Journal of Integer Sequences, Vol. 29 (2026),
Article 26.2.6

Rook Decomposition of the Partition Function

N. Guru Sharan
Department of Mathematics
Indian Institute of Technology Gandhinagar
Palaj, Gandhinagar, Gujarat 382355
India
sharanguru5@gmail.com

Abstract

The rook numbers are fairly well-studied in the literature. In this paper, we study the max-rook number of the Ferrers boards associated with integer partitions. We show its connections with the Durfee triangle of the partitions. The max-rook number gives a new decomposition of the partition function. We derive the generating functions of the partitions with the Durfee triangle of sizes 3, 4, and 5. We obtain their exact formula and further use it to show the periodicity modulo p for $p \in \mathbb{N}$ and $p \geq 2$. We also establish their parity and parity bias. We give the growth asymptotics of partitions with the Durfee triangle of sizes 3 and 4. We obtain a new rook analogue of the recurrence relation of the partition function.

1 Definitions and notation

A *partition* of a non-negative integer n is a non-increasing sequence of positive integers $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_r$ such that $\sum_{i=0}^{r} \lambda_i = n$. The individual λ_i are called the *parts* of the partition. For a fixed $n \in \mathbb{N}$, let $\mathscr{P}(n)$ be the set of all unrestricted partitions of n and let $P(n)$ be the cardinality of $\mathscr{P}(n)$. For example, $(5, 3, 2, 1)$ is a partition of 11, enumerated by $P(11)$. For convention, we take the empty partition to be the only partition of zero, i.e., $P(0) = 1$.

For each partition $\pi \in \mathscr{P}(n)$, as defined in [1, Section 1.3], we associate a graphical representation called *Ferrers diagram*, defined by the set of nodes with integer co-ordinates (i, j) such that if $\pi = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_r)$, then the node (i, j) is placed if and only if $-r+1 \leq$

$i \leq 0$ and $0 \leq j \leq \lambda_{|i|+1} - 1$. One must note that the set of nodes begin with the node $(0, 0)$ at the top left. For example, the Ferrers diagram of $(5, 3, 2, 1)$ is given in Figure 1a.

The largest square of nodes contained in the Ferrers diagram of a partition starting from the top-left node of the Ferrers diagram is called the *Durfee square* and its length is called the *size* of the Durfee square.

Similarly, let us call the largest right-angled isosceles triangle contained in the Ferrers diagram with the right-angle at the top-left node of the partition, the *Durfee triangle*. The number of nodes on the sides of triangle is called the *size* of the Durfee triangle. One such example of a Durfee square (size 2) and a Durfee triangle (size 4), of the partition $(5, 3, 2, 1)$ is given in Figure 1a. Durfee triangle of size n contains $\sum_{\ell=1}^{n} \ell = \frac{1}{2}n(n+1)$ many nodes, which is called the n^{th} *triangular number*, which we denote by t_n. The first few triangular numbers are $1, 3, 6, 10, 15, \ldots$

Formally, a *board* B is any finite sub-lattice of $\mathbb{Z}\times\mathbb{Z}$, containing squares of unit dimension. In the context of this study, we define a *Ferrers board*[1] associated with a partition π to be the board that is obtained by replacing each node in the Ferrers diagram of π by a unit square. From the defintion of Ferrers Diagram, it is clear that Ferrers Board is a finite connected sub-lattice of $\mathbb{Z} \times \mathbb{Z}$. See Figure 1b for clarity.

A *rook* is a chess piece that is only allowed to move horizontally or vertically on a board. Two rooks are said to be *non-intersecting* if they are not placed in a same row or a same column, which makes the rooks unable to capture each other in the game of chess. In the classical theory of rook numbers, the primary interest is the number of distinct placements of non-intersecting rooks on a board. *The rook number* $R(r, B)$ is the distinct number of ways of placing r many non-intersecting rooks in the Ferrers board B. For a fixed board B, we define the *rook polynomial* $f_{\mathrm{B}}(x)$ to be the generating function of the rook numbers $R(r, B)$, i.e.,

$$f_{\mathrm{B}}(x) := \sum_{n=1}^{\infty} R(n, B)x^n.$$

For a fixed board B, the quantity $R(n, B)$ vanishes for all large n, depending on the number of squares in the board, thus making $f_{\mathrm{B}}(x)$ a polynomial. This term was first coined by Kaplansky and Riordan [5]. The rook polynomial is well-studied. As shown before [9, Chapter 7], for a rectangular $m \times n$ board B, it is known that

$$f_{\mathrm{B}}(x) = n!x^n L_n^{(m-n)}(-x^{-1}), \quad (m \geq n)$$

where $L_n^{\alpha}(x)$, the *generalized Laguerre polynomial*, is

$$L_n^{\alpha}(x) := \frac{1}{n!}x^{-\alpha}e^x \frac{d^n}{dx^n}\left(x^{n+\alpha}e^{-x}\right),$$

as defined in [12, Equation (6.23)]. The rook polynomial can be seen as a specific case of the *matching polynomial* studied in Graph theory, which is the generating function of the k-edge

[1]The "Ferrers board" is usually referred to as a "Young diagram" in partition theory. The primary difference in this context is that the boxes are used for the placement of non-intersecting rooks.

matchings in a given graph. For a fixed board B, it has been shown that the rook numbers $R(n, B)$ are unimodal, i.e., they initially increase to a maximal value and then decrease. This has been proved as an implication of the fact that all the zeroes of the matching polynomial are real [4, Section 4]. The interested reader is referred to [3, Chapter 2] for finer details.

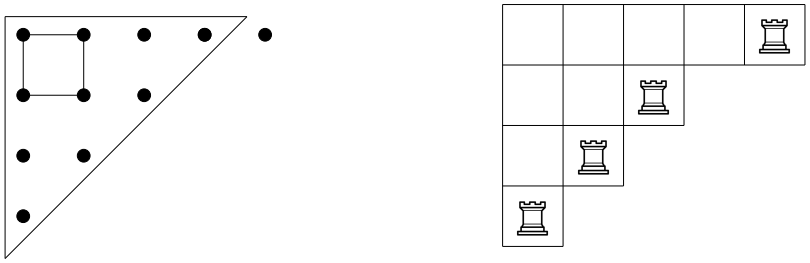
2 Introduction

We call it a *decomposition of the partition function* when $P(n)$ can be written in terms of a finite sum of other number-theoretic functions. Various decompositions of the partition function have been studied in the literature. For instance, recently, in [6] and [7], a decomposition of the partition function in terms of the Euler totient function and the Möbius function, respectively, has been derived.

In this paper, we study the highest possible rook number for a given Ferrers board (which is also the degree of the rook polynomial) and use it to get a new decomposition of the partition function in (3), which we call the *rook decomposition*. For a fixed partition $\pi \in \mathscr{P}(n)$, if B_π be the Ferrers board associated with π, let us define the *max-rook number* $R(\pi)$ as follows:

$$R(\pi) := \text{ The maximal number of non-intersecting rooks that can be placed in } B_\pi.$$

Clearly, the maximal number is unique and less than or equal to n, and hence $R(\pi)$ is well defined. One example is illustrated in Figure 1b.



(a) Ferrers diagram (b) Ferrers board

Figure 1: The partition $(5, 3, 2, 1)$ of 11 has a max-rook number 4.

Then, for $n \in \mathbb{N}$, let us define $F(n)$ as follows:

$$F(n) := \max \{R(\pi) \mid \pi \in \mathscr{P}(n)\}. \tag{1}$$

Clearly $R(\pi) \leq n$ for a fixed $n \in \mathbb{N}$ and $\pi \in \mathscr{P}(n)$. Hence the maximal element in the above set exists, and $F(n)$ is well-defined. For convention, let us define $F(0) := 1$.

Next, for fixed $n, k \in \mathbb{N}$, we define $R_k(n)$, which are the central objects of this study, by

$$R_k(n) := \#\{\pi \in \mathscr{P}(n) \mid R(\pi) = k\}. \tag{2}$$

Thus $R_k(n)$ is the number of partitions of n whose associated Ferrers board has a max-rook number k. Let us denote the generating function of $R_k(n)$ by $\mathscr{F}_k(q)$, i.e.,

$$\mathscr{F}_k(q) := \sum_{n=1}^{\infty} R_k(n) q^n.$$

As explained in Theorem 1, the number of partitions with a max-rook number k is equal to number of the partitions with a Durfee triangle of size k and hence $\mathscr{F}_k(q)$ is also the generating function of the number of partitions with a Durfee triangle of size k. Since each partition has a unique Durfee triangle, we get the rook decomposition of the partition function as follows: namely,

$$P(n) = \sum_{k=1}^{n} R_k(n), \tag{3}$$

for $n \in \mathbb{N}$, where $R_k(n)$ is as defined in (2). This decomposition is completely combinatorial in nature. Also note that, $R_k(n) = 0$ for all $n < t_k$, where t_n is the n^{th} triangular number. From this perspective, the family of functions $R_k(n)$ is the building block of the partition function $P(n)$. Hence, the properties of $R_k(n)$ can give an alternative way to see the properties of the partition function $P(n)$. The decomposition of the partitions up to $P(50)$ has been computed using a PARI algorithm and tabulated in rows and columns by Howroyd in A325188 [11]. The decomposition for $P(n)$ for $1 \leq n \leq 17$ can be seen in Table 1 below as the column sum.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$P(n)$	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297
$R_1(n)$	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
$R_2(n)$	–	–	1	3	5	8	9	12	13	16	17	20	21	24	25	28	29
$R_3(n)$	–	–	–	–	–	1	4	8	15	23	32	43	54	67	82	97	114
$R_4(n)$	–	–	–	–	–	–	–	–	–	1	5	12	24	42	66	98	135
$R_5(n)$	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	6	17

Table 1: Rook decomposition of the partition function $P(n)$.

The rows associated with $k = 1$ and $k = 2$ in Table 1 represented by $R_1(n)$ and $R_2(n)$, respectively, have been studied. The rows associated with $k \geq 3$ are yet to be studied and the generating functions of the same are not known. In this paper, we prove the generating functions of the rows $k = 3$, 4, and 5, their exact formulas, periodicity, congruences, and asymptotics.

It can be seen trivially that

$$R_1(n) = \begin{cases} 1, & \text{if } n = 1; \\ 2, & \text{if } n \geq 2, \end{cases} \tag{4}$$

since for $n \geq 2$, only the two partitions (n) and $(1, 1, \cdots, 1)$ of n take the value $R(\pi) = 1$. Barker A325168 [11] gave the following generating function of $R_2(n)$,

$$\mathscr{F}_2(q) = \sum_{n=1}^{\infty} R_2(n) q^n = \frac{q^3(1 + 2q + q^2 + q^3 - q^4)}{(1-q)^2(1+q)}.$$

$R_2(n)$ satisfies the recurrence relation A325168 [11] for $n > 7$ given by,

$$R_2(n) = R_2(n-1) + R_2(n-2) - R_2(n-3). \tag{5}$$

In Corollary 3, we prove that $R_3(n)$ satisfies the following analogous recursive relation for $n > 14$:

$$R_3(n) = 2R_3(n-1) - R_3(n-2) + R_3(n-3) - 2R_3(n-4) + R_3(n-5). \tag{6}$$

Further, as observed by Barker A325168 [11], we also know that, for $n > 4$,

$$R_2(n) = \begin{cases} 2n-4, & \text{when } n \text{ is even;} \\ 2n-5, & \text{when } n \text{ is odd.} \end{cases} \tag{7}$$

The explicit evaluation of $R_2(n)$ in (7) gives the parity for $n > 4$: it is even for even arguments and odd for odd arguments. Also, from (4) and (7), we can see that $R_1(n)$ is bounded and $R_2(n)$ grows linearly, respectively, as $n \to \infty$.

In Theorems 6 and 8, we find similar explicit formulas for $R_3(n)$ and $R_4(n)$, respectively, which look surprisingly simple. In principle, one should note that it is possible to use the same method to find explicit formula $R_k(n)$ for every $k \in \mathbb{N}$, but that it becomes cumbersome as k increases. In [2, Theorem 1.1], Bruinier and Ono obtain a closed-form expression for $P(n)$ as a finite sum of some algebraic numbers. Apart from this, there seems to be no known closed-form analytic expression of $P(n)$. The closed-form expressions for $R_k(n)$ can give an analytic closed-form expression for $P(n)$. Also, as a direct implication of Theorems 6 and 8, we obtain the asymptotic growth of $R_3(n)$ and $R_4(n)$ in Corollaries 7 and 9, respectively.

Ramanujan discovered and proved the following three congruence identities that are satisfied by the partition function, for $\ell \in \mathbb{N}$:

$$\begin{aligned} P(5\ell + 4) &\equiv 0 \pmod 5, \\ P(7\ell + 5) &\equiv 0 \pmod 7, \\ P(11\ell + 6) &\equiv 0 \pmod{11}. \end{aligned} \tag{8}$$

The above congruences (8) give us hints concerning the periodic property of $P(n)$ modulo 5, 7 and 11, respectively. We show that $R_k(n)$ for $k = 3$ and 4 exhibit such a periodic property modulo p, for $p \in \mathbb{N}$, $p \geq 2$. Ramanujan's congruences of $P(n)$ could possibly be seen as implications of the periodicity properties of the general $R_k(n) \pmod p$. In [8], Ono studied the distribution of $P(n) \pmod m$ for primes $m \geq 5$, which suggests that $R_k(n) \pmod p$ for non-primes p deserves an independent study.

Let $(a;q)_n$ denote the q-Pochhammer symbol defined by

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \text{ for } n \geq 1,$$

and similarly $(a;q)_\infty = (1-a)(1-aq)\cdots$. For convention, let us have $(a;q)_0 = 1$.

The generating function of the number of partitions with a Durfee square of size k, denoted by $\mathscr{G}_k(q)$, is well studied [1, p. 28] and is given by

$$\mathscr{G}_k(q) = \frac{q^{k^2}}{(q;q)_k^2}.$$

Unlike $\mathscr{G}_k(q)$, the generating functions $\mathscr{F}_k(q)$ are largely unexplored. As previously mentioned, we explained in Theorem 1 that the number of partitions with a max-rook number k is equal to the number of partitions with a Durfee triangle of size k. We exploit this equivalence to get the generating functions of $R_3(n), R_4(n)$ and $R_5(n)$, namely, $\mathscr{F}_3(q)$, $\mathscr{F}_4(q)$ and $\mathscr{F}_5(q)$ in Theorems 2, 4 and 5, respectively.

As a direct implication of Euler's pentagonal number theorem, we have the recurrence relation for the partition function $P(n)$ [1, Corollary 1.8] as follows,

$$P(n) = \sum_{k\in\mathbb{Z},\ k\neq 0} (-1)^{k-1} P(n-g_k), \tag{9}$$

where $g_k = \frac{1}{2}k(3k-1)$ for $k \in \mathbb{Z}\setminus\{0\}$ are called the *generalized pentagonal numbers*. The right-hand side of (9) is effectively a finite sum because $P(n-g_k) = 0$ for $k \notin (-\frac{1}{6}(\sqrt{24n+1}-1), \frac{1}{6}(\sqrt{24n+1}-1))$. Thus (9) is a finite recursive formula for the partition function $P(n)$, but with the number of non-zero terms on the right-hand side depending on n, the number being partitioned. In contrast with (9), the recursive relation for $R_2(n)$ (5) has a fixed number of terms (3 terms) on the right-hand side, which is independent of n. In Corollary 3, we obtain the recursive relation for $R_3(n)$ and further show that it, too, has a fixed number of terms (5 terms) on the right-hand side. These relations (5) and (6) can be seen as rook analogues of (9).

We explicitly prove the parity and parity bias of $R_3(n)$, $R_4(n)$ and $R_5(n)$ in Section 4.1. $R_k(n)$ also satisfies many congruences, apart from modulo 2.

3 Generating functions

The following is the generating function of $F(n)$ defined in (1).

Theorem 1. *For $|q| < 1$ we have*

$$\sum_{n=0}^{\infty} F(n)q^n = \frac{1}{1-q}\left((q^2;q^2)_\infty(-q;q)_\infty - q\right).$$

Proof. Fix an arbitrary $n \in \mathbb{N}$. There exists a unique $m \in \mathbb{N}$ such that $t_m \leq n < t_{m+1}$, where t_m is the m^{th} triangular number. Then there exists a partition of n that has a max-rook number m, for instance, $\left(n - \frac{m(m-1)}{2}, m-1, m-2, \ldots, 2, 1\right)$. Also observe that, for $n < t_{m+1}$, no partition of n can have a max-rook number of $m+1$, since the Durfee triangle of size $m+1$ is necessary for a partition to have a max-rook number $m+1$, which is not possible. Hence we see that

$$F(n) = m \quad \text{for } n \text{ such that } t_m \leq n < t_{m+1}.$$

Thus the generating function of $F(n)$ is

$$\sum_{n=1}^{\infty} F(n)q^n = \sum_{m=1}^{\infty}\sum_{r=0}^{m} mq^{\frac{m(m+1)}{2}+r} = \sum_{m=1}^{\infty} mq^{\frac{m(m+1)}{2}} \sum_{r=0}^{m} q^r = \sum_{m=1}^{\infty} mq^{\frac{m(m+1)}{2}} \times \frac{1-q^{m+1}}{1-q}, \quad (10)$$

where we use the sum of a geometric progression in the last step. Now we multiply both the sides of (10) by $1-q$ to get

$$(1-q)\sum_{n=1}^{\infty} F(n)q^n = \sum_{m=1}^{\infty} mq^{\frac{m(m+1)}{2}}(1-q^{m+1}). \quad (11)$$

We split the terms on both sides into separate series as follows:

$$\sum_{n=1}^{\infty} F(n)q^n - \sum_{n=1}^{\infty} F(n)q^{n+1} = \sum_{m=1}^{\infty} mq^{\frac{m(m+1)}{2}} - \sum_{m=1}^{\infty} mq^{\frac{(m+1)(m+2)}{2}}.$$

Replacing n by $n-1$ and m by $m-1$ in the second series of the left-hand side and right-hand side, respectively, we get

$$\sum_{n=1}^{\infty} F(n)q^n - \sum_{n=2}^{\infty} F(n-1)q^n = \sum_{m=1}^{\infty} mq^{\frac{m(m+1)}{2}} - \sum_{m=2}^{\infty} (m-1)q^{\frac{m(m+1)}{2}}.$$

Now combine the series to get

$$F(1)q + \sum_{n=2}^{\infty}(F(n) - F(n-1))q^n = q + \sum_{m=2}^{\infty} q^{\frac{m(m+1)}{2}} = q + \sum_{m=2}^{\infty} q^{t_m}, \quad (12)$$

where t_m is the m^{th} triangular number. Hence $F(1) = 1$, and for $n \geq 2$, we have

$$F(n) - F(n-1) = \begin{cases} 1, & \text{if } n = t_m \text{ for some } m \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

From (11) and (12), we have

$$\sum_{n=1}^{\infty} F(n)q^n = \frac{1}{1-q}\sum_{m=1}^{\infty} q^{\frac{m(m+1)}{2}}.$$

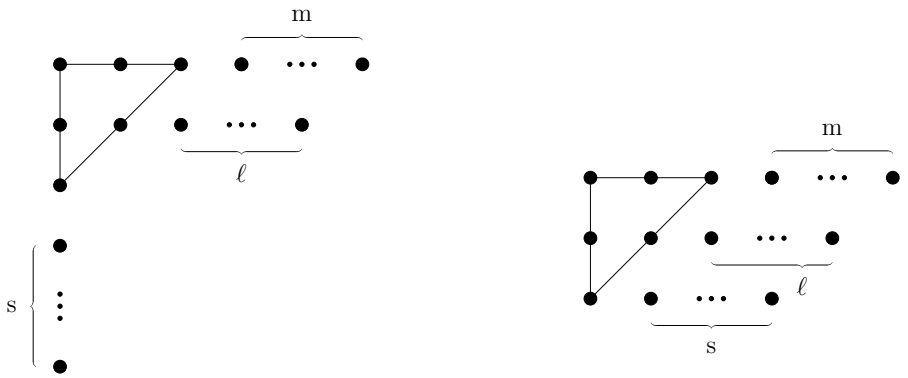
Use [1, Corollary 2.10] and add $F(0) = 1$ to both sides and simplify to get the required result. □

Now we give a combinatorial proof of the generating function of $R_3(n)$ as a Theorem.

Theorem 2. *For $|q| < 1$ we have*

$$\mathscr{F}_3(q) = \sum_{n=1}^{\infty} R_3(n) q^n = \frac{q^6(1 + 2q + q^2 + 2q^3 - q^4 - q^6 - q^7 + q^8)}{(1-q)^3(1+q+q^2)}. \tag{13}$$

Proof. The partitions of a number with the Durfee triangle of size 3 are either of the type A, B, or their conjugates A^c or B^c, as shown in Figure 2. We have the conditions $m, s \geq 0$ and $\ell \leq m+1$ for type A, and $s \leq \ell + 1$ and $\ell \leq m+1$, along with $s, \ell \geq 1$ for type B. We take $s, \ell \geq 1$ since any partition of type B or B^c with $\ell = 0$ or $s = 0$ is either a partition of type A or A^c. We find the generating functions of partitions of type A, B, A^c and B^c separately and add them. We then subtract off the partitions that are repeated to get the required generating function $\mathscr{F}_3(q)$.



(a) Type A (b) Type B

Figure 2: The two types of partitions with Durfee triangle of size 3.

The generating function of partitions of type A is

$$\begin{aligned}
&q^6 \sum_{m=0}^{\infty} q^m \sum_{\ell=0}^{m+1} q^\ell \sum_{s=0}^{\infty} q^s \\
&= \frac{q^6}{1-q} \sum_{m=0}^{\infty} \frac{q^m(1-q^{m+2})}{1-q} \\
&= \frac{q^6}{(1-q)^2} \sum_{m=0}^{\infty} q^m - \frac{q^8}{(1-q)^2} \sum_{m=0}^{\infty} q^{2m} \\
&= \frac{q^6}{(1-q)^3} - \frac{q^8}{(1-q)^2(1-q^2)}.
\end{aligned} \tag{14}$$

The generating function of partitions of type B is

$$
\begin{aligned}
&q^6 \sum_{m=0}^{\infty} q^m \sum_{\ell=1}^{m+1} q^\ell \sum_{s=1}^{\ell+1} q^s \\
&= q^7 \sum_{m=0}^{\infty} q^m \sum_{\ell=1}^{m+1} \frac{q^\ell(1-q^{\ell+1})}{1-q} \\
&= \frac{q^7}{1-q} \sum_{m=0}^{\infty} q^m \sum_{\ell=1}^{m+1} q^\ell - \frac{q^8}{1-q} \sum_{m=0}^{\infty} q^m \sum_{\ell=1}^{m+1} q^{2\ell} \\
&= \frac{q^8}{1-q} \sum_{m=0}^{\infty} \frac{q^m(1-q^{m+1})}{1-q} - \frac{q^{10}}{1-q} \sum_{m=0}^{\infty} \frac{q^m(1-q^{2m+2})}{1-q^2} \\
&= \frac{q^8}{(1-q)^2} \sum_{m=0}^{\infty} q^m - \frac{q^9}{(1-q)^2} \sum_{m=0}^{\infty} q^{2m} - \frac{q^{10}}{(1-q)(1-q^2)} \sum_{m=0}^{\infty} q^m + \frac{q^{12}}{(1-q)(1-q^2)} \sum_{m=0}^{\infty} q^{3m} \\
&= \frac{q^8}{(1-q)^3} - \frac{q^9}{(1-q)^2(1-q^2)} - \frac{q^{10}}{(1-q)^2(1-q^2)} + \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)}. \qquad (15)
\end{aligned}
$$

The generating functions of type A^c and B^c are the same as A and B, respectively. Hence we multiply (14) and (15) each by 2. The partitions that are both of type A and A^c—i.e., the ones that are counted twice—are exactly when $\ell = 0$, which we subtract. The only partitions that are of type B and B^c are $(3,3,2)$ and $(3,3,3)$. Hence we subtract q^8 and q^9 to get the generating function $\mathscr{F}_3(q)$. Thus

$$
\begin{aligned}
\mathscr{F}_3(q) = 2\left(\frac{q^6}{(1-q)^3} - \frac{q^8}{(1-q)^2(1-q^2)}\right) + 2\left(\frac{q^8}{(1-q)^3} - \frac{q^9}{(1-q)^2(1-q^2)}\right. \\
\left. - \frac{q^{10}}{(1-q)^2(1-q^2)} + \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)}\right) - q^6 \sum_{m=1}^{\infty} q^m \sum_{s=1}^{\infty} q^s - q^8 - q^9.
\end{aligned}
$$

Simplify to get (13). □

We use the generating function of $R_3(n)$ to obtain its recurrence relation analogous to that of $R_2(n)$ as stated in (5).

Corollary 3. *For $n > 14$, the following recurrence relation holds:*

$$R_3(n) = 2R_3(n-1) - R_3(n-2) + R_3(n-3) - 2R_3(n-4) + R_3(n-5). \qquad (16)$$

Proof. From Theorem 2, we have

$$\sum_{n=1}^{\infty} R_3(n) q^n = \frac{q^6(1+2q+q^2+2q^3-q^4-q^6-q^7+q^8)}{1-2q+q^2-q^3+2q^4-q^5}.$$

We cross-multiply to get

$$\left(\sum_{n=1}^{\infty} R_3(n)q^n\right)\left(1-2q+q^2-q^3+2q^4-q^5\right) = q^6(1+2q+q^2+2q^3-q^4-q^6-q^7+q^8).$$

Hence we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} R_3(n)q^n - \sum_{n=1}^{\infty} 2R_3(n)q^{n+1} + \sum_{n=1}^{\infty} R_3(n)q^{n+2} - \sum_{n=1}^{\infty} R_3(n)q^{n+3} + \sum_{n=1}^{\infty} 2R_3(n)q^{n+4} \\
&- \sum_{n=1}^{\infty} R_3(n)q^{n+5} \\
&= q^6(1+2q+q^2+2q^3-q^4-q^6-q^7+q^8).
\end{aligned}$$

Use the fact that $R_3(n) = 0$ for $1 \le n \le 5$ to get

$$\begin{aligned}
&\sum_{n=6}^{\infty} R_3(n)q^n - \sum_{n=5}^{\infty} 2R_3(n)q^{n+1} + \sum_{n=4}^{\infty} R_3(n)q^{n+2} - \sum_{n=3}^{\infty} R_3(n)q^{n+3} + \sum_{n=2}^{\infty} 2R_3(n)q^{n+4} \\
&- \sum_{n=1}^{\infty} R_3(n)q^{n+5} \\
&= q^6(1+2q+q^2+2q^3-q^4-q^6-q^7+q^8).
\end{aligned}$$

Making a suitable change of variables in each of the series on the left-hand side, we get

$$\begin{aligned}
&\sum_{n=6}^{\infty}(R_3(n) - 2R_3(n-1) + R_3(n-2) - R_3(n-3) + 2R_3(n-4) - R_3(n-5))q^n \\
&= q^6(1+2q+q^2+2q^3-q^4-q^6-q^7+q^8).
\end{aligned}$$

Hence, by comparing the coefficients of q^n on both sides of the equation, we get, for $n > 14$, that

$$R_3(n) - 2R_3(n-1) + R_3(n-2) - R_3(n-3) + 2R_3(n-4) - R_3(n-5) = 0.$$

Rearrange the terms to get the required recurrence relation (16). □

Now we give a combinatorial proof of the generating function of $R_4(n)$ as a Theorem.

Theorem 4. *For $|q| < 1$ we have*

$$\begin{aligned}
\mathscr{F}_4(q) &= \sum_{n=1}^{\infty} R_4(n)q^n \\
&= \frac{q^{10}(1+4q+6q^2+7q^3+6q^4+2q^5-5q^7-5q^8-5q^9+q^{11}+3q^{12}+2q^{13}-q^{16})}{(1-q)(1-q^2)(1-q^3)(1-q^4)}.
\end{aligned} \tag{17}$$

Proof. The partitions of a number with the Durfee triangle of size 4 are at least of one of the type A, B, C, or their conjugates A^c, B^c or C^c, as shown in Figure 3. For type A, we take $s, m \geq 0$, $r \leq \ell + 1$ and $\ell \leq m + 1$. For type B, we take $s \geq 0$, $m \geq 0$, $r \geq 1$, $\ell \geq 1$, $r \leq s+1$ and $\ell \leq m+1$, since $r = 0$ or $\ell = 0$ is already covered by type A and A^c. For type C, we take $m \geq 0$, $s \geq 1$, $r \geq 1$, $\ell \geq 1$, $s \leq r + 1$, $r \leq \ell + 1$ and $\ell \leq m + 1$, since $s = 0$, $r = 0$ and $\ell = 0$ is covered by type A, type B and type A^c, respectively. We now find the generating functions of the types separately.

The generating function of partitions of type A is

$$\begin{aligned} &q^{10} \sum_{s=0}^{\infty} q^s \sum_{m=0}^{\infty} q^m \sum_{\ell=0}^{m+1} q^\ell \sum_{r=0}^{\ell+1} q^r \\ &= \frac{q^{10}(1 + 2q - q^3 - 2q^4 + q^6)}{(1-q)^4(1+q)(1+q+q^2)}. \end{aligned} \tag{18}$$

The generating function of partitions of type B is

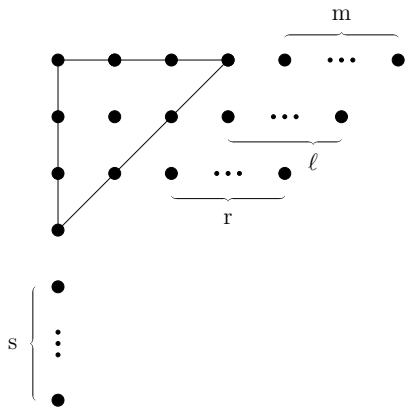
$$\begin{aligned} &q^{10} \sum_{s=0}^{\infty} q^s \sum_{r=1}^{s+1} q^r \sum_{m=0}^{\infty} q^m \sum_{\ell=1}^{m+1} q^\ell \\ &= q^{10} \left(\sum_{s=0}^{\infty} q^s \sum_{r=1}^{s+1} q^r \right)^2 \\ &= \frac{q^{12}}{(1-q)^4(1+q)^2}. \end{aligned} \tag{19}$$

The generating function of partitions of type C is

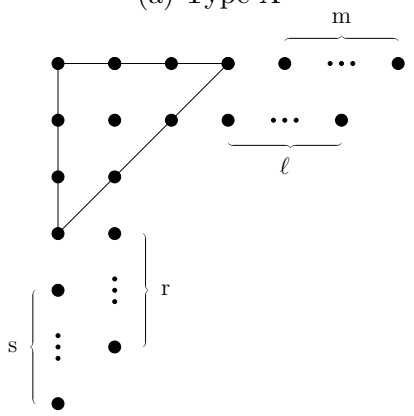
$$\begin{aligned} &q^{10} \sum_{m=0}^{\infty} q^m \sum_{\ell=1}^{m+1} q^\ell \sum_{r=1}^{\ell+1} q^r \sum_{s=1}^{r+1} q^s \\ &= \frac{q^{13}(1 + 2q + q^2 - 2q^4 - 2q^5 - q^6 + q^7 + q^8)}{(1-q)^4(1+q)^2(1+q^2)(1+q+q^2)}. \end{aligned} \tag{20}$$

Simplifications similar to the ones done in Theorem 2 are done to get (18), (19) and (20). The generating functions of type A^c and C^c are the same as that of type A and C, so we multiply (18) and (20) each by 2. Since B^c is the same type as B, we ignore B^c. The partitions that are of type both A and A^c are exactly when $\ell = 0$ in (18), and hence we subtract them. The only partitions that are of type C and C^c are $(4, 4, 3, 2)$, $(4, 4, 4, 2)$, $(4, 4, 3, 3)$, $(4, 4, 4, 3)$ and $(4, 4, 4, 4)$. Hence we subtract off q^{13}, $2q^{14}$, q^{15} and q^{16} to get the required generating function. Thus we have

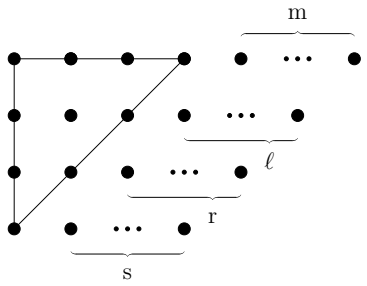
$$\mathscr{F}_4(q) = 2\left(\frac{q^{10}(1 + 2q - q^3 - 2q^4 + q^6)}{(1-q)^4(1+q)(1+q+q^2)}\right) - \frac{q^{10}(1+q)}{(1-q)^2} + \left(\frac{q^{12}}{(1-q)^4(1+q)^2}\right)$$



(a) Type A



(b) Type B



(c) Type C

Figure 3: The three types of partitions with Durfee triangle of size 4.

$$+ 2\left(\frac{q^{13}(1+2q+q^2-2q^4-2q^5-q^6+q^7+q^8)}{(1-q)^4(1+q)^2(1+q^2)(1+q+q^2)}\right) - q^{13} - 2q^{14} - q^{15} - q^{16}.$$

Simplify to get (17). □

Now we give a combinatorial proof of the generating function of $R_5(n)$, with some details curbed to reduce repetition.

Theorem 5. *For $|q| < 1$ we have*

$$\begin{aligned}\mathscr{F}_5(q) &= \sum_{n=1}^{\infty} R_5(n) q^n \\ &= \frac{q^{15}}{(1-q)^2(1-q^3)(1-q^4)(1-q^5)}\big(1 + 4q + 6q^2 + 8q^3 + 8q^4 + 4q^5 + 4q^6 - 5q^7 - 5q^8 \\ &\quad - 10q^9 - 7q^{10} - 5q^{11} + 4q^{13} + 6q^{14} + 7q^{15} + 2q^{17} - 4q^{18} - q^{19} - q^{20} - q^{21} + q^{22} - q^{23} + q^{24}\big).\end{aligned} \tag{21}$$

Proof. The partitions of a number with the Durfee triangle of size 4 are at least of one of the type A, B, C, or their conjugates A^c, B^c or C^c, as shown in Figure 4. For type A, we take $m \geq 0$, $\ell \leq m+1$, $r \leq \ell+1$, $s \leq r+1$ and $p \leq s+1$. For type B, we take $p \geq 1$, $m \geq 1$, $\ell \leq m+1$, $r \leq \ell+1$ and $s \leq r+1$, since $p = 0$ or $m = 0$ gives a partition of type A or A^c, respectively. For type C, we take $m \geq 1$, $1 \leq \ell \leq m+1$, $r \leq \ell+1$, $p \geq 1$ and $1 \leq s \leq p+1$, since $\ell = 0$, $p = 0$, $m = 0$, and $s = 0$ gives a partition of type B^c, A, A^c, and B, respectively. We now find the generating functions of the types separately.

The generating function of partitions of type A is

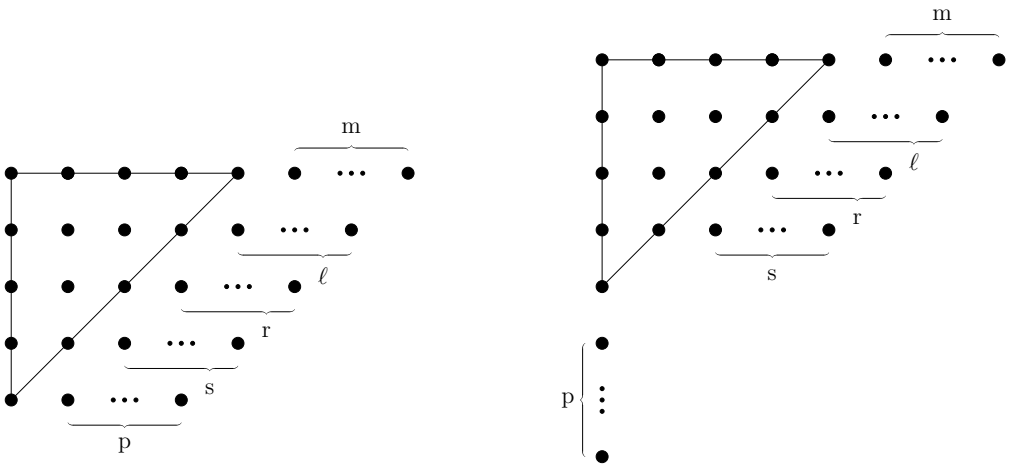
$$\begin{aligned}&q^{15} \sum_{m=0}^{\infty} q^m \sum_{\ell=0}^{m+1} q^{\ell} \sum_{r=0}^{\ell+1} q^r \sum_{s=0}^{r+1} q^s \sum_{p=0}^{s+1} q^p \\ &= \frac{q^{15}}{(1-q)^3(1+q)^2(1+q^2)(1-q^3)(1-q^5)}\big(1 + 4q + 5q^2 + 3q^3 - q^4 - 8q^5 - 9q^6 \\ &\quad - 8q^7 - 2q^8 + 5q^9 + 9q^{10} + 7q^{11} + 3q^{12} - q^{13} - 4q^{14} - 2q^{15} - 2q^{16} + q^{20}\big).\end{aligned} \tag{22}$$

The generating function of partitions of type B is

$$\begin{aligned}&q^{15} \sum_{m=1}^{\infty} q^m \sum_{\ell=0}^{m+1} q^{\ell} \sum_{r=0}^{\ell+1} q^r \sum_{s=0}^{r+1} q^s \sum_{p=1}^{\infty} q^p \\ &= \frac{q^{16}(q + 3q^2 + 3q^3 + q^4 - 3q^5 - 5q^6 - 4q^7 - q^8 + 2q^9 + 3q^{10} + q^{11} + q^{12} - q^{16})}{(1-q)^4(1+q)^2(1+q^2)(1-q^3)}.\end{aligned} \tag{23}$$

The generating function of partitions of type C is

$$\begin{aligned}&q^{15} \sum_{m=1}^{\infty} q^m \sum_{\ell=1}^{m+1} q^{\ell} \sum_{r=0}^{\ell+1} q^r \sum_{p=1}^{\infty} q^p \sum_{s=1}^{p+1} q^s \\ &= \frac{q^{17}(q^2 + 3q^3 + 2q^4 - 2q^5 - 4q^6 - 2q^7 + q^8 + 2q^9 + q^{10} - q^{11})}{(1-q)^4(1+q)^2(1-q^3)}.\end{aligned} \tag{24}$$



(a) Type A

(b) Type B

(c) Type C

Figure 4: The three types of partitions with Durfee triangle of size 5.

Simplifications similar to the ones done in Theorem 2 are done to get (22), (23) and (24). The partitions that are of type both A and A^c are exactly when $m = 0$ in type A. The partitions that are of type both B and B^c are exactly when $\ell = 0$ in type B. The partitions that are of type both C and C^c are exactly when $r = 0$ in type C. There are no possible repetitions apart from these three cases due to the definitions of the three types.

The generating function of partitions of type both A and A^c is

$$q^{15}\left(1+4q+6q^2+7q^3+7q^4+5q^5+5q^6+3q^7+2q^8+q^9+q^{10}\right). \tag{25}$$

The generating function of partitions of type both B and B^c is $\dfrac{q^{17}(1+q-2q^2-q^3+q^6)}{(1-q)^4(1+q)}$. (26)

The generating function of partitions of type both C and C^c is $\dfrac{q^{19}(1+q-q^2)^2}{(1-q)^4(1+q)^2}$. (27)

The generating functions of type A^c, B^c and C^c are the same as that of type A, B and C, respectively, so we multiply (22), (23) and (24) each by 2. The generating functions of the partitions that are repeated are (25), (26) and (27). Hence we subtract them off to get the required generating function (21). □

4 Exact formulas and periodicity

In this Section, we first derive the exact formulas for $R_3(n)$ and $R_4(n)$, and then further obtain their growth as $n \to \infty$ as a consequence. Due to the absence of infinite products in the generating functions, their exact formulas can be obtained, which can be used directly to obtain their growth rates without using the classical circle method. Both $R_3(n)$ and $R_4(n)$ have a polynomial growth rate unlike the partition function which grows sub-exponentially. We prove the polynomial growth rates of $R_3(n)$ and $R_4(n)$ in Corollaries 7 and 9, respectively. The exact formula also gives the periodicity modulo p for $p \in \mathbb{N}$ and $p \geq 2$.

Let us define the basic notation before proving the Theorems. Let $f(x)$ and $g(x)$ be two real-valued functions. We say $f(x) \sim g(x)$ as $x \to \infty$ if

$$\lim_{x\to\infty} \left(\frac{f(x)}{g(x)}\right) = 1.$$

Let $f_1(q)$ and $f_2(q)$ be the generating functions of $a_1(n)$ and $a_2(n)$, respectively. Then, we say f_1 and f_2 are congruent modulo p (i.e., $f_1(q) \equiv f_2(q) \pmod p$) if and only if $a_1(n) \equiv a_2(n) \pmod p$ for all n. We use this notation in this paper from now on. We also use the exact formulas to obtain the periodic congruences modulo p for $p \in \mathbb{N}$ and $p \geq 2$.

Theorem 6. *Suppose $n > 9$. If $n = 3m + a$ for $0 \leq a \leq 2$, then*

$$R_3(n) = \begin{cases} 6m^2 - 15m + 7, & \text{if } a = 0; \\ 6m^2 - 11m + 2, & \text{if } a = 1; \\ 6m^2 - 7m - 1, & \text{if } a = 2. \end{cases} \tag{28}$$

Further, for $n > 9$ and fixed $p \in \mathbb{N}$ and $p \geq 2$, we have that $R_3(n) \pmod p$ is periodic in n with period $3p$, i.e.,

$$R_3(n + 3p) \equiv R_3(n) \pmod p.$$

Proof. From (13), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} R_3(n)q^n \\
&= \frac{q^6(1+2q+q^2+2q^3-q^4-q^6-q^7+q^8)}{(1-q)^3(1+q+q^2)} \\
&= -7-2q+q^2+2q^3+3q^4+2q^5-q^8-q^9+\frac{4}{3(1-q)^3}-\frac{7}{(1-q)^2}+\frac{110}{9(1-q)} \\
&\quad +\frac{2(2+q)(1-q)}{9(1-q^3)}.
\end{aligned} \tag{29}$$

From the binomial theorem, for every $r \in \mathbb{N}$ and $|x|<1$ we have

$$(1-x)^{-r} = \sum_{k=0}^{\infty} \frac{(r)_k}{k!} x^k, \tag{30}$$

where $(r)_k$ is the Pochhammer symbol defined by $(r)_k := (r)(r+1)\cdots(r+k-1) = \frac{\Gamma(r+k)}{\Gamma(r)}$. Using (30) repeatedly with appropriate arguments, we get

$$\begin{aligned}
&\frac{4}{3(1-q)^3}-\frac{7}{(1-q)^2}+\frac{110}{9(1-q)}+\frac{2(2+q)(1-q)}{9(1-q^3)} \\
&= \frac{2}{3}\sum_{k=0}^{\infty}(k+2)(k+1)q^k - 7\sum_{k=0}^{\infty}(k+1)q^k + \frac{110}{9}\sum_{k=0}^{\infty}q^k - \frac{2}{9}(q^2+q-2)\sum_{k=0}^{\infty}q^{3k} \\
&= \sum_{k=0}^{\infty}\left(\frac{2}{3}(k+2)(k+1)-7(k+1)+\frac{110}{9}\right)q^k - \frac{2}{9}(q^2+q-2)\sum_{k=0}^{\infty}q^{3k} \\
&= \sum_{k=0}^{\infty}\left(\frac{2}{3}k^2-5k+\frac{59}{9}\right)q^k - \frac{2}{9}(q^2+q-2)\sum_{k=0}^{\infty}q^{3k} \\
&= \sum_{m=0}^{\infty}\left(\frac{2}{3}(3m)^2-5(3m)+\frac{59}{9}\right)q^{3m} + \sum_{m=0}^{\infty}\left(\frac{2}{3}(3m+1)^2-5(3m+1)+\frac{59}{9}\right)q^{3m+1} \\
&\quad + \sum_{m=0}^{\infty}\left(\frac{2}{3}(3m+2)^2-5(3m+2)+\frac{59}{9}\right)q^{3m+2} - \frac{2}{9}(q^2+q-2)\sum_{k=0}^{\infty}q^{3k} \\
&= \sum_{m=0}^{\infty}\left(6m^2-15m+\frac{59}{9}\right)q^{3m} + \sum_{m=0}^{\infty}\left(6m^2-11m+\frac{20}{9}\right)q^{3m+1} \\
&\quad + \sum_{m=0}^{\infty}\left(6m^2-7m-\frac{7}{9}\right)q^{3m+2} - \frac{2}{9}(q^2+q-2)\sum_{k=0}^{\infty}q^{3k} \\
&= \sum_{m=0}^{\infty}\left(6m^2-15m+7\right)q^{3m} + \sum_{m=0}^{\infty}\left(6m^2-11m+2\right)q^{3m+1} + \sum_{m=0}^{\infty}\left(6m^2-7m-1\right)q^{3m+2}.
\end{aligned} \tag{31}$$

From (29) and (31), we get the required exact formula for $R_3(n)$ for $n > 9$. For any polynomial with integer coefficients, say $g(x) := a_0 + a_1x + \cdots a_mx^m$, we can see that

$$\begin{aligned} g(x+p) &= a_0 + a_1(x+p) + \cdots a_m(x+p)^m \\ &\equiv (a_0 + a_1x + \cdots a_mx^m) \pmod{\text{p}} \\ &\equiv g(x) \pmod{\text{p}}, \end{aligned}$$

since by the binomial theorem, we know that $(x+p)^k \equiv x^k \pmod{p}$ for $p \in \mathbb{N}$ such that $p \geq 2$ and $k \in \mathbb{N}$. From (29) and (31), for $m > 4$, we have seen that $R_3(3m) = 6m^2 - 15m + 7$ is a polynomial, and hence $R_3(3(m+p)) \equiv R_3(3m) \pmod{\text{p}}$ for $p \in \mathbb{N}$, $p \geq 2$. The same argument can be used for the other congruence classes to show that $R_3(n) \pmod{\text{p}}$ is a periodic function in n with a period $3p$. □

Corollary 7. *As $n \to \infty$, we have*

$$R_3(n) \sim \frac{2}{3}n^2.$$

Proof. From Theorem 6, we have

$$R_3(3m) = 6m^2 - 15m + 7.$$

Replace m by $\frac{n}{3}$ in the above equation to get the required result for the case $n = 3m$. Then, repeat the above step for the other congruence classes modulo 3 in Theorem 6 accordingly. Hence $R_3(n)$ grows like $\frac{2}{3}n^2$ asymptotically, since all the congruence classes have the identical asymptotic growth. □

Next we prove the analogous exact formula for $R_4(n)$.

Theorem 8. *Suppose $n > 16$. If $n = 12m + a$ where $0 \leq a \leq 11$, then*

$$R_4(n) = \begin{cases} 192m^3 - 264m^2 + 87m - 3, & \text{if } a = 0; \\ 192m^3 - 216m^2 + 44m + 5, & \text{if } a = 1; \\ 192m^3 - 168m^2 + 15m + 5, & \text{if } a = 2; \\ 192m^3 - 120m^2 - 12m + 7, & \text{if } a = 3; \\ 192m^3 - 72m^2 - 25m + 4, & \text{if } a = 4; \\ 192m^3 - 24m^2 - 36m + 3, & \text{if } a = 5; \\ 192m^3 + 24m^2 - 33m - 2, & \text{if } a = 6; \\ 192m^3 + 72m^2 - 28m - 3, & \text{if } a = 7; \\ 192m^3 + 120m^2 - 9m - 5, & \text{if } a = 8; \\ 192m^3 + 168m^2 + 12m - 5, & \text{if } a = 9; \\ 192m^3 + 216m^2 + 47m - 3, & \text{if } a = 10; \\ 192m^3 + 264m^2 + 84m + 3, & \text{if } a = 11. \end{cases} \tag{32}$$

Further, for $n > 16$ and fixed $p \in \mathbb{N}$ and $p \geq 2$, the function $R_4(n) \pmod{\mathrm{p}}$ is periodic in n with period $12p$, i.e.,

$$R_4(n + 12p) \equiv R_4(n) \pmod{\mathrm{p}}.$$

Proof. From (17), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} R_4(n)q^n \\
&= \frac{q^{10}(1 + 4q + 6q^2 + 7q^3 + 6q^4 + 2q^5 - 5q^7 - 5q^8 - 5q^9 + q^{11} + 3q^{12} + 2q^{13} - q^{16})}{(1-q)^4(1+q)^2(1+q^2)(1+q+q^2)} \\
&= 3 - 5q - 5q^2 - 7q^3 - 4q^4 - 3q^5 + 2q^6 + 3q^7 + 5q^8 + 5q^9 + 4q^{10} + 2q^{11} - q^{13} - 2q^{14} \\
&\quad - q^{15} - q^{16} - \frac{193}{18(1-q)} + \frac{965}{72(1-q)^2} - \frac{5}{(1-q)^3} + \frac{2}{3(1-q)^4} - \frac{3}{2(1+q)} + \frac{1}{8(1+q)^2} \\
&\quad + \frac{1}{4(1+q^2)} - \frac{2(1-q^2)}{9(1-q^3)}.
\end{aligned}$$

Using (30) repeatedly with appropriate arguments, we get

$$\begin{aligned}
&- \frac{193}{18(1-q)} + \frac{965}{72(1-q)^2} - \frac{5}{(1-q)^3} + \frac{2}{3(1-q)^4} - \frac{3}{2(1+q)} + \frac{1}{8(1+q)^2} \\
&\quad + \frac{1}{4(1+q^2)} - \frac{2(1-q^2)}{9(1-q^3)} \\
&= -\frac{193}{18}\sum_{k=0}^{\infty} q^k + \frac{965}{72}\sum_{k=0}^{\infty}(k+1)q^k - \frac{5}{2}\sum_{k=0}^{\infty}(k+2)(k+1)q^k + \frac{1}{9}\sum_{k=0}^{\infty}(k+3)(k+2)(k+1)q^k \\
&\quad - \frac{3}{2}\sum_{k=0}^{\infty}(-1)^k q^k + \frac{1}{8}\sum_{k=0}^{\infty}(-1)^k(k+1)q^k + \frac{1}{4}\sum_{k=0}^{\infty}(-1)^k q^{2k} - \frac{2}{9}(1-q^2)\sum_{k=0}^{\infty} q^{3k} \\
&= \sum_{k=0}^{\infty}\left(-\frac{193}{18} + \frac{965}{72}(k+1) - \frac{5}{2}(k+2)(k+1) + \frac{1}{9}(k+3)(k+2)(k+1) - \frac{3}{2}(-1)^k\right. \\
&\quad \left. + \frac{1}{8}(-1)^k(k+1)\right) q^k + \frac{1}{4}\sum_{k=0}^{\infty}(-1)^k q^{2k} - \frac{2}{9}(1-q^2)\sum_{k=0}^{\infty} q^{3k} \\
&= \sum_{k=0}^{\infty}\frac{1}{72}\left(8k^3 - 132k^2 - (-1)^k 99 + 9(57 + (-1)^k)k - 119\right) q^k + \frac{1}{4}\sum_{k=0}^{\infty}(-1)^k q^{2k} \\
&\quad - \frac{2}{9}(1-q^2)\sum_{k=0}^{\infty} q^{3k}.
\end{aligned}$$

Similarly, as done in Theorem 6, split the sums on the right-hand side of the above equation modulo 12 and simplify to get the required result (32). Further, from (32), for $m \geq 2$, we have seen that $R_4(12m) = 192m^3 - 264m^2 + 87m - 3$ is a polynomial and hence $R_4(12(m +$

$p)) \equiv R_4(12m) \pmod{p}$. This same argument can be used for the other cases to show that $R_4(n) \pmod{p}$ is a periodic function in n with a period $12p$. □

Corollary 9. *As $n \to \infty$, we have*

$$R_4(n) \sim \frac{1}{9}n^3.$$

Proof. From Theorem 8, we have

$$R_4(12m) = 192m^3 - 264m^2 + 87m - 3.$$

Replace m by $\frac{n}{12}$ in the above equation to get the required result for the case $n = 12m$. Then, repeat the above step for the other congruence classes modulo 12 in Theorem 8 accordingly. Hence, $R_4(n)$ grows like $\frac{1}{9}n^3$ asymptotically, since, all the congruence classes have the identical asymptotic growth. □

4.1 Parity and Congruences

In this Section, we study the parity of $R_k(n)$ for $k = 3$, 4, and 5 using their generating functions. The same results can be obtained using their exact formulas too, which could be hard to handle due to their nature. We obtain the parity properties of $R_5(n)$ without discovering its exact formula.

Theorem 10. *For $n > 9$ we have*

$$R_3(n) \equiv \begin{cases} 0 \pmod 2, & \text{if } n \text{ is odd;} \\ 1 \pmod 2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. From (13), we know

$$\begin{aligned} \sum_{n=1}^{\infty} R_3(n)q^n &\equiv \frac{q^6(1+q^2+q^4+q^6+q^7+q^8)}{(1-q)^2(1-q^3)} \pmod 2 \\ &\equiv \frac{q^6(1-q^6)}{(1-q)^2(1-q^3)} + \frac{q^6(q^2-q^8)}{(1-q)^2(1-q^3)} + \frac{q^6(q^4-q^7)}{(1-q)^2(1-q^3)} \pmod 2 \\ &\equiv \frac{q^6(1+q^3)}{(1-q)^2} + \frac{q^8(1+q^3)}{(1-q)^2} + \frac{q^{10}}{(1-q)^2} \pmod 2 \\ &\equiv \frac{q^6+q^9+q^8+q^{10}+q^{11}}{1-q^2} \pmod 2, \end{aligned}$$

since $\frac{1}{(1-q)^2} \equiv \frac{1}{(1-q^2)} \pmod 2$. For an even n, q^6, q^8 and q^{10} contribute one each to the parity of $R_3(n)$ to make it odd. For an odd n, q^9 and q^{11} contribute one each to the parity of $R_3(n)$ to make it even. □

Theorem 11. *For $n > 16$ we have*

$$R_4(n) \equiv \begin{cases} 0 \pmod 2, & \text{if } n \equiv 4, 6 \pmod 8; \\ 1 \pmod 2, & \text{if } n \equiv 0, 1, 2, 3, 5, 7 \pmod 8. \end{cases}$$

Proof. From (17), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} R_4(n) q^n \\
&\equiv \frac{q^{10}+q^{13}+q^{17}+q^{18}+q^{19}+q^{21}+q^{22}+q^{26}}{(1-q)^4(1+q)^2(1+q^2)(1+q+q^2)} \pmod 2 \\
&\equiv \frac{q^{10}+q^{13}+q^{17}+q^{18}+q^{19}+q^{21}+q^{22}+q^{26}}{(1-q)^3(1+q)^2(1+q^2)(1-q^3)} \pmod 2 \\
&\equiv \frac{q^{10}(1-q^3)+q^{17}(1-q^3)+q^{18}(1-q^3)+q^{19}(1-q^3)+q^{20}(1-q^3)+q^{23}(1-q^3)}{(1-q)^3(1+q)^2(1+q^2)(1-q^3)} \pmod 2 \\
&\equiv \frac{q^{10}+q^{17}+q^{18}+q^{19}+q^{20}+q^{23}}{(1-q)^2(1+q)(1-q^4)} \pmod 2.
\end{aligned}$$

Use the fact $\frac{1}{1-q^m} \equiv \frac{1}{1+q^m} \pmod 2$ for any $m \in \mathbb{N}$ and then multiply both the numerator and the denominator of the right-hand side by $1+q$ in the next step to get

$$\begin{aligned}
\sum_{n=1}^{\infty} R_4(n) q^n &\equiv \frac{q^{10}+q^{17}+q^{18}+q^{19}+q^{20}+q^{23}}{(1-q)^2(1+q)(1+q^4)} \pmod 2 \\
&\equiv \frac{(q^{10}+q^{17}+q^{18}+q^{19}+q^{20}+q^{23})(1+q)}{(1-q)(1+q)(1+q^2)(1+q^4)} \pmod 2 \\
&\equiv \frac{q^{10}+q^{11}+q^{17}+q^{21}+q^{23}+q^{24}}{1-q^8} \pmod 2.
\end{aligned}$$

For $n \equiv 0, 1, 2, 3, 5, 7 \pmod 8$ and $n > 16$, the terms q^{24}, q^{17}, q^{10}, q^{11}, q^{21}, and q^{23} contribute one each to the parity of $R_4(n)$, respectively, to make it odd. For $n \equiv 4, 6 \pmod 8$ and $n > 16$, $R_4(n)$ are even since they do not appear on the right-hand side of the last equation. □

Remark 12. For $n > 16$ and *odd*, $R_4(n)$ is always odd. This is evident from the observation in Theorem (11), that $R_4(n)$ is odd for all the four cases $n \equiv 1, 3, 5, 7 \pmod 8$.

Theorem 13. *For $n > 25$ we have*

$$R_5(n) \equiv \begin{cases} 0 \pmod 2, & \text{if } n \equiv 5, 7 \pmod 8; \\ 1 \pmod 2, & \text{if } n \equiv 0, 1, 2, 3, 4, 6 \pmod 8. \end{cases}$$

Proof. Reducing both sides of (21) modulo 2, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} R_5(n)q^n \\
&\equiv \frac{q^{15}(1+q^7+q^8+q^{10}+q^{11}+q^{15}+q^{19}+q^{20}+q^{21}+q^{22}+q^{23}+q^{24})}{(1-q)^2(1-q^3)(1-q^4)(1-q^5)} \pmod 2 \\
&\equiv \frac{q^{15}((1-q^{15})+q^5(1-q^5)+q^7(1-q^5)+q^8(1-q^5)+q^{11}(1-q^5)+q^{12}(1-q^5))}{(1-q)^2(1-q^3)(1-q^4)(1-q^5)} \\
&\quad + \frac{q^{15}(q^{13}(1-q^5)+q^{15}(1-q^5)+q^{16}(1-q^5)+q^{17}(1-q^5)+q^{18}(1-q^5)+q^{19}(1-q^5))}{(1-q)^2(1-q^3)(1-q^4)(1-q^5)} \\
&\quad \pmod 2 \\
&\equiv \frac{q^{15}(1+q^5+q^7+q^8+q^{11}+q^{12}+q^{13}+q^{15}+q^{16}+q^{17}+q^{18}+q^{19})}{(1-q)^2(1-q^3)(1-q^4)} \pmod 2.
\end{aligned}$$

Use the facts $\frac{1}{(1-q)} \equiv \frac{1}{(1+q)} \pmod 2$ and $\frac{1}{(1-q^4)} \equiv \frac{1}{(1+q^4)} \pmod 2$ and multiply the numerator and denominator by $(1+q^2)$ to get

$$\begin{aligned}
&\sum_{n=1}^{\infty} R_5(n)q^n \\
&\equiv \frac{q^{15}(1+q^5+q^7+q^8+q^{11}+q^{12}+q^{13}+q^{15}+q^{16}+q^{17}+q^{18}+q^{19})(1+q^2)}{(1-q)(1+q)(1-q^3)(1+q^4)(1+q^2)} \pmod 2 \\
&\equiv \frac{q^{15}(1+q^2+q^5+q^8+q^9+q^{10}+q^{11}+q^{12}+q^{14}+q^{16}+q^{20}+q^{21})}{(1-q^8)(1-q^3)} \pmod 2 \\
&\equiv \frac{q^{15}((1-q^9)+(q^2-q^8)+(q^5-q^{11})+(q^{10}-q^{16})+(q^{12}-q^{21})+(q^{14}-q^{20}))}{(1-q^8)(1-q^3)} \pmod 2.
\end{aligned}$$

We use the identities $(1-q^6)=(1-q^3)(1+q^3)$ and $(1-q^9)=(1-q^3)(1+q^3+q^6)$ accordingly and simplify to get

$$\sum_{n=1}^{\infty} R_5(n)q^n \equiv \frac{1}{1-q^8}\left(q^{15}+q^{17}+q^{18}+q^{21}+q^{23}+q^{25}+q^{27}+q^{28}+q^{29}+q^{30}+q^{32}+q^{33}\right).$$

For $n \equiv 0,2,3,4,6 \pmod 8$ and $n > 25$, the terms q^{32}, q^{18}, q^{27}, q^{28} and q^{30} contribute one each to the parity of $R_4(n)$, respectively, to make it odd. For $n \equiv 1 \pmod 8$ and $n > 25$, the term $q^{17}+q^{25}+q^{33}$ contributes three to the parity of $R_5(n)$, to make it odd. For $n \equiv 5,7 \pmod 8$ and $n > 25$, $q^{21}+q^{29}$ and $q^{15}+q^{23}$ contribute two each to the parity of $R_5(n)$, to make it even. □

Remark 14. For $n > 25$ and *even*, $R_5(n)$ is always odd. This is evident from the observation in Theorem 13, that $R_5(n)$ is odd for all the four cases $n \equiv 0,2,4,6 \pmod 8$.

5 Concluding remarks

We end the paper with the following concluding remarks:

1. Unlike $\mathscr{G}_k(q)$-the generating function of partitions with Durfee square of size k, $\mathscr{F}_k(q)$-the generating function of partitions with Durfee triangle of size k seems to be non-trivial, as can be seen from the increasing complexities of $\mathscr{F}_3(q), \mathscr{F}_4(q)$ and $\mathscr{F}_5(q)$, as shown in Section 3. Straub and the author address this question in an upcoming paper [10].

2. It is very curious and unclear as to why the parity congruences for $R_k(n)$ seem to hold for $n > k^2$. This suggests a *combinatorial connection* between the Durfee triangle and the Durfee square of a partition.

3. It should be possible to obtain the recurrence relations for $R_k(n)$, in general, like the initial cases in (5) and (16).

4. The parity of $P(n)$, which is largely unexplored combinatorially, and could be studied using the parity of $R_k(n)$.

5. The parity bias of $R_2(n)$ is same as that of $R_3(n)$, while the parity bias of $R_4(n)$ is same as that of $R_5(n)$. It is not evident why this is the case.

6. From the initial few cases, it seems like the following conjecture holds.

   Conjecture 15. For any fixed $k \in \mathbb{N}$, as $n \to \infty$, we have

   $$R_k(n) \sim c_k n^{k-1},$$

   for some constant c_k.

7. The exact formula for $R_k(n)$ is the polynomial expression for the partition function $P(n)$ which has been of interest for many reasons, including faster computability. One must investigate if there could be an alternative way to obtain them.

6 Acknowledgments

The author sincerely thanks the anonymous referee for the valuable suggestions. The author thanks Atul Dixit for his continuous support. The author thanks James Sellers for his insight to obtain the exact formulas in (28) and (32). The author thanks Krishnaswami Alladi for informing the work of James Haglund, including [3]. The author thanks Aviral Srivastava and Rajat Gupta for general discussions. The author thanks Armin Straub for pointing out a calculation error in one of the results.

References

[1] G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, 1976. Reissued, Cambridge University Press, 1998.

[2] J. H. Bruinier and K. Ono, Algebraic formulas for the coefficients of half-integral weight harmonic weak Maass forms, *Adv. Math.* **246** (2013), 198–219.

[3] F. Butler, J. Haglund, and J. B. Remmel, *Notes on rook polynomials*, unpublished manuscript, 2026. Available at https://www2.math.upenn.edu/~jhaglund/books/rook.pdf.

[4] O. J. Heilmann and E. H. Lieb, Theory of monomer-dimer systems, *Comm. Math. Phys.* **25** (1972), 190–232.

[5] I. Kaplansky and J. Riordan, The problem of the rooks and its applications, *Duke Math. J.* **13** (1946), 259–268.

[6] M. Merca and M. D. Schmidt, A partition identity related to Stanley's theorem, *Amer. Math. Monthly* **125** (2018) 929–933.

[7] M. Merca and M. D. Schmidt, The partition function p(n) in terms of the classical Möbius function, *Ramanujan J.* **49** (2019) 87–96.

[8] K. Ono, Distribution of the partition function modulo m, *Ann. of Math.* (2) **151** (2000), no. 1, 293–307.

[9] J. Riordan, *Introduction to Combinatorial Analysis*, Wiley Publication, 1958. Reissued, Dover Publications, 2002.

[10] N. G. Sharan and A. Straub, Partitions with Durfee triangles of fixed size, *J. Combin. Theory Ser. A* **220** (2026), 106158.

[11] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, 2026. Available at https://oeis.org/.

[12] N. M. Temme, *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*, Wiley-Interscience Publication, 1996.

2020 *Mathematics Subject Classification*: Primary 05A17; Secondary 11P83, 11N56.
Keywords: decomposition, rook number, Durfee triangle, partition function, asymptotics, periodicity.

(Concerned with sequences A325168 and A325188.)

Received June 2 2025; revised version received June 4 2025; April 23 2026. Published in *Journal of Integer Sequences*, April 30 2026.

Return to Journal of Integer Sequences home page.