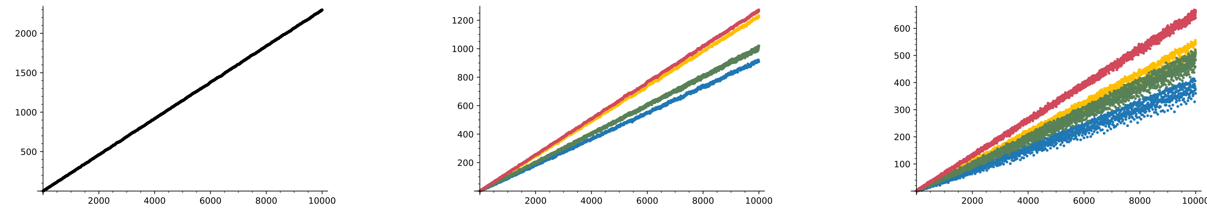
Problem 1. Improve the bound for the error term for $s(n)$ in Theorem 1.

For the iterated remainder sets, recall that Theorem 3 says that

$$\frac{1}{(j+2)!} \leq \liminf_{n\to\infty} \frac{s_j(n)}{n} \leq \limsup_{n\to\infty} \frac{s_j(n)}{n} \leq \frac{1}{j+2}.$$

In particular, there is a gap between our lower and our upper bound. Indeed, numerical experiments strongly suggest that $\lim_{n\to\infty} s_j(n)/n$ does not exist for $j \geq 2$. The three plots in Figure 2 show the values of $s_1(n), s_2(n), s_3(n)$, respectively, for $n \leq 10^4$. For $s_2(n)$ and $s_3(n)$ we see some "bands" of values. The blue points correspond to $n \equiv 0 \pmod 6$, the green points correspond to $n \equiv 2, 4 \pmod 6$, the yellow points to $n \equiv 3 \pmod 6$, and the red points to $n \equiv 1, 5 \pmod 6$. It seems that really the precise divisibility properties of n determine the value of $s_2(n), s_3(n)$. To support this further, in Figure 3 we have plotted the values of $s_2(n)$ only for $n \equiv 1 \pmod 6$ in the range $[6 \cdot 10^4, 6 \cdot 10^4 + 10^3]$. In particular, we only consider numbers n not divisible by 2 and 3. Indeed, the numbers n that are divisible by 5 (points colored red) yield the smallest relative values. In any case, the easiest problem in this context might be Problem 2.

Problem 2. Prove that for $j \geq 2$ the limit $\lim_{n\to\infty} s_j(n)/n$ does not exist.



(a) $s_1(n)$ (b) $s_2(n)$ (c) $s_3(n)$

Figure 2: Plots of $s_j(n)$ for $n = 1, 2, 3$; colors according to divisibility by 2 and 3.

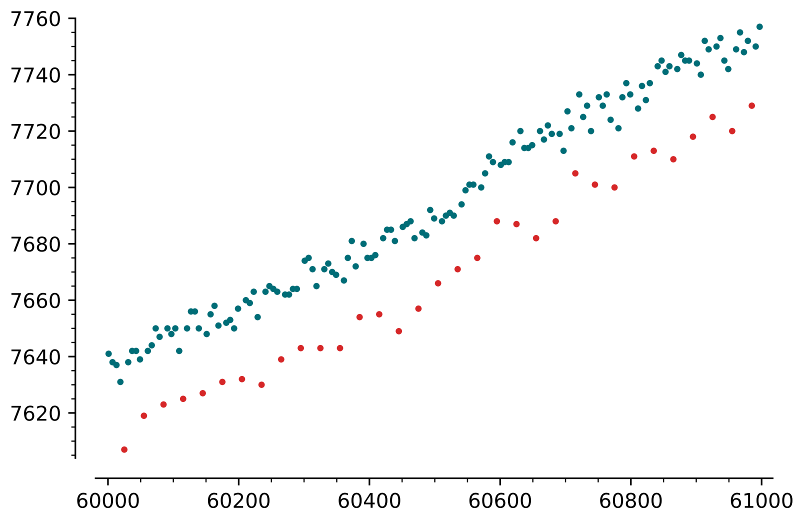


Figure 3: $s_2(n)$ for $n \equiv 1 \pmod{6}$; the points where n is divisible by 5 are colored red.

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