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A Note on Non-Integrality of the (k, l)-Göbel Sequences

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Abstract

The (k, l)-Göbel sequences defined by Ibstedt remain integers for the first (in some cases, many) terms, but for selected values of (k, l), computations show that the terms eventually stop being integers. It is still unresolved whether the integrality of these sequences breaks down for all $k, l \geq 2$. In this article, we prove the non-integrality for a specific class of (k, l) values. Our proof is based on geometric arguments related to the distribution of quadratic residues modulo a prime.

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1 Introduction

Göbel's sequence $(g_n)_{n\geq 0}$ is defined by the following recurrence relation:

$$g_n = \frac{2 + g_1^2 + \cdots + g_{n-1}^2}{n}$$

with $g_1 = 2$ (A003504). This sequence remains an integer up to g_{42}, but in 1975, Lenstra found that g_{43} is no longer an integer. Since then, there has been little research on Göbel's sequence. However, after Göbel's sequence was featured in the Japanese manga *Seisu-tan* [4] written by the second author of this article and Doom Kobayashi in 2023, the manga inspired two recent articles by Matsusaka and colleagues [6, 1]. For the history of Göbel's sequence, refer to the article by Matsuhira et al. [6].

Ibstedt [2] generalized Göbel's sequence to one with two parameters, k and l. For integers $k \geq 2$ and $l \geq 0$, the sequence $(g_{k,l}(n))_{n\geq 1}$, which is called the (k,l)-*Göbel sequence*, is defined by the recurrence relation

$$g_{k,l}(n) = \frac{1}{n}\left(l + \sum_{i=1}^{n-1} g_{k,l}(i)^k\right).$$

We can check that $g_{2,2}(n) = g_n$ holds for all n and that the recurrence relation for (k,l)-Göbel sequence can be rewritten as

$$(n+1)g_{k,l}(n+1) = g_{k,l}(n) \cdot \left(n + g_{k,l}(n)^{k-1}\right) \tag{1}$$

with the initial condition $g_{k,l}(1) = l$. In particular, we refer to the sequence $(g_{k,2}(n))_{n\geq 1}$ as the k-*Göbel sequence*.

We are interested in whether the (k,l)-Göbel sequences stop being an integer, as in the case of Göbel's original sequence. We define

$$N_{k,l} \coloneqq \inf\{n \in \mathbb{Z}_{\geq 1} \mid g_{k,l}(n) \notin \mathbb{Z}\},$$

where $N_{k,l} = \infty$ if $g_{k,l}(n)$ remains an integer for all n. Also, let $N_k \coloneqq N_{k,2}$ (A108394). We are particularly interested in the behavior of sequences $(N_{k,l})_{k,l\geq 2}$ or $(N_k)_{k\geq 2}$. We note that the cases $l = 0$ and $l = 1$ are included in the definition for convenience, where we have $g_{k,0}(n) = 0$ and $g_{k,1}(n) = 1$ for all n, and in particular, we have $N_{k,0} = N_{k,1} = \infty$.

Matsuhira, Matsusaka, and Tsuchida [6] proved that

$$\min_{k\geq 2} N_k = 19$$

and $N_k = 19$ if and only if $k \equiv 6, 14 \pmod{18}$. Gima, Matsusaka, Miyazaki, and Yara [1] proved

$$\min_{k,l\geq 2} N_{k,l} = 7$$

and $N_{k,l} = 7$ if and only if $k \equiv 2 \pmod 6$ and $l \equiv 3 \pmod 7$.

Whereas the minimum values have been determined, the fundamental problem of the finiteness of N_k for all $k \geq 2$ remains unresolved. In fact, no approach independent of computer searches has been found so far. By considering the general initial value case, we are led to the following weaker problem:

For each $k \geq 2$, does there exist some $l \geq 2$ such that $N_{k,l}$ is finite?

We partially solve this problem for k belonging to an infinite class of even integers that satisfy certain conditions (see Corollary 2).

As we are interested in the non-integrality of (k,l)-Göbel sequences, we here recall the method of proving it. Since $g_{43} \approx 5.4 \times 10^{178485291567}$, fully calculating g_{43} with a computer is highly impractical. (See the article by Gima et al. [1, Theorem 4] for the asymptotic expansion formula of general (k,l)-Göbel sequences.) However, by using the recurrence relation $(n+1)g_{n+1} = g_n(n+g_n)$, we can iteratively calculate the congruences modulo 43 in $\mathbb{Z}_{(43)}$ as follows: $g_1 = 2$, $g_2 = 3$, $g_3 = 5$, $g_4 = 10$, $g_5 = 28$,

$$\begin{aligned} g_6 &\equiv 28(5+28) \cdot 6^{-1} \equiv 28 \cdot 33 \cdot 36 \equiv 25 \pmod{43}, \\ g_7 &\equiv 25(6+25) \cdot 7^{-1} \equiv 25 \cdot 31 \cdot 37 \equiv 37 \pmod{43}, \\ &\vdots \\ g_{42} &\equiv 23(41+23) \cdot 42^{-1} \equiv 23 \cdot 21 \cdot 42 \equiv 33 \pmod{43}, \end{aligned}$$

and

$$43g_{43} \equiv 33 \cdot (42+33) \equiv 33 \cdot 32 \equiv 24 \pmod{43}.$$

If $g_{43} \in \mathbb{Z}$, then we have $43g_{43} \equiv 0 \pmod{43}$ in $\mathbb{Z}_{(43)}$. Therefore, it follows that $g_{43} \notin \mathbb{Z}$. Here, for a prime number p, the ring $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the ideal (p). Similarly, if, based on such iterated calculations, we find a prime number p such that $pg_{k,l}(p) \not\equiv 0 \pmod{p}$ in $\mathbb{Z}_{(p)}$, we can conclude the non-integrality of the (k,l)-Göbel sequence (and more strongly, that $N_{k,l} \leq p$).

In the article by Gima et al. [1, Section 4], the authors posed the following question:

> "*for any pair of integers k, $l \geq 2$, does there exist (infinitely many) $p \in \mathcal{P}$ such that $g_{k,l}(p) \notin \mathbb{Z}_{(p)}$?*"

Here, their $\mathcal{P}$ is the set of all prime numbers. The answer to this question remains unknown, and whether such primes exist — and if so, which ones — cannot be determined until we compute it explicitly. Another question can also be posed by switching the roles of (k,l) and p as follows:

For a given prime p, which k, $l \geq 2$ satisfy $g_{k,l}(p) \notin \mathbb{Z}_{(p)}$?

Because it depends only on the residue class $pg_{k,l}(p) \bmod p$, it is sufficient, by Fermat's little theorem, to consider only the cases where $0 \leq k \leq p-2$ and $0 \leq l \leq p-1$. Thus, for

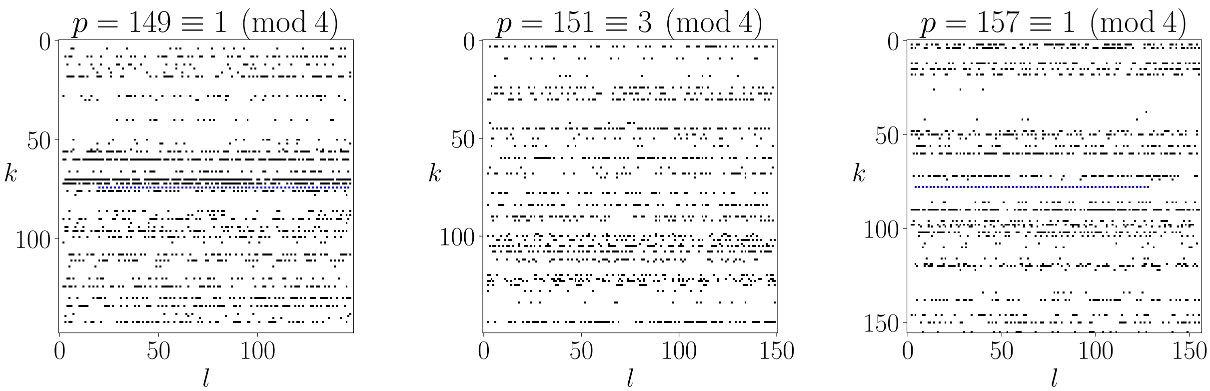

Figure 1: Blue ($k = (p-1)/2$) and black ($k \neq (p-1)/2$) dots represent the pair (k, l) for which $g_{k,l}(p) \notin \mathbb{Z}_{(p)}$.

each given p, the values of k and l that satisfy $g_{k,l}(p) \notin \mathbb{Z}_{(p)}$ can be completely determined through calculations using a computer.

While performing this calculation, we discovered a general phenomenon: for primes p satisfying $p \equiv 1 \pmod 4$, when $k = (p-1)/2$, even values of l for which $g_{k,l}(p) \notin \mathbb{Z}_{(p)}$ appear consecutively (see Figure 1). In this article, we prove that this phenomenon indeed holds true.

Theorem 1. *Let p be an odd prime number.*

(a) *Assume that $p \equiv 3 \pmod 4$. For every integer l satisfying $0 \leq l < p$, we have*

$$g_{\frac{p-1}{2},l}(p) \in \mathbb{Z}_{(p)}.$$

(b) *Assume that $p \equiv 1 \pmod 4$. For every odd integer l satisfying $0 \leq l < p$, we have*

$$g_{\frac{p-1}{2},l}(p) \in \mathbb{Z}_{(p)}.$$

Furthermore, if $p \geq 13$, there exists a unique decomposition

$$\{0, 2, 4, \ldots, p-1\} = I_p^{\mathrm{L}} \cup J_p \cup I_p^{\mathrm{R}}$$

such that for even integers l_{L} and l_{R} with $2 \leq l_{\mathrm{L}} < l_{\mathrm{R}} \leq p-1$, the following assertions are true:

$$\begin{aligned} I_p^{\mathrm{L}} &= \{l \in 2\mathbb{Z} \mid 0 \leq l < l_{\mathrm{L}}\}, \\ J_p &= \{l \in 2\mathbb{Z} \mid l_{\mathrm{L}} \leq l < l_{\mathrm{R}}\}, \\ I_p^{\mathrm{R}} &= \{l \in 2\mathbb{Z} \mid l_{\mathrm{R}} \leq l \leq p-1\} \end{aligned}$$

and for every even integer l satisfying $0 \leq l < p$, we have

$$g_{\frac{p-1}{2},l}(p) \in \mathbb{Z}_{(p)} \text{ if and only if } l \in I_p^{\mathrm{L}} \cup I_p^{\mathrm{R}}.$$

We prove this theorem through elementary geometric arguments based on the distribution of quadratic residues modulo p.

As an application of our result, we obtain the following corollary, which provides a partial answer to the question posed by Gima, Matsusaka, Miyazaki, and Yara.

Corollary 2. *Let k be a positive even integer. Assume that there exists a prime $p \geq 13$ such that $p \equiv 1 \pmod 4$ and k is a positive odd multiple of $(p-1)/2$. Then there exists an even integer l satisfying $2 \leq l \leq p-3$ such that for every positive integer l' with $l' \equiv l \pmod p$, we have $g_{k,l'}(p) \notin \mathbb{Z}_{(p)}$, and hence $N_{k,l'} \leq p$.*

Proof. Since $l_{\mathrm{L}} < l_{\mathrm{R}}$ in Theorem 1, it follows that $J_p \neq \varnothing$. Therefore, for $l \in J_p$, we have $g_{\frac{p-1}{2},l}(p) \notin \mathbb{Z}_{(p)}$. By periodicity, we have the conclusion. □

We prepare a lemma regarding the strength of this corollary.

Lemma 3. *The union*

$$\bigcup_{\substack{p \equiv 1 \pmod 4, \\ p \geq 13}} \frac{p-1}{2} \cdot (2\mathbb{Z}_{\geq 0} + 1)$$

cannot be covered by finitely many of the sets $\frac{p-1}{2} \cdot (2\mathbb{Z}_{\geq 0} + 1) = \{\frac{p-1}{2} \cdot (2m+1) \mid m \in \mathbb{Z}_{\geq 0}\}$. In the union above, the prime p varies over primes satisfying $p \equiv 1 \pmod 4$ and $p \geq 13$.

Proof. Assume that there exist finitely many primes $p_1, \dots, p_t$ such that

$$\bigcup_{\substack{p \equiv 1 \pmod 4, \\ p \geq 13}} \frac{p-1}{2} \cdot (2\mathbb{Z}_{\geq 0} + 1) = \bigcup_{1 \leq i \leq t} \frac{p_i - 1}{2} \cdot (2\mathbb{Z}_{\geq 0} + 1)$$

holds. Let s be the product of odd prime factors of $(p_i - 1)/2$ for all $1 \leq i \leq t$. Take a positive odd integer b such that $2b \equiv -1 \pmod s$. Let q be a prime of the form $q = 8sn + 4b + 1$ for some positive integer n. Such a prime exists by Dirichlet's theorem on arithmetic progressions. Since $(q-1)/2 = 2(2sn + b)$ is not a multiple of $(p_i - 1)/2$ for any $1 \leq i \leq t$, we have a contradiction. □

By Lemma 3, the set of values of k to which Corollary 2 can be applied is broader than the set that can be covered by finite computer calculations with previously proposed algorithms checking whether $g_{k,l}(p) \notin \mathbb{Z}_{(p)}$.

We present the numerical data for $\#J_p$, l_{L}, and l_{R} in Section 5.2. Here, for a set A, the notation $\#A$ denotes the cardinality of A. These data (especially Figure 5) support the following conjecture:

Conjecture 4. For primes $p \geq 13$ satisfying $p \equiv 1 \pmod 4$, let J_p be the set defined as in Theorem 1. The ratio $\#J_p/p$, which is clearly less than $1/2$ by definition, tends to $1/2$ as p to infinity, that is,

$$\lim_{p \to \infty} \frac{\#J_p}{p} = \frac{1}{2},$$

where p varies over primes satisfying $p \equiv 1 \pmod 4$.

This article is organized as follows. In Section 2, we prove a relaxed version of Theorem 1 that allows the possibility of $l_{\mathrm{L}} = l_{\mathrm{R}}$ and provide a criterion for when $l_{\mathrm{L}} < l_{\mathrm{R}}$ holds (Theorem 15). In Section 3, we investigate fundamental properties of two kinds of special sequences whose values are in $\{+1, -1\}$. In Section 4, we prove $l_{\mathrm{L}} < l_{\mathrm{R}}$ by using results obtained in Section 3. In Section 5, we include the data obtained from our numerical experiments, with pseudocode for the programs used in those experiments.

2 Geometric arguments involving Legendre symbols

Throughout this section, we fix an odd prime p. Let the symbol $\left(\frac{\cdot}{p}\right)$ denote the Legendre symbol. That is, for an integer a relatively prime to p, we define

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p; \\ -1, & \text{if } a \text{ is a quadratic non-residue modulo } p. \end{cases}$$

Definition 5. Let l be an integer such that $0 \leq l \leq p-1$. We define a sequence $(\tilde{g}_l(n))_{1 \leq n \leq p}$ with $\tilde{g}_l(1) = l$ and the recurrence relation:

$$\tilde{g}_l(n+1) := \begin{cases} \tilde{g}_l(n) + \left(\frac{n}{p}\right)\left(\frac{\tilde{g}_l(n)}{p}\right), & \text{if } 0 < \tilde{g}_l(n) < p; \\ 0, & \text{if } \tilde{g}_l(n) = 0; \\ p, & \text{if } \tilde{g}_l(n) = p. \end{cases}$$

Lemma 6. *For every integer n with $1 \leq n \leq p$, the following congruence holds in $\mathbb{Z}_{(p)}$:*

$$n g_{\frac{p-1}{2},l}(n) \equiv \tilde{g}_l(n) \pmod{p}.$$

In particular, we have $g_{\frac{p-1}{2},l}(p) \in \mathbb{Z}_{(p)}$ if and only if $\tilde{g}_l(p) \in \{0, p\}$.

Proof. By the recurrence relation (1), we see that if $g_{\frac{p-1}{2},l}(n) \equiv 0 \pmod{p}$, then we have $(n+1)g_{\frac{p-1}{2},l}(n+1) \equiv 0 \pmod{p}$. We assume that $n g_{\frac{p-1}{2},l}(n)$ is prime to p. Since

$$(n+1)g_{\frac{p-1}{2},l}(n+1) = n g_{\frac{p-1}{2},l}(n) + n^{-\frac{p-1}{2}}\left(n g_{\frac{p-1}{2},l}(n)\right)^{\frac{p-1}{2}}$$

holds by (1), we have

$$(n+1)g_{\frac{p-1}{2},l}(n+1) \equiv n g_{\frac{p-1}{2},l}(n) + \left(\frac{n}{p}\right)^{-1}\left(\frac{n g_{\frac{p-1}{2},l}(n)}{p}\right) \pmod{p}$$

by Euler's criterion. Hence, we obtain the conclusion from $\left(\frac{n}{p}\right)^{-1} = \left(\frac{n}{p}\right)$. □

Definition 7. We define a set $\mathcal{L}$ of point sets as

$$\mathcal{L} := \{\{(n, a(n))\}_{1 \leq n \leq p} \mid a(n) \in \mathbb{Z}, 0 \leq a(n) \leq p \text{ for } 1 \leq n \leq p\}.$$

Here, the notation $\{(n, a(n))\}_{1 \leq n \leq p}$ is an abbreviation of the set $\{(n, a(n)) \mid 1 \leq n \leq p\}$. For $\mathsf{A} = \{(n, a(n))\}_{1 \leq n \leq p}$ and $\mathsf{B} = \{(n, b(n))\}_{1 \leq n \leq p}$ in $\mathcal{L}$, we use the following notation:

- $\mathsf{A} \leq \mathsf{B} \overset{\text{def}}{\iff} a(n) \leq b(n)$ for all $1 \leq n \leq p$,
- $\mathsf{A} \preceq \mathsf{B} \overset{\text{def}}{\iff}$ there exists an integer m with $0 \leq m \leq p$ such that $a(n) < b(n)$ holds for $1 \leq n \leq m$, and $a(n) = b(n)$ holds for $m < n \leq p$.

We also introduce symbols G_l for $0 \leq l \leq p-1$, diag, and $\overline{\mathsf{diag}}$ as elements of $\mathcal{L}$, defined respectively as follows:

$$\mathsf{G}_l \coloneqq \{(n, \tilde{g}_l(n))\}_{1 \leq n \leq p}, \quad \mathsf{diag} \coloneqq \{(n,n)\}_{1 \leq n \leq p}, \quad \overline{\mathsf{diag}} \coloneqq \{(n, p-n)\}_{1 \leq n \leq p}.$$

Lemma 8. *Let m be an integer such that $1 \leq m \leq p$. If $\tilde{g}_l(m) = m$, then for all n such that $m \leq n \leq p$, we have $\tilde{g}_l(n) = n$.*

Proof. If $\tilde{g}_l(m) = m$ for $m < p$, then

$$\tilde{g}_l(m+1) = m + \left(\frac{m}{p}\right)^2 = m+1.$$

We can repeat this process to obtain the conclusion. □

Proposition 9. *For every odd integer l such that $1 \leq l < p$, we have $\mathsf{diag} \preceq \mathsf{G}_l$.*

Proof. First, we claim that $\mathsf{diag} \leq \mathsf{G}_l$. If $\tilde{g}_l(n) = p$, then clearly $n \leq \tilde{g}_l(n)$. Therefore, we may consider the case where $\tilde{g}_l(n) < p$. At the initial point, the difference $\tilde{g}_l(1) - 1 = l - 1$ is a non-negative even integer, and we have either $\tilde{g}_l(n+1) - (n+1) = \tilde{g}_l(n) - n$ or $\tilde{g}_l(n+1) - (n+1) = (\tilde{g}_l(n) - n) - 2$, since the values of the Legendre symbols are either $+1$ or -1. Therefore, if $\tilde{g}_l(n) - n < 0$ for some n, then there exists an $m < n$ such that $\tilde{g}_l(m) - m = 0$. However, if $\tilde{g}_l(m) - m = 0$ for $m \leq p-1$, then by Lemma 8, we have $\tilde{g}_l(n) - n = 0$, which leads to a contradiction. This proves the claim made in the beginning. Since $\tilde{g}_l(p) = p$, we can take m such that $1 \leq m \leq p$, $\tilde{g}_l(1) > 1, \ldots, \tilde{g}_l(m-1) > m-1$, and $\tilde{g}_l(m) = m$. Then, by Lemma 8, we have $\tilde{g}_l(m) - m = \tilde{g}_l(m+1) - (m+1) = \cdots = \tilde{g}_l(p) - p = 0$. This proves $\mathsf{diag} \preceq \mathsf{G}_l$. □

Proposition 10. *Assume that $p \equiv 3 \pmod 4$. For every even integer l such that $0 \leq l < p$, we have $\mathsf{G}_l \preceq \overline{\mathsf{diag}}$.*

Proof. If $\tilde{g}_l(m) = p - m$ for $1 \leq m < p$, then

$$\tilde{g}_l(m+1) = p - m + \left(\frac{m}{p}\right)\left(\frac{p-m}{p}\right) = p - m + \left(\frac{-1}{p}\right) = p - (m+1),$$

by the first supplement to the law of quadratic reciprocity and the assumption $p \equiv 3 \pmod 4$. Based on this fact, we can prove that $\mathsf{G}_l \preceq \overline{\mathsf{diag}}$, in a manner similar to the proof of Proposition 9. □

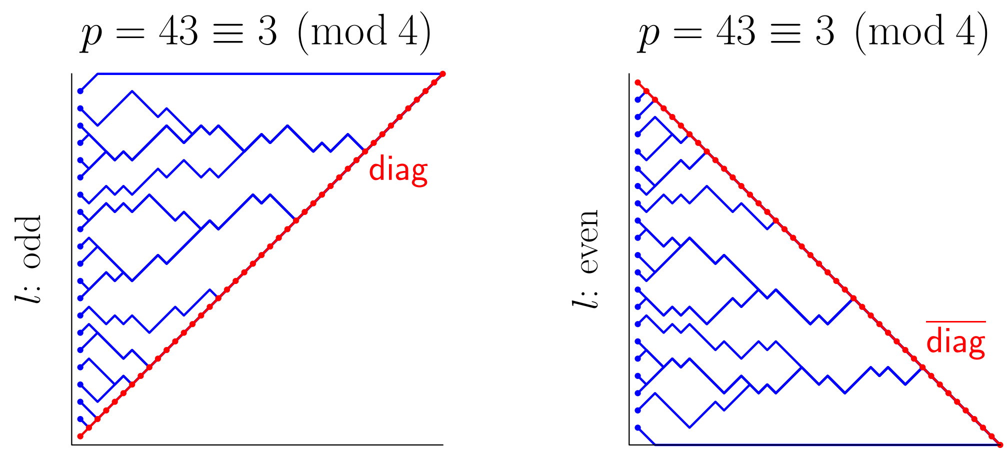

Figure 2: Plot of points belonging to sets G_l, diag, and $\overline{\mathsf{diag}}$.

Proposition 11. *Assume that* $p \equiv 1 \pmod 4$*. Then, we have*

$$\mathsf{G}_0 \preceq \mathsf{G}_2 \preceq \cdots \preceq \mathsf{G}_{p-3} \preceq \mathsf{G}_{p-1}.$$

Proof. Let l be an even integer such that $0 \leq l \leq p-3$. If $0 < \tilde{g}_l(m) = \tilde{g}_{l+2}(m) < p$ for an integer m such that $1 \leq m < p$, then

$$\tilde{g}_l(m+1) = \tilde{g}_l(m) + \left(\frac{m}{p}\right)\left(\frac{\tilde{g}_l(m)}{p}\right) = \tilde{g}_{l+2}(m) + \left(\frac{m}{p}\right)\left(\frac{\tilde{g}_{l+2}(m)}{p}\right) = \tilde{g}_{l+2}(m+1).$$

If $\tilde{g}_l(m) = \tilde{g}_{l+2}(m) = 0$ or $\tilde{g}_l(m) = \tilde{g}_{l+2}(m) = p$, then we similarly have $\tilde{g}_l(m+1) = \tilde{g}_{l+2}(m+1)$. By using this fact, we can prove that $\mathsf{G}_l \preceq \mathsf{G}_{l+2}$, in a manner similar to the proofs of Proposition 9 and Proposition 10. □

Proof of Theorem 1 except for $l_{\mathrm{L}} < l_{\mathrm{R}}$. If l is an odd integer such that $1 \leq l < p$, then by Proposition 9, we have $\tilde{g}_l(p) = p$. If $p \equiv 3 \pmod 4$ and l is an even integer such that $0 \leq l < p$, then by Proposition 10, we have $\tilde{g}_l(p) = 0$. Now, we assume that $p \equiv 1 \pmod 4$ and l is an even integer such that $0 \leq l < p$. Set

$$l_{\mathrm{L}} \coloneqq \max\{l \in 2\mathbb{Z} \mid \tilde{g}_l(p) = 0\} + 2 \quad \text{and} \quad l_{\mathrm{R}} \coloneqq \min\{l \in 2\mathbb{Z} \mid \tilde{g}_l(p) = p\}.$$

Using l_{L} and l_{R}, we define the sets I_p^{L} and I_p^{R} as in the statement of Theorem 1. Since

$$\tilde{g}_0(p) \leq \tilde{g}_2(p) \leq \cdots \leq \tilde{g}_{p-3}(p) \leq \tilde{g}_{p-1}(p)$$

holds by Proposition 11, we see that $\tilde{g}_l(p) = 0$ if and only if $l \in I_p^{\mathrm{L}}$, and that $\tilde{g}_l(p) = p$ if and only if $l \in I_p^{\mathrm{R}}$. By Lemma 6, we have $g_{\frac{p-1}{2},l}(p) \in \mathbb{Z}_{(p)}$ if and only if $l \in I_p^{\mathrm{L}} \cup I_p^{\mathrm{R}}$. □

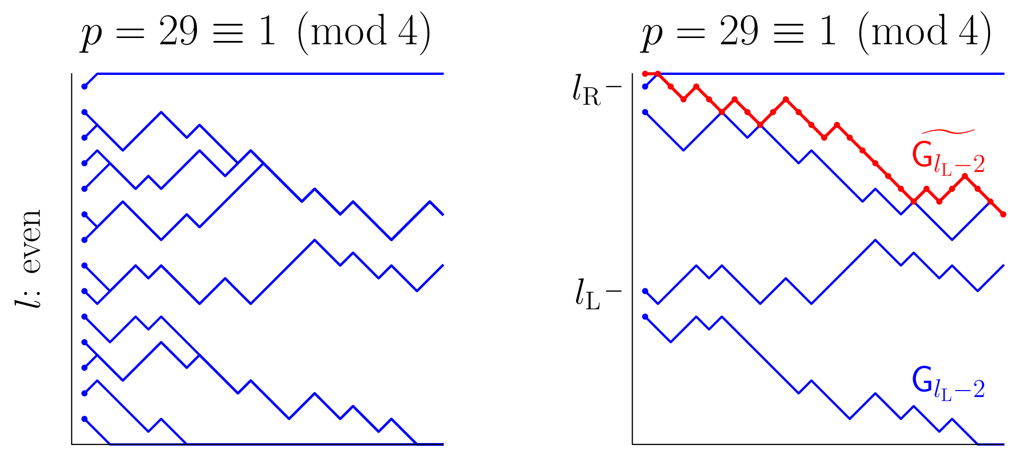

Figure 3: Plot of points belonging to sets G_l for even l (left), and the barrier $\widetilde{\mathsf{G}_{l_{\mathrm{L}}-2}}$ (right).

Remark 12. Let p denote a prime satisfying $p \equiv 1 \pmod 4$. In the above proof, the integers l_{L} and l_{R} exist because $\tilde{g}_0(p) = 0$ and $\tilde{g}_{p-1}(p) = p$. The latter equality follows from

$$\tilde{g}_{p-1}(2) = (p-1) + \left(\frac{1}{p}\right)\left(\frac{p-1}{p}\right) = p,$$

since $p \equiv 1 \pmod 4$. Let $J_p := \{l \in 2\mathbb{Z} \mid l_{\mathrm{L}} \le l < l_{\mathrm{R}}\}$ as in the statement of Theorem 1. For small primes p, we have $2 \in I_p^{\mathrm{L}}$, but we have also found the following primes p for which $2 \in J_p$: 313, 1873, 2081, 2089, 2377, 4481, 5281, 6361, 6961, 7681, 8161, 8209, 8521, 8929, 9001, ... Such primes may find some applications in the study of the non-integrality of k-Göbel sequences, but their distribution is not yet understood. There are 502 of them below 10^6.

Question 13. *Do infinitely many primes p exist, satisfying $p \equiv 1 \pmod 4$, such that $2 \in J_p$?*

Definition 14. For $\mathsf{A} = \{(n, a(n))\}_{1 \le n \le p} \in \mathcal{L}$, we set

$$\mathsf{A}^+ := \{(n, a(n)) \in \mathsf{A} \mid a(n+1) = a(n) + 1\},$$
$$\mathsf{A}^- := \{(n, a(n)) \in \mathsf{A} \mid a(n+1) = a(n) - 1\}.$$

We say that $\mathsf{A} \in \mathcal{L}$ is *zigzag* if both A^+ and A^- are non-empty.

The following is the key criterion for determining whether $l_{\mathrm{L}} < l_{\mathrm{R}}$.

Theorem 15. *Assume that $p \equiv 1 \pmod 4$. The equality $l_{\mathrm{L}} = l_{\mathrm{R}}$ holds if and only if the following conditions hold for some even l such that $0 \le l \le p-3$:*

(a) $\left(\frac{n}{p}\right) = -\left(\frac{l+1-n}{p}\right)$ *holds for $1 \le n \le l/2$;*

(b) $\left(\frac{n}{p}\right) = \left(\frac{l+1+n}{p}\right)$ *holds for* $1 \leq n \leq p - l - 2$.

Proof. We can easily check that the condition (a) is equivalent to G_l not being zigzag and $l \in I_p^{\mathrm{L}}$, while the condition (b) is equivalent to G_{l+2} not being zigzag and $l + 2 \in I_p^{\mathrm{R}}$. In particular, if both conditions (a) and (b) hold, then we have $l + 2 = l_{\mathrm{L}} = l_{\mathrm{R}}$. Therefore, it is sufficient to show that if at least one of $\mathsf{G}_{l_{\mathrm{L}}-2}$ and $\mathsf{G}_{l_{\mathrm{R}}}$ is zigzag, then we have $l_{\mathrm{L}} < l_{\mathrm{R}}$.

We define $\widetilde{\mathsf{G}_{l_{\mathrm{L}}-2}}$ by

$$\widetilde{\mathsf{G}_{l_{\mathrm{L}}-2}} \coloneqq \{(p - n, p - \tilde{g}_{l_{\mathrm{L}}-2}(n))\}_{0 \leq n \leq p-1}.$$

Here, we set $\tilde{g}_{l_{\mathrm{L}}-2}(0) \coloneqq l_{\mathrm{L}} - 1$. First, assume that $\mathsf{G}_{l_{\mathrm{L}}-2}$ is zigzag. We prove that $\widetilde{\mathsf{G}_{l_{\mathrm{L}}-2}}$ is a *barrier*, meaning that

$$\mathsf{G}_{l_{\mathrm{L}}} \leq \widetilde{\mathsf{G}_{l_{\mathrm{L}}-2}}. \tag{2}$$

Let $n^* \coloneqq \min\{n \in \mathbb{Z} \mid 1 \leq n \leq p, \tilde{g}_{l_{\mathrm{L}}-2}(n) = 0\}$. Since l_{L} is even, the integer n^* is odd. Because $\mathsf{G}_{l_{\mathrm{L}}-2}$ is zigzag, we have $\tilde{g}_{l_{\mathrm{L}}-2}(n) > 0$ for $1 \leq n \leq l_{\mathrm{L}} - 1$ and $n^* \geq l_{\mathrm{L}} + 1$. Since $\tilde{g}_{l_{\mathrm{L}}-2}(p-1) = 0$ holds by $p \equiv 1 \pmod 4$, we have $n^* \leq p - 2$. Therefore, for $1 \leq n \leq p - n^*$,

$$\tilde{g}_{l_{\mathrm{L}}}(n) \leq l_{\mathrm{L}} + (p - n^* - 1) \leq p - 2.$$

Define $h(n) \coloneqq (p - \tilde{g}_{l_{\mathrm{L}}-2}(p-n)) - \tilde{g}_{l_{\mathrm{L}}}(n)$. By the parity of l_{L} and n^*, we see that $h(p-n^*) \geq 2$ is even. For $p - n^* \leq n < p$, by definition, we have

$$h(n+1) - h(n) \in \{+2, 0, -2\},$$

unless $\tilde{g}_{l_{\mathrm{L}}}(n) = p$. Assume that $h(n) < 0$ for some n such that $p - n^* + 2 \leq n \leq p$. Then, we can take some m and m', with $3 \leq m \leq m' < p$, such that

$$h(m-1) = +2, \quad h(m) = \cdots = h(m') = 0, \quad h(m'+1) = -2.$$

In this situation, since $\tilde{g}_{l_{\mathrm{L}-2}}(p-m+1) = \tilde{g}_{l_{\mathrm{L}-2}}(p-m) - 1$, we see that $(p-m, \tilde{g}_{l_{\mathrm{L}}-2}(p-m)) \in \mathsf{G}^-_{l_{\mathrm{L}}-2}$. Hence, the equality

$$\left(\frac{p-m}{p}\right)\left(\frac{\tilde{g}_{l_{\mathrm{L}}-2}(p-m)}{p}\right) = -1$$

must hold, and from $\tilde{g}_{l_{\mathrm{L}}}(m) = p - \tilde{g}_{l_{\mathrm{L}}-2}(p-m)$, we have

$$\left(\frac{m}{p}\right)\left(\frac{\tilde{g}_{l_{\mathrm{L}}}(m)}{p}\right) = -1.$$

This implies that $(m, \tilde{g}_{l_{\mathrm{L}}}(m)) \in \mathsf{G}^-_{l_{\mathrm{L}}}$. By repeating a similar argument, we see that for all $m \leq n \leq m'$, we have $(n, \tilde{g}_{l_{\mathrm{L}}}(n)) \in \mathsf{G}^-_{l_{\mathrm{L}}}$. In particular, we have $(m', \tilde{g}_{l_{\mathrm{L}}}(m')) \in \mathsf{G}^-_{l_{\mathrm{L}}}$; however, it is impossible that $h(m'+1) = -2$. Thus, we have $h(n) \geq 0$ for every n, and we get (2). In particular, we have $\tilde{g}_{l_{\mathrm{L}}}(p) \leq p - l_{\mathrm{L}} + 1 < p$, and hence $l_{\mathrm{L}} < l_{\mathrm{R}}$.

Next, assume that $\mathsf{G}_{l_{\mathrm{L}}-2}$ is not zigzag and $\mathsf{G}_{l_{\mathrm{R}}}$ is zigzag. In this case, we have

$$\widetilde{\mathsf{G}_{l_{\mathrm{L}}-2}} = \{(n, p) \mid 1 \leq n \leq p - l_{\mathrm{L}} + 1\} \cup \{(p - n, p - l_{\mathrm{L}} + n + 1) \mid 0 \leq n \leq l_{\mathrm{L}} - 2\}.$$

If $l_{\mathrm{L}} = l_{\mathrm{R}}$, since $\mathsf{G}_{l_{\mathrm{L}}}$ is zigzag, we can check, by a similar argument as before, that $\mathsf{G}_{l_{\mathrm{L}}} \preceq \widetilde{\mathsf{G}_{l_{\mathrm{L}}-2}}$ and $\tilde{g}_{l_{\mathrm{L}}}(p) < p$, which contradicts the definition of l_{R}. Therefore, we have $l_{\mathrm{L}} < l_{\mathrm{R}}$. □

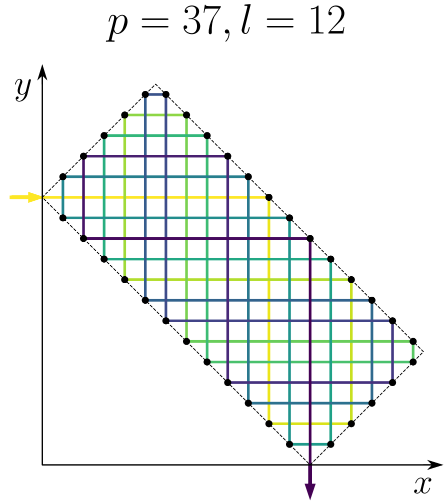

Figure 4: An example of Lemma 16.

3 Special sequences $(a_{p,l}(n))_{1\le n\le p-1}$ and $(b_{l,s}(n))_{1\le n\le l+1}$

In this section, we study special sequences related to the conditions in Theorem 15.

Lemma 16 (Arithmetic billiards)**.** *Let p be a prime number such that $p \equiv 1 \pmod 4$, and let l an even integer such that $0 \le l \le p-3$. First, we consider the case where $l \le (p-5)/2$. On an x-y plane, let $R_{p,l}$ denote the rectangle with vertices $(0, l+1)$, $(l+1, 0)$, $(p/2, p/2-(l+1))$, and $(p/2-(l+1), p/2)$, and let $L_{p,l}$ denote the set of all lattice points on the boundary of $R_{p,l}$ where both coordinates are integers, excluding $(0, l+1)$ and $(l+1, 0)$. From the vertex $(0, l+1)$, draw a path horizontally along the x-axis. When it reaches the edge of $R_{p,l}$, reflect it in the direction parallel to the y-axis and continue drawing the path. When it reaches the edge of $R_{p,l}$ again, reflect it in the direction parallel to the x-axis and continue drawing. By repeating this process, the path passes through each of the $p-2$ elements of $L_{p,l}$ exactly once and reaches the vertex $(l+1, 0)$. Moreover, the entire path is symmetric with respect to the line $y = x$. For the case where $l \ge (p-1)/2$, let $R_{p,l}$ denote the rectangle with vertices $(0, p-(l+1))$, $(p-(l+1), 0)$, $(p/2, (l+1)-p/2)$, and $((l+1)-p/2, p/2)$. Then, a similar statement holds.*

Proof. For the case where $l \le (p-5)/2$, since $p-2(l+1)$ and $2(l+1)$ are coprime, the result can be proved by the standard argument regarding *arithmetic billiards*, and thus we omit the details. See the article by Perucca et al. [7] and the references therein. For $l \ge (p-1)/2$, the result similarly follows because $2(l+1)-p$ and $2(p-(l+1))$ are coprime. □

Proposition 17. *Let p be a prime number such that $p \equiv 1 \pmod 4$, and let l be an even integer such that $0 \le l \le p-3$. Then, there exists a unique ± 1 sequence $(a_n)_{1\le n\le p-1}$ ($a_n \in \{+1, -1\}$) satisfying the following conditions:*

(a) $a_1 = 1$;

(b) $a_n = a_{p-n}$ *holds for* $1 \leq n \leq (p-1)/2$;

(c) $a_n = -a_{l+1-n}$ *holds for* $1 \leq n \leq l/2$;

(d) $a_n = a_{l+1+n}$ *holds for* $1 \leq n \leq p-l-2$.

Proof. First, we consider the case where $l \leq (p-5)/2$. We define a mapping $\psi_{p,l}\colon L_{p,l} \to \{+1,-1\}$ as follows: assign -1 to the points of $L_{p,l}$ on the line segment connecting $(0, l+1)$ and $(l+1, 0)$, and assign $+1$ to all the other points. Note that the condition (c) is equivalent to $a_n a_{l+1-n} = -1$ for $1 \leq n \leq l$. The condition (d) is equivalent to $a_n a_{l+1+n} = +1$ for $1 \leq n \leq p-l-2$, and furthermore, this is equivalent to $a_n a_m = +1$ for all $(n,m) \in L_{p,l} \setminus \{(1,l),(2,l-1),\ldots,(l,1)\}$ under the condition (b). Therefore, it is sufficient to show that a ± 1 sequence $(a_n)_{1\leq n\leq (p-1)/2}$ is uniquely determined by the conditions $a_1 = 1$ and $\psi_{p,l}(n,m) = a_n a_m$ for all $(n,m) \in L_{p,l}$.

In the setting of Lemma 16, considering symmetry, list the first half of the lattice points in $L_{p,l}$ that the path passes through, in the order they are visited, as follows:

$$\begin{aligned}
&(0,\sigma(1)) \longrightarrow (\tau(1),\sigma(1))\\
&\longrightarrow (\tau(1),\sigma(2)) \longrightarrow (\tau(2),\sigma(2))\\
&\longrightarrow (\tau(2),\sigma(3)) \longrightarrow (\tau(3),\sigma(3))\\
&\longrightarrow \cdots\\
&\longrightarrow (\tau((p-5)/4),\sigma((p-1)/4)) \longrightarrow (\tau((p-1)/4),\sigma((p-1)/4))\\
&\longrightarrow ((p-(l+1))/2,(p-(l+1))/2),
\end{aligned}$$

where σ and τ are permutations on $\{1,2,\ldots,(p-1)/2\}$ with $\sigma(1) = l+1$, $\tau(1) = p-2(l+1)$, and $\tau((p+1)/2-n) = \sigma(n)$ for $1 \leq n \leq (p-1)/2$. Since $a_{\tau(1)}a_{\sigma(1)} = \psi_{p,l}(\tau(1),\sigma(1))$, we have $a_{\tau(1)} = \psi_{p,l}(\tau(1),\sigma(1)) \cdot a_{\sigma(1)}$. Next, since $a_{\tau(1)}a_{\sigma(2)} = \psi_{p,l}(\tau(1),\sigma(2))$, we have

$$a_{\sigma(2)} = \psi_{p,l}(\tau(1),\sigma(2)) \cdot a_{\tau(1)} = \psi_{p,l}(\tau(1),\sigma(2))\psi_{p,l}(\tau(1),\sigma(1)) \cdot a_{\sigma(1)}.$$

By repeating this argument, the values of $a_{\tau(1)},\ldots,a_{\tau((p-1)/4)}$ and $a_{\sigma(2)},\ldots,a_{\sigma((p-1)/4)}$ are uniquely determined up to the value of $a_{\sigma(1)}$. In this process, we obtain the equality $a_1 = \pm a_{\sigma(1)}$ and the value of $a_{\sigma(1)}$ is also determined by the condition $a_1 = 1$.

For the case where $l \geq (p-1)/2$, we define a mapping $\psi_{p,l}\colon L_{p,l} \to \{+1,-1\}$ by the following: assign $+1$ to the points of $L_{p,l}$ on the line segment connecting $(0, p-(l+1))$ and $(p-(l+1), 0)$, and assign -1 to all the other points. Then, the rest of the argument proceeds in exactly the same manner. □

Example 18. For example, in the case of Figure 4, we have

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 13 & 2 & 9 & 17 & 6 & 5 & 16 & 10 & 1 & 12 & 14 & 3 & 8 & 18 & 7 & 4 & 15 & 11 \end{pmatrix},$$

and the values of a_n up to a_{13} are determined sequentially as follows:

$$\begin{array}{cccccc} a_{11}=a_{13}, & a_2=-a_{13}, & a_{15}=-a_{13}, & a_9=-a_{13}, & a_4=a_{13}, & a_{17}=a_{13}, \\ a_7=a_{13}, & a_6=-a_{13}, & a_{18}=-a_{13}, & a_5=-a_{13}, & a_8=a_{13}, & a_{16}=a_{13}, \\ a_3=a_{13}, & a_{10}=-a_{13}, & a_{14}=-a_{13}, & a_1=-a_{13}, & a_{12}=a_{13}. & \end{array}$$

It follows from $a_1=-a_{13}$ that $a_{13}=-1$, and that the sequence $(a_{37,12}(n))_{1\le n\le 18}$ is given by

$$(+1,+1,-1,-1,+1,+1,-1,-1,+1,+1,-1,-1,-1,+1,+1,-1,-1,+1).$$

Definition 19. For each p and l, the sequence $(a_{p,l}(n))_{1\le n\le p-1}$ denotes the uniquely existing sequence in Proposition 17.

Proposition 20. *Let l be a non-negative even integer and s a non-negative integer such that $0\le s\le l$. We assume that $2s+1$ is prime to $l+1$. Then, there exists a unique ± 1 sequence $(b_n)_{1\le n\le l+1}$ ($b_n\in\{+1,-1\}$) satisfying the following conditions:*

(a) $b_1=1$;

(b) $b_n=-b_{l+1-n}$ *holds for* $n\not\equiv 0 \pmod{l+1}$;

(c) $b_{s-n}=b_{s+1+n}$ *holds for all* n.

Here, the notation b_n is extended to every integer n, where if $n\equiv m \pmod{l+1}$, then $b_n=b_m$.

Proof. Take a prime number $p\ge l+3$ such that $p\equiv 1 \pmod 4$ and

$$\frac{p-1}{2}\equiv s \pmod{l+1}.$$

Since $\gcd(2s+1,l+1)=1$, such a prime exists by Dirichlet's theorem on arithmetic progressions. Then, we can check that the subsequence $(a_{p,l}(n))_{1\le n\le l+1}$ satisfies the conditions of $(b_n)_{1\le n\le l+1}$. For the condition (c), take integers m and t such that $1\le m,t\le l+1$, $s+1+n\equiv m \pmod{l+1}$, and $p-m\equiv t \pmod{l+1}$ hold. Since $p-m\equiv s-n \pmod{l+1}$, we calculate

$$b_{s+1+n}=b_m=a_{p,l}(m)=a_{p,l}(p-m)=a_{p,l}(t)=b_t=b_{s-n}.$$

The uniqueness of $(b_n)_{1\le n\le l+1}$ follows from that of $(a_{p,l}(n))_{1\le n\le p-1}$, because the extended sequence $(b_n)_{1\le n\le p-1}$ satisfies all the conditions for $(a_{p,l}(n))_{1\le n\le p-1}$. □

Definition 21. For each l and s, the sequence $(b_{l,s}(n))_{1\le n\le l+1}$ denotes the uniquely existing sequence in Proposition 20. Furthermore, we extend the definition of $b_{l,s}(n)$ to every integer n as follows: if $n\equiv m \pmod{l+1}$, then we set $b_{l,s}(n)=b_{l,s}(m)$.

From the proof of Proposition 20, we see that $a_{p,l}(n)=b_{l,s}(n)$ holds for all $1\le n\le p-1$ when $(p-1)/2\equiv s \pmod{l+1}$.

Lemma 22. *Let l and s be integers as in Proposition 20, and assume that $l \geq 2$. Then, for all integers n, we have*

$$b_{l,s}(n) = \begin{cases} b_{l,l-s}(n), & \text{if } n \not\equiv 0 \pmod{l+1}; \\ -b_{l,l-s}(n), & \text{if } n \equiv 0 \pmod{l+1}. \end{cases}$$

Proof. If both $s - n$ and $s + 1 + n$ are not divisible by $l + 1$, then

$$b_{l,s}(l - s - n) = -b_{l,s}(s + 1 + n) = -b_{l,s}(s - n) = b_{l,s}(l - s + 1 + n).$$

If $s \equiv n \pmod{l + 1}$, then $s + 1 + n$ is not divisible by $l + 1$ and

$$b_{l,s}(l - s - n) = -b_{l,s}(s + 1 + n) = -b_{l,s}(s - n) = -b_{l,s}(l - s + 1 + n).$$

If $s + 1 + n \equiv 0 \pmod{l + 1}$, then $s - n$ is not divisible by $l + 1$ and

$$b_{l,s}(l - s - n) = b_{l,s}(s + 1 + n) = b_{l,s}(s - n) = -b_{l,s}(l - s + 1 + n).$$

By these observations and the uniqueness of $(b_{l,l-s}(n))_{1\leq n\leq l+1}$, we obtain the conclusion. □

Lemma 23. *Let l and s be integers as in Proposition 20, and assume that $l \geq 2$. Let n be an integer.*

(a) *If both n and $n+2s+1$ are not divisible by $l+1$, then we have $b_{l,s}(n) = -b_{l,s}(n+2s+1)$.*

(b) *Consider the case where $s < l/2$. Set $t \coloneqq l/2 - s$. If both n and $n+2t$ are not divisible by $l + 1$, then we have $b_{l,s}(n) = -b_{l,s}(n + 2t)$.*

(c) *Consider the case where $s < l/2$. Set $t \coloneqq l/2 - s$. If both $t - n$ and $t + n$ are not divisible by $l + 1$, then we have $b_{l,s}(t - n) = b_{l,s}(t + n)$.*

Proof. For (a), using Lemma 22, we calculate

$$\begin{aligned} b_{l,s}(n) &= b_{l,l-s}(n) = b_{l,l-s}(l - s - (l - s - n)) \\ &= b_{l,l-s}(l - s + 1 + (l - s - n)) \\ &= b_{l,l-s}(l + 1 - (n + 2s + 1)) \\ &= -b_{l,l-s}(n + 2s + 1) \\ &= -b_{l,s}(n + 2s + 1). \end{aligned}$$

For (b), using (a), we have

$$b_{l,s}(n + 2t) = -b_{l,s}(n + 2t + 2s + 1) = -b_{l,s}(n + l + 1) = -b_{l,s}(n).$$

For (c), using (b), we have

$$b_{l,s}(t - n) = -b_{l,s}(l + 1 + n - t) = -b_{l,s}(n - t) = b_{l,s}(n - t + 2t) = b_{l,s}(n + t).$$ □

4 Proof of $l_{\mathrm{L}} < l_{\mathrm{R}}$

Proof. Let p be a prime number such that $p \equiv 1 \pmod 4$. Since the Legendre symbol has the symmetry $\left(\frac{n}{p}\right) = \left(\frac{p-n}{p}\right)$, in order to prove $l_{\mathrm{L}} < l_{\mathrm{R}}$, by Theorem 15, it is sufficient to show that

$$\left(\left(\frac{n}{p}\right)\right)_{1 \le n \le p-1} \neq (a_{p,l}(n))_{1 \le n \le p-1} \quad \text{(as sequences)}$$

for all even integers $0 \le l \le p-3$.

In the following, we assume that l is an even integer such that $0 \le l \le p-3$ and that $p \ge 13$. Moreover, we assume that

$$\left(\left(\frac{n}{p}\right)\right)_{1 \le n \le p-1} = (a_{p,l}(n))_{1 \le n \le p-1} \quad \text{(as sequences).}$$

If $l = 0$, then we have $a_{p,0}(n) = 1$ for all $1 \le n \le p-1$, which contradicts the fact that the half of non-zero residues modulo p are not quadratic residues. Therefore, we may assume that $l \ge 2$. Let s be the unique integer satisfying $(p-1)/2 \equiv s \pmod{l+1}$, with $0 \le s \le l$. Then, under our assumption, we have

$$\left(\frac{n}{p}\right) = b_{l,s}(n) \quad \text{for all } 1 \le n \le p-1.$$

Other than in a few exceptional cases, a contradiction can be derived from the information for $1 \le n \le l$ alone. By Lemma 22, we may assume that $s < l/2$. (Since $2 \cdot (l/2) + 1 = l + 1$, the equality $s = l/2$ is impossible.)

Case I: Assume that $p \equiv 1 \pmod 8$ and $s < l/4$. Then, we have $4s + 2 \le l$. By (a) of Lemma 23, we have

$$b_{l,s}(2s+1) = -b_{l,s}(4s+2).$$

Therefore, by our assumption, we have

$$\left(\frac{2s+1}{p}\right) = -\left(\frac{4s+2}{p}\right) = -\left(\frac{2}{p}\right)\left(\frac{2s+1}{p}\right)$$

or $\left(\frac{2}{p}\right) = -1$, which contradicts the second supplement to the law of quadratic reciprocity.

Case II: Assume that $p \equiv 1 \pmod 8$ and $l/4 \le s < l/2$. Set $t := l/2 - s$. Then, we have $0 < 2t < 4t \le l$. By (b) of Lemma 23, we have

$$b_{l,s}(2t) = -b_{l,s}(4t).$$

Therefore, by our assumption, we have

$$\left(\frac{2t}{p}\right) = -\left(\frac{4t}{p}\right) = -\left(\frac{2}{p}\right)\left(\frac{2t}{p}\right)$$

or $\left(\frac{2}{p}\right) = -1$.

Case III: Assume that $p \equiv 5 \pmod 8$ and $s < l/4$. Then, we have $4s+2 \leq l$. If $s = 0$, then we can easily check that

$$(b_{l,0}(n))_{1 \leq n \leq l+1} = (+1, -1, +1, -1, \dots, +1, -1, +1).$$

So, if $l \geq 4$, then we have $b_{l,0}(4) = -1 \neq \left(\frac{4}{p}\right)$, which is a contradiction. If $l = 2$, then

$$(b_{2,0}(n))_{1 \leq n \leq p-1} = (+1, -1, +1, +1, -1, +1, \dots),$$

in particular, the equality $b_{2,0}(2)b_{2,0}(3) = -b_{2,0}(6)$ holds, and this contradicts $\left(\frac{2}{p}\right)\left(\frac{3}{p}\right) = \left(\frac{6}{p}\right)$. (Recall $p \geq 13$.) Thus, we may assume that $s > 0$. Since

$$b_{l,s}(2s) = b_{l,s}(s+1+(s-1)) = b_{l,s}(s-(s-1)) = b_{l,s}(1) = 1,$$

we have

$$b_{l,s}(s) = \left(\frac{s}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{2s}{p}\right) = -b_{l,s}(2s) = -1.$$

Hence, by (a) of Lemma 23, we have

$$\begin{aligned} b_{l,s}(2s-1) &= -b_{l,s}(2s-1+(2s+1)) = -b_{l,s}(4s) \\ &= -\left(\frac{4s}{p}\right) = -\left(\frac{4}{p}\right)\left(\frac{s}{p}\right) \\ &= -b_{l,s}(s) = 1. \end{aligned}$$

On the other hand,

$$b_{l,s}(2s-1) = b_{l,s}(s+1+(s-2)) = b_{l,s}(s-(s-2)) = b_{l,s}(2) = \left(\frac{2}{p}\right) = -1.$$

Therefore, we arrive at a contradiction.

Case IV: Assume that $p \equiv 5 \pmod 8$ and $l/4 \leq s < l/2$. Set $t := l/2 - s$. Then, we have $0 < t < 4t \leq l$. If $t = 1$, then we can easily check that

$$\begin{aligned} &(b_{l,l/2-1}(n))_{1 \leq n \leq l+1} \\ &= \begin{cases} (\underline{+1, +1, -1, -1}, \dots, \underline{+1, +1, -1, -1}, -1), & \text{if } l \equiv 0 \pmod 4; \\ (+1, \underline{-1, -1, +1, +1}, \dots, \underline{-1, -1, +1, +1}, -1, +1), & \text{if } l \equiv 2 \pmod 4). \end{cases} \end{aligned}$$

Here, the underlined parts indicate the repeated patterns. For the case where $l \equiv 0 \pmod 4$, the equality $b_{l,l/2-1}(2) = +1$ contradicts the second supplement. For the case where $l \equiv 2$

(mod 4), we have $b_{l,l/2-1}(2)b_{l,l/2-1}(3) = -b_{l,l/2-1}(6)$, and this contradicts $\left(\frac{2}{p}\right)\left(\frac{3}{p}\right) = \left(\frac{6}{p}\right)$. (Note that $l \geq 4t = 4$.) If $t = 2$ and $l \geq 10$, then by (b) of Lemma 23, we have

$$b_{l,l/2-2}(10) = -b_{l,l/2-2}(6) = b_{l,l/2-2}(2) = \left(\frac{2}{p}\right) = -1$$

and $b_{l,l/2-2}(5) = -b_{l,l/2-2}(1) = -1$. These lead to $\left(\frac{2}{p}\right)\left(\frac{5}{p}\right) \neq \left(\frac{10}{p}\right)$. We can easily check that

$$(b_{8,2}(n))_{1\leq n\leq 9} = (+1, +1, +1, +1, -1, -1, -1, -1, -1),$$

which leads to $\left(\frac{2}{p}\right) = +1$. (Note that $l \geq 4t = 8$.) Thus, we may assume that $t > 2$. By (c) of Lemma 23, we have

$$b_{l,s}(2t-2) = b_{l,s}(t-(2-t)) = b_{l,s}(t+(2-t)) = b_{l,s}(2) = \left(\frac{2}{p}\right) = -1$$

and

$$\left(\frac{t-1}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{2t-2}{p}\right) = -b_{l,s}(2t-2) = 1.$$

Therefore,

$$b_{l,s}(4t-4) = \left(\frac{4t-4}{p}\right) = \left(\frac{4}{p}\right)\left(\frac{t-1}{p}\right) = 1.$$

On the other hand, by (b) and (c) of Lemma 23,

$$\begin{aligned} b_{l,s}(4t-4) &= b_{l,s}(2t-4+2t) = -b_{l,s}(2t-4) \\ &= -b_{l,s}(t+(t-4)) = -b_{l,s}(t-(t-4)) \\ &= -b_{l,s}(4) = -\left(\frac{4}{p}\right) = -1, \end{aligned}$$

which leads to a contradiction.

Since we see that contradictions arise in all cases, the proof of the main theorem has been completed. □

Remark 24. In the proof for the case $l \geq 2$, we have not fully utilized the fact that the Legendre symbol is a completely multiplicative function, and only used its multiplicativity within a limited range. Neither is the proof critically dependent on any other properties of the Legendre symbol. In fact, we have used the second supplement to the law of quadratic reciprocity only for the case analysis to determine whether $a_{p,l}(2)$ is $+1$ or -1. Therefore, we have actually obtained the following theorem concerning the property of the sequence $(a_{p,l}(n))_{1\leq n\leq p-1}$.

Theorem 25. *Let $p \geq 13$ be a prime number such that $p \equiv 1 \pmod 4$. Let l be an even integer such that $2 \leq l \leq p-3$. Then there exists a positive integer m such that $2m \leq p-3$ and $a_{p,l}(2m) \neq a_{p,l}(2)a_{p,l}(m)$.*

5 Computational results

In this section, we present the details of the computations regarding the N_k values along with the computational results that support Conjecture 4. After we determined the exact N_k values for k up to 10^7 (Theorem 26), we further explored the finiteness of N_k using a sieving algorithm to obtain the upper bound of N_k for k up to 10^{14} (Theorem 29). We can easily generalize these algorithms to the case of arbitrary l, and the latter algorithm for general (k, l) particularly inspired the discovery of our main result.

5.1 Determining exact values and upper bounds of N_k

We extended the calculations of N_k, which was previously only listed up to $k \le 61$ in A108394, to obtain the following results:

Theorem 26. *For all $2 \le k \le 10^7$, the values of N_k are explicitly determined (see our accompanying repository [3]), and the largest N_k in this range is $N_{2600725} = 9011$. Furthermore, the value N_k is a prime number for 8649270 (86.49%) of 9999999 possible values of $k \le 10^7$.*

We computed the exact values of N_k according to Algorithm 1, which is explained in detail in Section 5.3.

Remark 27. After the first version of our article had been posted to arXiv, a short Mathematica code for calculating N_k by Buck, Motley, and Wagon was made publicly available at A108394. The exact values of N_k for k up to 10^5 are also available at the same site as of the final revision of our article.

We tabulate the first 360 terms of N_k in Table 1. The full data of computed N_k values are available at our accompanying repository [3].

It appears that the typical values of N_k are larger for $k \equiv 1 \pmod 2$ than for the other case, and the value N_k seems to get even larger for $k \equiv 1 \pmod 6$. Here we let $\overline{N}_{a \bmod d}$ denote the arithmetic mean of the exact values of N_k with $2 \le k \le 10^7$ and $k \equiv a \pmod d$. We found that $\overline{N}_{0 \bmod 2} \approx 47.4$, $\overline{N}_{1 \bmod 2} \approx 263.2$, and $\overline{N}_{1 \bmod 6} \approx 383.1$. Under the condition $k \equiv 1 \pmod 6$, we found almost no correlation between the values of N_k and whether k is a prime number. When N_k values are grouped by whether k is a prime, the arithmetic mean of N_k for $k \equiv 1 \pmod 6$ with $2 \le k \le 10^7$ is approximately 382.2 for 332194 prime values of k and 383.3 for 1334472 composite values of k. From these numerical results, several questions naturally arise. First, does $\overline{N}_{1 \bmod d} \ge \overline{N}_{a \bmod d}$ hold for every a? We found that it is not true: for integer d with $2 \le d \le 10^3$, there are only 204 values of d such that $\overline{N}_{1 \bmod d} = \max_{0 \le a < d} \overline{N}_{a \bmod d}$. However, for the remaining 795 cases, we still have $\overline{N}_{1 \bmod d} > 0.9486 \times \max_{0 \le a < d} \overline{N}_{a \bmod d}$. Therefore, although the rigorous answer to the question is false, the quantity $\overline{N}_{1 \bmod d}$ is still close to the maximum in many cases. Second, does $\overline{N}_{1 \bmod a} < \overline{N}_{1 \bmod b}$ hold for every pair (a, b) of integers greater than 1, with $a \ne b$ and $a \mid b$? The answer is again negative in a strict sense: among the 5070 pairs of (a, b) with $a, b \le 10^3$ that satisfy the conditions above, the inequality in question is false in 201 cases.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	∞	∞	43	89	97	214	19	239	37	79	83	239	31	431	19	79	23	827
18	43	173	31	103	94	73	19	243	141	101	53	811	47	1077	19	251	29	311
36	134	71	23	86	43	47	19	419	31	191	83	337	59	1559	19	127	109	163
54	67	353	83	191	83	107	19	503	29	191	47	83	51	1907	19	131	37	137
72	31	214	31	127	47	443	19	173	31	227	23	337	83	563	19	47	166	487
90	29	89	83	79	137	73	19	2039	62	218	59	127	31	81	19	239	37	71
108	46	167	31	457	101	179	19	173	37	179	29	191	67	563	19	86	43	151
126	23	101	43	81	59	139	19	47	31	249	46	101	83	647	19	179	25	103
144	43	486	29	83	23	167	19	167	37	331	53	167	47	167	19	25	59	326
162	31	191	31	79	43	73	19	479	23	79	47	359	29	359	19	71	37	47
180	97	839	61	431	46	227	19	827	37	241	159	118	23	167	19	103	97	179
198	47	131	31	127	29	254	19	251	46	137	43	331	79	479	19	239	23	163
216	47	214	47	347	83	307	19	251	31	47	173	101	43	83	19	229	173	751
234	113	191	23	101	53	73	19	1149	61	79	47	103	59	71	19	79	37	173
252	31	191	31	251	83	201	19	233	31	499	47	313	47	359	19	89	46	139
270	43	47	46	151	59	151	19	863	25	223	23	614	31	191	19	163	29	173
288	53	431	31	81	43	311	19	179	37	103	101	129	113	1559	19	127	59	331
306	34	227	47	179	47	73	19	227	29	158	47	47	46	179	19	79	37	167
324	23	491	109	79	141	131	19	479	37	86	43	193	47	101	19	223	47	129
342	29	137	31	311	23	103	19	563	31	169	47	127	34	89	19	337	37	167
$\vdots$						$\vdots$							$\vdots$					

Table 1: Computed values of N_k for k-Göbel sequences with k up to 359. Each entry represents the value of N_k for k equal to the sum of the numbers at the top and left. Note that $N_k = 19$ for $k \equiv 6, 14 \pmod{18}$.

Nevertheless, for all the 201 cases where the inequality does not hold, it is still true that $0.9824 \times \overline{N}_{1 \bmod a} < \overline{N}_{1 \bmod b}$. Even when we add the condition that b/a should be a prime number for a pair (a, b), the inequality is false in just 159 cases among 1958 possible pairs of (a, b). Therefore, we may conclude that our second question seems true in an approximate sense. The pairs (a, b) for which $\overline{N}_{1 \bmod a} > \overline{N}_{1 \bmod b}$ with $a \mid b$ include $(8, 16)$, $(16, 32)$, $(24, 48)$, and $(243, 729)$. Based on the hypothesis that a greater variety of prime factors in d leads to a larger N_k value, we calculated $\overline{N}_{1 \bmod p\#}$ for prime p up to 13 and obtained the following:

$$\overline{N}_{1 \bmod 3\#} < \overline{N}_{1 \bmod 5\#} < \overline{N}_{1 \bmod 7\#} < \overline{N}_{1 \bmod 11\#} < \overline{N}_{1 \bmod 13\#} \approx 600.3,$$

where $p\#$ denotes the primorial of p. Within the range $2 \leq k \leq 10^7$, we have only 333 values of k satisfying $k \equiv 1 \pmod{13\#}$. We additionally calculated $N_{c(p\#)+1}$ with p prime up to 29 for $1 \leq c \leq 3000$ by Algorithm 1. Letting $\overline{\overline{N}}_{1 \bmod p\#}$ denote the arithmetic mean of these 3000 terms, we obtained the following:

$$\overline{\overline{N}}_{1 \bmod p\#} < \overline{\overline{N}}_{1 \bmod q\#}$$

for all primes p and q with $2 \leq p < q \leq 29$, and we have $\overline{\overline{N}}_{1 \bmod 29\#} \approx 753.5$. We are currently unable to explain this phenomenon theoretically. Nevertheless, it could be a topic of future research to explore the set of k for which the value of N_k is likely to be large.

Remark 28. After we had posted the first version of our article to arXiv, we [5] succeeded in

k	N_k		k	N_k		k	N_k	
2	43	prime	49	1559	prime	107161	4463	prime
3	89	prime	67	1907	prime	121801	5507	prime
4	97	prime	97	2039	prime	707197	5879	prime
5	214	composite	1441	2339	prime	832321	7127	prime
7	239	prime	4189	2589	composite	1412161	7883	prime
13	431	prime	5581	2687	prime	2600725	9011	prime
17	827	prime	8209	2939	prime			
31	1077	composite	13201	4139	prime			

Table 2: All values of k with $2 \leq k \leq 10^7$ such that $N_{k'} < N_k$ holds for all k' satisfying $2 \leq k' < k$, along with the corresponding values of N_k and whether N_k is a prime number.

proving that $\sup_{k \geq 2} N_k = \infty$. In the proof, we consider congruences modulo $m!/m\#$, rather than modulo $p\#$, and show that if $k \equiv 1 \pmod{m!/m\#}$, then $N_k > m$.

We found that N_k is a prime number for 8649270 (86.49%) of 9999999 possible values of k in this range. Matsuhira, Matsusaka, and Tsuchida [6, Section 3] stated that $\{N_k \mid k \geq 2\} = \{19, 23, 31, 37, 43, \dots\}$ based on the data in A108394, and observed that $N_{142} = 25$. We also found $N_{306} = 34$ and a wide variety of composite N_k values: among the 621 values of $N_k \leq 1000$ for $2 \leq k \leq 10^7$, only 160 are prime, while the other 461 are composite. We note that 99 of these 461 composite numbers occur less than 10 times in nearly 10^7 data, and 27 of them occur only once. It is possible that additional composite numbers less than or equal to 1000, which did not occur in our data, may start to appear as we calculate more N_k values.

In Table 2, we present the "new records" of N_k that have appeared as k increments. From the table, we can see that the growth of the largest N_k value as k increases is rather slow. Additionally, we found $N_{1984 \times 29\# + 1} = 10007$ in the exact determination of $\overline{\overline{N}}_{1 \bmod 29\#}$. We note that $1984 \times 29\# = 2^6 \times 31\#$.

If we limit ourselves to the upper bound estimation of N_k, we can obtain further results regarding the non-integrality of the k-Göbel sequences. Once we identify the values of k for a given prime p that result in non-integrality of the k-Göbel sequences, we can sieve out the values of k' that satisfy $k' \equiv k \pmod{p-1}$ by Fermat's little theorem. If a range of integers are all sieved out by a finite number of congruence classes with guaranteed non-integrality, we can verify that N_k is bounded for that range of k. We conclude the following from our computation in this approach.

Theorem 29. *For all $2 \leq k \leq 10^{14}$, we have $N_k \leq 29363$.*

Refer to Section 5.3 for an explanation of our algorithm for this result.

p	$\#J_p$	l_L	l_R	p	$\#J_p$	l_L	l_R	p	$\#J_p$	l_L	l_R	p	$\#J_p$	l_L	l_R
13	3	4	10	89	38	8	84	173	80	4	164	269	122	16	260
17	3	8	14	97	43	8	94	181	78	14	170	277	114	22	250
29	8	12	28	101	42	4	88	193	85	6	176	281	116	16	248
37	12	10	34	109	45	12	102	197	91	14	196	293	127	30	284
41	14	8	36	113	41	30	112	229	103	12	218	313	136	2	274
53	21	10	52	137	60	8	128	233	102	14	218	317	149	10	308
61	22	12	56	149	64	20	148	241	106	24	236	337	161	6	328
73	27	16	70	157	63	4	130	257	119	8	246	349	162	20	344

Table 3: First 32 values of $\#J_p$, l_{L}, and l_{R} for primes $p \equiv 1 \pmod 4$ with $p \geq 13$.

5.2 Computation of $\#J_p$, l_{L}, and l_{R} for varying $p \equiv 1 \pmod 4$

In Table 3, we present the value of $\#J_p$, l_{L}, and l_{R} as defined in Theorem 1 for the first 32 prime numbers p with $p \geq 13$ and $p \equiv 1 \pmod 4$. Although $l_{\mathrm{L}} > 2$ for the first few dozen primes $p \equiv 1 \pmod 4$, there also exist prime numbers for which $l_{\mathrm{L}} = 2$, as we have noted in Remark 12. In addition, we see the general trend that $\#J_p$ tends to increase with p, although it does not increase strictly monotonically.

In order to investigate the behavior of $\#J_p$ for larger values of p, we computed the exact values of $\#J_p$ for p up to 10^6 based on the algorithm described in Section 5.3. The full data of computed values of $\#J_p$, l_{L}, and l_{R} for $p \equiv 1 \pmod 4$ with $13 \leq p \leq 10^6$ are available at our accompanying repository [3].

In the left panel of Figure 5, we can clearly see that $\#J_p$ steadily increases with p. Since both the axes of the plot are in logarithmic scale in the left panel, the curve of $\#J_p$ against p for larger values of p, which is nearly parallel to the identity line, indicates that $\#J_p$ is approximately proportional to p. By calculating and plotting $\#J_p/p$ against p, we can see the overall level of the constant of proportionality for this relationship. Consequently, we obtained the right panel of Figure 5, which visualizes the tendency of $\#J_p/p$ approaching 0.5 — the highest possible supremum — as p increases, and thus supports Conjecture 4.

5.3 On algorithms

In Algorithm 1, we present the pseudocode for calculating the value of N_k for given values of k. The main part of our algorithm (lines 31–42) works by iteratively determining whether $g_{k,2}(n) \in \mathbb{Z}$ or not under the assumption that $g_{k,2}(i) \in \mathbb{Z}$ for all $1 \leq i < n$. In order to determine the exact value of N_k, it is sufficient to start this procedure from $n = 2$ (line 32), as $g_{k,2}(1) = 2 \in \mathbb{Z}$ holds by definition.

For each iteration of $n_{\max}$, the `CUMULATIVE_PRODUCT` function is used at first to obtain the appropriate initial modulus P (see lines 1–18) with $P = \prod_{p|n} p^{\nu_p(n!)}$, where ν_p denotes the p-adic valuation for a prime number p. Once we obtain the appropriate initial modulus P for $g_{k,2}(1)$, we can determine the residue class modulo n that $ng_{k,2}(n) \in \mathbb{Z}$ belongs to.

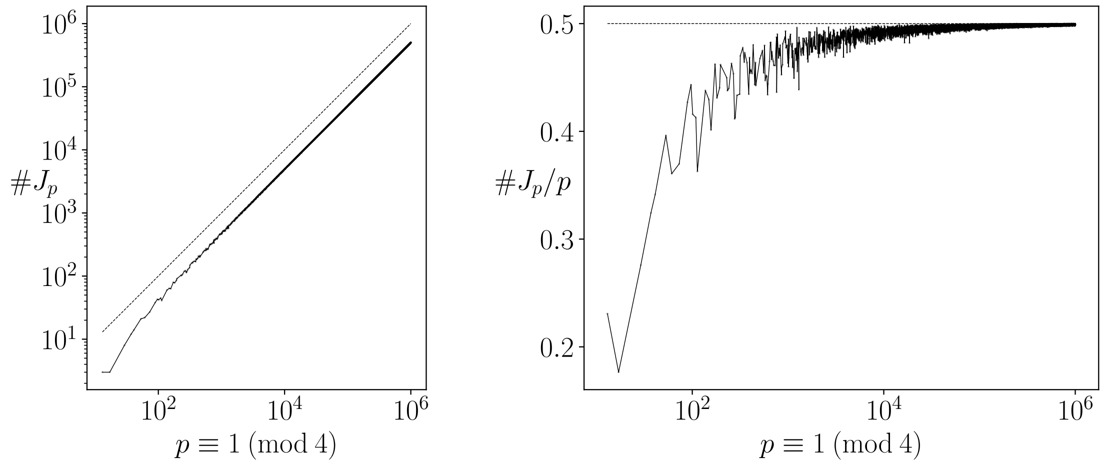

Figure 5: Plots of $\#J_p$ and $\#J_p/p$ against varying prime numbers $p \equiv 1 \pmod 4$. In the left panel, the dashed line represents the identity line, where the values along the horizontal and vertical axes are equal to each other. Note that both the axes are in logarithmic scale. In the right panel, the horizontal dashed line represents the value 0.5 on the vertical axis. Note that the horizontal axis is in logarithmic scale.

This is carried out step-by-step using the algorithm outlined as the `GOBEL_PROCEED` function (lines 20–29). Note that the modulus decreases as the iteration proceeds due to the $(n+1)^{-1}$ term in the recurrence relation. In this iterative process, we can assume the integrality of $g_{k,2}(i)$ for i up to $n-1$, because the sequence has already been confirmed to be integral up to the $(n-1)$-st term when the for-loop of lines 32–41 is executed for $n_{\max} = n$. If the variable g_{mult}, which is a representative of $(n+1)g_{k,2}(n+1)$ modulo d, is not divisible by the greatest common divisor of d and $n+1$ (line 24), the function returns a null result to indicate that the integrality breaks at $n+1$.

The implementation of the function `powmod` is critical to the efficiency of Algorithm 1. Note that this function is called $O((N_k)^2)$ times as $n_{\max}$ increments in the for-loop of lines 32–41. In the previous implementations of calculation of $N_{k,l}$ values by Matsusaka and colleagues [6, 1], the modulo operation was performed *after* the exponentiation. However, because we only need the residue class modulo d, the explicit exponentiation is unnecessary. Instead, we used the built-in `pow` function in Python (version 3.8 or later), which employs left-to-right binary exponentiation and left-to-right k-ary sliding window exponentiation algorithms, depending on whether the bit length of the power exponent exceeds 60. The `invmod` function is also readily available as the `pow` function in Python (version 3.8 or later) by setting the exponent argument to a negative integer and we used it in our implementation. For `factorint` and `gcd`, we used the `ntheory.factorint` function from the `SymPy` package (version 1.12) and the `gcd` function of the built-in `math` module in Python, respectively. For future coding, we also note that the implementation of `CUMULATIVE_PRODUCT` using the big integer type in Python could be more efficient if we instead used the dictionary that

Algorithm 1 Pseudocode for computing N_k for given values of k.

Require: `factorint`(i) returns a dictionary {prime: exponent}, the prime factorization of i
Require: `powmod`(x, y, m) returns the modular exponentiation $x^y \bmod m$ for $x \in \mathbb{Z}, y, m \in \mathbb{Z}_{>0}$
Require: `invmod`(x, m) returns the modular multiplicative inverse mod m of $x \in \mathbb{Z}$
Require: `gcd`(x, y) returns the greatest common divisor of integers x and y

```
L_f, L_c ← List of (pre-determined) length i_max
D ← Empty dictionary
for i ← 1 to i_max do
    L_f[i] ← factorint(i)
    Add to D every key in L_f[i] that is not in D, with the corresponding value 0
    for every key p of L_f[i] do
        D[p] ← D[p] + L_f[i][p]
    end for
    L_c[i] ← D
end for

function CUMULATIVE_PRODUCT(n, L_f, L_c)
    P ← 1
    for every p in keys of L_f[n] do
        P ← P * p^(L_c[n][p])
    end for
    return P
end function

function GOBEL_PROCEED(g, d, n, k)
    g_mult ← ng + powmod(g, k, d)
    m_gcd ← gcd(d, n + 1)
    d_next ← d/m_gcd
    if g_mult is not divisible by m_gcd then
        return Null
    end if
    g_next ← invmod((n + 1)/m_gcd, d_next) * (g_mult/m_gcd)
    return (g_next mod d_next, d_next)
end function

for every k for which N_k is to be determined do          ▷ Main part of the algorithm
    for n_max ← 2 to the length of L_f do
        (g, d) ← (2, CUMULATIVE_PRODUCT(n_max, L_f, L_c))
        for n ← 1 to (n_max − 1) do
            (g, d) ← GOBEL_PROCEED(g, d, n, k)
        end for
        if (g, d) is Null then
            Print (k, n_max)                               ▷ The value of n_max here should be N_k
            break
        end if
    end for
end for
```

represents the prime factorization as an object that represents an integer.

If we are interested only in the non-integrality of (k, l)-Göbel sequences and not in the exact value of $N_{k,l}$, we can perform the numerical search more efficiently. We first describe the case $l = 2$. Recall that $N_k = 19$ for all $k \equiv 6, 14 \pmod{18}$, and the proof consists of two parts: first, the integrality of k-Göbel sequences for all $k \geq 2$ does not break for $n < 19$ and second, $19g_{k,2}(19) \not\equiv 0 \pmod{19}$ for $k \equiv 6, 14 \pmod{18}$. Omitting the first part of the proof, we still obtain $N_k \leq 19$ for all $k \equiv 6, 14 \pmod{18}$. In general cases, the appropriate initial modulus P can become very large for composite n. However, because $P = n$ for prime n, it is comparatively easy to complete all the possible cases modulo P. Moreover, the computational result $N_k \leq p$ for $k = a$ extends to all $k \equiv a \pmod{p-1}$ by Fermat's little theorem. Therefore, for a given range of integers k, we can progressively sieve out $k \equiv a \pmod{p-1}$ for a such that $pg_{a,2}(p) \not\equiv 0 \pmod{p}$. We can also make the algorithm more efficient in the case of prime n, as it is guaranteed that $m_{\text{gcd}} = 1$ for $n < n_{\max}$ at line 22 of Algorithm 1.

In the actual computation, we precomputed the residue classes to be sieved out from k for prime $p < 30000$. We first sieved $k \leq 10^{14}$ by the residue classes modulo $(p-1)$ for prime $p < 15000$, which left 132603 integers greater than 1. This calculation took approximately 22 days with a single NVIDIA RTX A6000 GPU and the `Tensor` class from `PyTorch` package. We then subjected the resulting integers to sequential sieving to obtain 2810 and 56 remaining integers after sieving by the residue classes modulo $(p-1)$ for prime $p < 20000$ and $p < 25000$, respectively. The last integer, 26626531900321, was sieved out by 26059 modulo 29362. The full data and actual codes are available at our accompanying repository [3].

We took almost the same approach to study the (k, l)-Göbel sequences computationally. In this case, it is sufficient to replace the initial value 2 for g with the parameter l at line 33 of Algorithm 1. We used this modified algorithm to obtain the values of (k, l) such that $pg_{k,l}(p) \not\equiv 0 \pmod{p}$ for prime p up to 3000. The result was used to produce Figure 1.

Lastly, we shortly describe the algorithm for computing l_{L} and l_{R}. According to Theorem 1, if there exists an $l \in J_p$, then $l_{\mathrm{L}} \leq l < l_{\mathrm{R}}$. Furthermore, $0 \in I_p^{\mathrm{L}}$ and $p - 1 \in I_p^{\mathrm{R}}$ always hold. Therefore, we can apply the binary search to determine l_{L} and l_{R} efficiently once we find an $l \in J_p$. We computed the actual values of l_{L} and l_{R} for $p \equiv 1 \pmod{4}$ up to 10^6. We calculated the cardinality of J_p by $\#J_p = (l_{\mathrm{R}} - l_{\mathrm{L}})/2$.

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