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Determinantal Identities for Exceedances and Fixed Points in Permutations

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Abstract

We develop a determinantal approach to study permutation statistics involving exceedances and fixed points. A matrix encoding allows us to express a signed generating function over the symmetric group as a determinant. We derive an exact recurrence relation for this determinant, provide explicit formulas, and obtain several identities. Special cases include fixed points, exceedances, and derangements. While the resulting closed-form polynomials connect with known cycle-weight specializations (i.e., polynomial evaluations tracking cycle statistics alongside exceedances and fixed points), our approach provides a direct determinantal construction and a systematic framework for signed factorial-moment sums (sums over the symmetric group weighted by permutation signatures and falling factorials of joint permutation statistics).

1 Introduction

The study of permutation statistics is a cornerstone of enumerative combinatorics, rooted in the classical works of Euler and MacMahon [3]. Among these statistics, the number of fixed

points and the number of exceedances play a primary role, appearing in various contexts ranging from the theory of derangements to the study of Eulerian polynomials.

For a permutation σ in the symmetric group $\mathfrak{S}_n$ with $n \in \mathbb{N}^*$ ($\mathbb{N}$ denoting the set of non-negative integers), a fixed point occurs when an element's value remains unchanged by the transformation:

$$\mathrm{fix}(\sigma) = |\{i \in \{1, \ldots, n\} : \sigma(i) = i\}|,$$

while an exceedance represents a position where the value increases:

$$\mathrm{exc}(\sigma) = |\{i \in \{1, \ldots, n\} : \sigma(i) > i\}|.$$

The statistics exc and fix are classical and have been extensively studied [2, 13]. While the distributions of these statistics are well-known individually, their joint distribution often involves complex algebraic manipulations. Signed enumeration of permutation statistics, where permutations are weighted by their signature $\varepsilon(\sigma) = (-1)^{\mathrm{inv}(\sigma)} = (-1)^{n-\mathrm{cyc}(\sigma)}$ (with $\mathrm{cyc}(\sigma)$ denoting the number of disjoint cycles), has also been studied in connection with Eulerian statistics and derangements (see [7, 9]). Generating function methods play a central role in this area (see [4]). Traditionally, generating functions for such statistics are derived using combinatorial bijections or symbolic methods. Determinantal approaches to signed exceedance enumeration have previously been considered, notably by Sivasubramanian [11], where determinant expansions encode exceedance statistics through refined weightings on permutation matrices.

The present work considers a bivariate determinant whose entries encode exceedances and fixed points simultaneously. Elementary row operations yield a first-order recurrence, from which we derive an explicit formula and identities for mixed factorial moments (i.e., expectations or sums weighted by falling factorials of the statistics, such as $(\mathrm{exc}(\sigma))_k(\mathrm{fix}(\sigma))_\ell$). The relationship with earlier determinant-based approaches to signed exceedance enumeration, as well as with cycle-weighted enumerators (polynomials tracking cycles, exceedances, and fixed points simultaneously), provides the context for these results. The contribution of the paper is the particular bivariate encoding and the systematic derivation of the resulting identities in a unified framework for signed joint factorial moments (i.e., signed sums involving falling factorials of multiple joint statistics over the symmetric group). Although the approach successfully recovers known results, the present work extends beyond the general methodology to include novel identities (especially in Section 4) which, to the best of our knowledge, have never been published before.

The paper is organized as follows. In Section 2, we introduce the determinantal encoding and show that the signed bivariate generating function for exceedances and fixed points can be expressed as the determinant of the matrix $M_{x,y}$. We then derive a recurrence relation satisfied by $D_n(x, y) = \det(M_{x,y})$ through elementary row operations. In Section 3, we study the associated exponential generating function and obtain explicit closed formulas for $D_n(x, y)$, together with several specializations related to fixed points, exceedances, and derangements. In Section 4, we derive identities obtained by differentiation of the determinantal expression, providing a systematic framework for signed factorial-moment sums

and higher factorial moments (i.e., factorial moments of order greater than one, measuring higher-order joint variations and correlations of exceedances and fixed points).

2 Determinantal encoding and reduction of the matrix

For real parameters x and y, and for $n \in \mathbb{N}^*$, define the matrix $M_{x,y} = (a_{i,j})_{1 \le i,j \le n}$ by

$$a_{i,j} = \begin{cases} y, & \text{if } j = i; \\ x, & \text{if } j > i; \\ 1, & \text{if } j < i. \end{cases}$$

Lemma 1. *Let $n \in \mathbb{N}^*$ and let (x, y) be a pair of real numbers. We have*

$$\det(M_{x,y}) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) x^{\mathrm{exc}(\sigma)} y^{\mathrm{fix}(\sigma)}.$$

Proof. By the Leibniz formula, we can write

$$\det(M_{x,y}) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

For each index i, the factor $a_{i,\sigma(i)}$ equals x if $\sigma(i) > i$, equals y if $\sigma(i) = i$, and equals 1 if $\sigma(i) < i$. Therefore,

$$\prod_{i=1}^{n} a_{i,\sigma(i)} = x^{\mathrm{exc}(\sigma)} y^{\mathrm{fix}(\sigma)},$$

which proves the result. □

We perform the elementary row operations denoted by $L_i \leftarrow L_i - L_{i+1}$ (where L_i represents the i-th row of the matrix)

$$L_i \leftarrow L_i - L_{i+1}, \quad \text{for } i = 1, \ldots, n-1.$$

These operations do not change the determinant. After these transformations, the matrix becomes

$$\begin{pmatrix} y-1 & x-y & 0 & \cdots & 0 \\ 0 & y-1 & x-y & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & y-1 & x-y \\ 1 & 1 & \cdots & 1 & y \end{pmatrix}.$$

Let $D_n(x, y) = \det(M_{x,y})$. We derive a recurrence for $D_n(x, y)$. Throughout the article, we sometimes omit the dependence on (x, y) for the sake of clarity.

Theorem 2. *Let x and y be real numbers. For all natural numbers $n \geq 2$, we have*

$$D_n = (y-1)D_{n-1} + (y-x)^{n-1}, \tag{1}$$

with initial conditions $D_0 = 1$ and $D_1 = y$.

Proof. We expand the transformed matrix along the first column. The first entry contributes $(y-1)D_{n-1}$ since the corresponding minor has the same structure. The last row contributes a cofactor of sign $(-1)^{n+1}$ multiplied by the determinant of a lower triangular matrix with diagonal entries $x - y$. Hence this contribution is $(y-x)^{n-1}$. Summing the contributions gives the recurrence. □

The recurrence of Theorem 2 shows that $D_n(x, y)$ is a polynomial of degree n in y and degree $n-1$ in x.

3 Exponential generating function and analytic structure

3.1 Differential equation for the generating function

Theorem 3. *Let $x, y \in \mathbb{R}$ and let t be a real variable. Define*

$$F(t) = \sum_{n \geq 0} D_n(x, y) \frac{t^n}{n!},$$

where $D_0 = 1$, to be the exponential generating function of the sequence $\big(D_n(x, y)\big)_{n \geq 0}$. Then $F(t)$ satisfies the first-order linear differential equation

$$F'(t) = (y-1)F(t) + e^{(y-x)t},$$

with the initial condition $F(0) = 1$.

Proof. We start from the recurrence relation (see Eq. (1)) established in Theorem 2:

$$D_n = (y-1)D_{n-1} + (y-x)^{n-1}, \quad n \geq 2,$$

with $D_0 = 1$ and $D_1 = y$. The recurrence also holds for $n = 1$, since

$$D_1 = (y-1)D_0 + (y-x)^0 = (y-1) + 1 = y.$$

Hence, for all $n \geq 1$,

$$D_n = (y-1)D_{n-1} + (y-x)^{n-1}.$$

Multiplying by $\frac{t^{n-1}}{(n-1)!}$ and summing over $n \geq 1$ yields

$$\sum_{n\geq 1} D_n \frac{t^{n-1}}{(n-1)!} = (y-1)\sum_{n\geq 1} D_{n-1}\frac{t^{n-1}}{(n-1)!} + \sum_{n\geq 1}(y-x)^{n-1}\frac{t^{n-1}}{(n-1)!}.$$

The left-hand side is $F'(t)$. The first term on the right is $(y-1)F(t)$. For the second term in the right-hand side, we have

$$\sum_{n\geq 1} \frac{(-(x-y)t)^{n-1}}{(n-1)!} = \sum_{m\geq 0} \frac{((y-x)t)^m}{m!} = e^{(y-x)t}.$$

Thus, the equation holds for all $n \geq 1$, leading to

$$F'(t) = (y-1)F(t) + e^{(y-x)t}.$$

□

3.2 Explicit solution

Proposition 4. *Let $x, y \in \mathbb{R}$ and let t be a real variable. For $x \neq 1$, the exponential generating function $F(t) = \sum_{n\geq 0} D_n(x,y)\frac{t^n}{n!}$ is equal to*

$$F(t) = \frac{xe^{(y-1)t} - e^{(y-x)t}}{x-1}.$$

Proof. From Theorem 3, the exponential generating function satisfies

$$F'(t) - (y-1)F(t) = e^{(y-x)t},$$

with initial condition

$$F(0) = 1.$$

This is a first-order linear differential equation. Multiplying both sides by the integrating factor $e^{-(y-1)t}$, we obtain

$$e^{-(y-1)t}F'(t) - (y-1)e^{-(y-1)t}F(t) = e^{(y-x)t}e^{-(y-1)t}.$$

Hence

$$\frac{d}{dt}\left(F(t)e^{-(y-1)t}\right) = e^{(1-x)t}.$$

Integrating from 0 to t gives

$$F(t)e^{-(y-1)t} - F(0) = \int_0^t e^{(1-x)u}\,du.$$

Since $F(0) = 1$, we get

$$F(t)e^{-(y-1)t} = 1 + \frac{e^{(1-x)t} - 1}{1-x},$$

that is,

$$F(t) = \frac{xe^{(y-1)t} - e^{(y-x)t}}{x-1}.$$

□

This explicit form matches the specialization at $\beta = -1$ of the trivariate cycle enumerator $P_n(x, y, \beta) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{exc}(\sigma)} y^{\mathrm{fix}(\sigma)} \beta^{\mathrm{cyc}(\sigma)}$ obtained by Ksavrelof and Zeng [5] (page 2, equation (3)), noting that $D_n(x, y) = (-1)^n P_n(x, y, -1)$. More generally, Ma [6] gives (page 416, equation (1.2)) the multivariate generating function:

$$\sum_{n \geq 0} P_n(x, y, q) \frac{t^n}{n!} = \left(\frac{e^{t(x-y)} - xe^{t(1-y)}}{1-x} \right)^{-q}.$$

Setting $q = -1$ and replacing t by $-t$ produces the generating function of Proposition 4. The explicit expression shows that $x = 1$ corresponds to a removable singularity, and the exponential generating function extends smoothly to this point, yielding $(1 + t)e^{(y-1)t}$.

3.3 A few specializations

The explicit formula of the exponential generating function presented in Proposition 4 allows us to recover several classical signed statistics for integers $n \geq 1$:

- **Signed fixed-point enumerator ($x = 1$):**

  Taking the limit $x \to 1$ using L'Hôpital's rule gives

  $$F(t) = (1 + t)e^{(y-1)t}.$$

  Expanding the exponential generating function, we obtain

  $$(1 + t)e^{(y-1)t} = 1 + \sum_{n \geq 1} (y-1)^{n-1}(y + n - 1) \frac{t^n}{n!}.$$

  Therefore, for $n \geq 1$,

  $$D_n(1, y) = (y-1)^{n-1}(y + n - 1).$$

- **Signed exceedance generating function ($y = 1$):**

  Setting $y = 1$ yields

  $$F(t) = \frac{x - e^{(1-x)t}}{x-1}.$$

Expanding the exponential generating function gives

$$F(t) = 1 + \sum_{n\geq 1} (-1)^{n+1}(x-1)^{n-1}\frac{t^n}{n!}.$$

Therefore, for $n \geq 1$, as studied in connection with classical signed Eulerian distributions [9]:

$$D_n(x,1) = (-1)^{n+1}(x-1)^{n-1}.$$

- **Alternating exceedance sum ($(x,y) = (-1,1)$):**

  We obtain

$$F(t) = \frac{1+e^{2t}}{2} = 1 + \sum_{n\geq 1} 2^{n-1}\frac{t^n}{n!},$$

  which immediately gives

$$D_n(-1,1) = 2^{n-1}.$$

3.4 Closed-form for the coefficients

Theorem 5. *Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. For $n = 0$, $D_0(x,y) = 1$. For $n \geq 1$ and $x \neq 1$:*

$$D_n(x,y) = \frac{x(y-1)^n - (y-x)^n}{x-1}. \tag{2}$$

If $x = 1$ and $n \geq 1$, $D_n(1,y) = (y-1)^{n-1}(y+n-1)$.

For small values of n, one obtains

$$D_1(x,y) = y, \quad D_2(x,y) = y^2 - x, \quad D_3(x,y) = y^3 - 3xy + x^2 + x,$$

$$D_4(x,y) = y^4 - 6xy^2 + 4x^2y + 4xy - x^3 - x^2 - x.$$

Proof. Let $a = y - 1$ and $b = y - x$. We first assume that $x \neq 1$ and $y \neq 1$, so that $a = y - 1 \neq 0$ and $a - b = x - 1 \neq 0$. The recurrence from Theorem 2 becomes

$$D_n = aD_{n-1} + b^{n-1}, \qquad n \geq 2,$$

with $D_1 = y$. Dividing both sides by a^n yields

$$\frac{D_n}{a^n} - \frac{D_{n-1}}{a^{n-1}} = \frac{b^{n-1}}{a^n} = \frac{1}{a}\left(\frac{b}{a}\right)^{n-1}.$$

Summing this telescoping relation from $k = 2$ to n, we obtain

$$\frac{D_n}{a^n} - \frac{D_1}{a} = \frac{1}{a}\sum_{k=1}^{n-1}\left(\frac{b}{a}\right)^k,$$

which gives

$$D_n(x,y) = ya^{n-1} + \frac{ba^{n-1} - b^n}{x-1} = \frac{y(x-1)a^{n-1} + ba^{n-1} - b^n}{x-1}.$$

Substituting a and b back gives

$$D_n(x,y) = \frac{xa^n - b^n}{x-1} = \frac{x(y-1)^n - (y-x)^n}{x-1}.$$

The resulting identity extends to $y = 1$ by continuity. For $x = 1$, taking the limit as $x \to 1$ in the closed formula gives

$$D_n(1,y) = (y-1)^n + n(y-1)^{n-1} = (y-1)^{n-1}(y+n-1), \qquad n \geq 1.$$

This completes the proof. □

3.5 Sum rules for special values of the arguments

Specializing x and/or y using Eq. (2) yields the following identities for $n \in \mathbb{N}^*$:

1. **Total signed sum:**

   The expression of $D_n(1,1)$ gives

   $$\sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n \geq 2. \end{cases}$$

2. **Signed fixed-point enumerator:**

   Similarly, the expression of $D_n(1,y)$ yields

   $$\sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\, y^{\mathrm{fix}(\sigma)} = (y-1)^{n-1}(y+n-1).$$

3. **Exceedance generating function (signed):**

   For $y = 1$ and any value of x, $D_n(x,1)$ enables one to write

   $$\sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\, x^{\mathrm{exc}(\sigma)} = (-1)^{n+1}(x-1)^{n-1},$$

   which is related to the combinatorial distribution studies of anti-exceedances (for a permutation σ of the set $[n] = \{1, 2, \ldots, n\}$, an anti-exceedance is an index $i \in [n]$ such that $\sigma(i) < i$) and parities by Mantaci [8] and Rakotondrajao [10].

4. **Zero-exceedance contribution:**

   The quantity $D_n(0,1)$, which is equal to

$$\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \mathrm{exc}(\sigma)=0}} \varepsilon(\sigma) = 1,$$

   counts signed permutations with no exceedances (this is a trivial consequence, as the identity permutation is the unique permutation with zero exceedances).

5. **Derangement-weighted sum:**

   For $n \geq 2$ and $x \neq 1$, $D_n(x,0)$ provides the sum (strengthened in the broader context of derangement involutions by Alexandersson and Kebede [1]):

$$\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \mathrm{fix}(\sigma)=0}} \varepsilon(\sigma)\, x^{\mathrm{exc}(\sigma)} = (-1)^n \frac{x - x^n}{x-1} = (-1)^{n+1} x(1 + x + \cdots + x^{n-2}).$$

   The expression extends polynomially to $x = 1$.

6. **Alternating exceedance sum:**

   Finally, we have the alternating exceedance sum $D_n(-1,1)$:

$$\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)(-1)^{\mathrm{exc}(\sigma)} = 2^{n-1},$$

   which is a corollary of item 3 above (taking $x = -1$).

4 Derivatives and factorial moments

4.1 General mixed derivatives

Proposition 6 (Mixed derivatives of $D_n(x,y)$ and factorial moments)**.** *For all integers $k, \ell \geq 0$, the mixed derivatives of $D_n(x,y)$ satisfy*

$$\frac{\partial^{k+\ell} D_n}{\partial x^k \partial y^\ell}(x,y) = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \mathrm{exc}(\sigma)\geq k\\ \mathrm{fix}(\sigma)\geq \ell}} \varepsilon(\sigma)\, (\mathrm{exc}(\sigma))_k\, (\mathrm{fix}(\sigma))_\ell\, x^{\mathrm{exc}(\sigma)-k} y^{\mathrm{fix}(\sigma)-\ell},$$

where $(m)_k = m(m-1)\cdots(m-k+1)$ denotes the falling factorial.

4.2 Joint sum rules for $\mathrm{exc}(\sigma)$ and $\mathrm{fix}(\sigma)$

Proposition 7 (General sum rule from mixed derivatives)**.** *Let $n \geq 1$ and $k, \ell \geq 0$ be integers, and let $x \in \mathbb{R} \setminus \{1\}$ and $y \in \mathbb{R}$. We have*

$$\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) \geq k \\ \mathrm{fix}(\sigma) \geq \ell}} \varepsilon(\sigma)\,(\mathrm{exc}(\sigma))_k\,(\mathrm{fix}(\sigma))_\ell\, x^{\mathrm{exc}(\sigma)-k} y^{\mathrm{fix}(\sigma)-\ell} = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{j!}{(x-1)^{j+1}} \times \Big(A^{(n)}_{k-j,\ell}(x,y) - B^{(n)}_{k-j,\ell}(x,y) \Big),$$

where the two contributions are defined as follows:

$$A^{(n)}_{r,\ell}(x,y) = \begin{cases} x(n)_\ell (y-1)^{n-\ell}, & \text{if } r = 0 \text{ and } \ell \leq n; \\ (n)_\ell (y-1)^{n-\ell}, & \text{if } r = 1 \text{ and } \ell \leq n; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$B^{(n)}_{r,\ell}(x,y) = \begin{cases} (-1)^r (n)_{r+\ell} (y-x)^{n-r-\ell}, & \text{if } r + \ell \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We write

$$D_n(x,y) = \frac{G(x,y)}{x-1}, \qquad G(x,y) = x(y-1)^n - (y-x)^n.$$

By Leibniz's formula, we get

$$\frac{\partial^k}{\partial x^k}\left(\frac{G(x,y)}{x-1}\right) = \sum_{j=0}^{k} \binom{k}{j} \frac{\partial^j}{\partial x^j}\left(\frac{1}{x-1}\right) \frac{\partial^{k-j}}{\partial x^{k-j}} G(x,y).$$

Moreover,

$$\frac{\partial^j}{\partial x^j}\left(\frac{1}{x-1}\right) = (-1)^j \frac{j!}{(x-1)^{j+1}}.$$

For the term $x(y-1)^n$, taking into account the condition $\ell \leq n$, we have

$$\frac{\partial}{\partial x}(x(y-1)^n) = (y-1)^n, \qquad \frac{\partial^2}{\partial x^2}(x(y-1)^n) = 0,$$

hence

$$\frac{\partial^r}{\partial x^r}\frac{\partial^\ell}{\partial y^\ell}[x(y-1)^n] = \begin{cases} x(n)_\ell (y-1)^{n-\ell}, & \text{if } r = 0 \text{ and } \ell \leq n; \\ (n)_\ell (y-1)^{n-\ell}, & \text{if } r = 1 \text{ and } \ell \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

For the term $(y-x)^n$, the chain rule gives the non-zero contribution $(-1)^r (n)_{r+\ell} (y-x)^{n-r-\ell}$ provided $r + \ell \leq n$, and zero otherwise. Combining both contributions in Leibniz's formula yields the desired result. □

This yields, simply setting $x = a \neq 1$ and $y = 1$, the following corollary:

Corollary 8 (Specialized weighted exceedance and fixed-point identity)**.** *Let $n \geq 1$, $k \geq 0$, and $0 \leq \ell < n$, and let $a \in \mathbb{R}$. Then*

$$\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{exc}(\sigma) \geq k \\ \operatorname{fix}(\sigma) \geq \ell}} \varepsilon(\sigma)\,(\operatorname{exc}(\sigma))_k\,(\operatorname{fix}(\sigma))_\ell\,a^{\operatorname{exc}(\sigma)-k}$$
$$= \begin{cases} (-1)^{n-\ell+1}(n)_\ell(n-\ell-1)_k(a-1)^{n-k-\ell-1}, & k+\ell \leq n-1, \\ 0, & k+\ell \geq n. \end{cases}$$

Here $(m)_r = m(m-1)\cdots(m-r+1)$ denotes the falling factorial, with $(m)_0 = 1$.

Proof. By Proposition 6, the sum on the left-hand side is

$$\left.\frac{\partial^{k+\ell} D_n(x,y)}{\partial x^k\,\partial y^\ell}\right|_{x=a,\,y=1}.$$

We first assume that $a \neq 1$. From Theorem 5, for $x \neq 1$,

$$D_n(x,y) = \frac{x(y-1)^n - (y-x)^n}{x-1}.$$

Since $\ell < n$, we have

$$\left.\frac{\partial^\ell}{\partial y^\ell}(y-1)^n\right|_{y=1} = 0.$$

Consequently,

$$\begin{aligned}\left.\frac{\partial^\ell D_n(x,y)}{\partial y^\ell}\right|_{y=1} &= -\frac{(n)_\ell(1-x)^{n-\ell}}{x-1} \\ &= (-1)^{n-\ell+1}(n)_\ell(x-1)^{n-\ell-1}.\end{aligned}$$

If $k \leq n-\ell-1$, differentiating k times with respect to x gives

$$\left.\frac{\partial^{k+\ell} D_n(x,y)}{\partial x^k\,\partial y^\ell}\right|_{y=1} = (-1)^{n-\ell+1}(n)_\ell(n-\ell-1)_k(x-1)^{n-k-\ell-1}.$$

Evaluating at $x = a$ yields the first case of the stated identity. If $k > n-\ell-1$, the polynomial

$$(-1)^{n-\ell+1}(n)_\ell(x-1)^{n-\ell-1}$$

has degree less than k, so its k-th derivative is zero. This gives the second case. Finally, both sides of the identity are polynomials in a. Since the identity holds for every $a \neq 1$, it also holds at $a = 1$ by polynomial continuity. This completes the proof. □

Remark 9. Equivalently, the factor $(n)_\ell(n-\ell-1)_k$ in the preceding corollary 8 can be expressed as the following alternating sum:

$$(n)_\ell(n-\ell-1)_k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j j!\,(n)_{k+\ell-j}.$$

Corollary 10. *Let $n, k, \ell \in \mathbb{N}^*$ with $\ell < n$. Then we have*

$$\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\,(\mathrm{exc}(\sigma))_k\,(\mathrm{fix}(\sigma))_\ell = \begin{cases} \dfrac{(-1)^k n!}{k+1}, & \text{if } k+\ell = n-1; \\ 0, & \text{if } k+\ell \neq n-1. \end{cases}$$

Proof. Recall that

$$D_n(x,y) = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) x^{\mathrm{exc}(\sigma)} y^{\mathrm{fix}(\sigma)} = \frac{x(y-1)^n - (y-x)^n}{x-1}.$$

Set $u = x-1$ and $v = y-1$. Substituting $x = 1+u$ and $y = 1+v$ gives

$$D_n(1+u, 1+v) = \frac{(1+u)v^n - (v-u)^n}{u}.$$

Expanding $(v-u)^n$ by the binomial theorem, we obtain the key-shifted expansion

$$D_n(1+u,1+v) = v^n + \sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} u^j v^{n-1-j}. \tag{3}$$

On the other hand, differentiating the generating polynomial $D_n(x,y)$ yields

$$\left.\frac{\partial^{k+\ell} D_n(x,y)}{\partial x^k\,\partial y^\ell}\right|_{x=y=1} = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)(\mathrm{exc}(\sigma))_k(\mathrm{fix}(\sigma))_\ell.$$

Since $u = x-1$ and $v = y-1$, this derivative equals $k!\ell!$ times the coefficient of $u^k v^\ell$ in the expansion (3). The term v^n contributes only when $k=0$ and $\ell = n$. Because the corollary assumes $\ell < n$, this term does not contribute. In the remaining sum, a term $u^k v^\ell$ occurs precisely when

$$j = k, \qquad n-1-j = \ell,$$

or equivalently, when $k+\ell = n-1$. In that case, the coefficient is $(-1)^k\binom{n}{k+1}$. Consequently,

$$k!\ell!(-1)^k \binom{n}{k+1} = \frac{(-1)^k n!}{k+1},$$

where we used $k+\ell = n-1$. If $k+\ell \neq n-1$, the coefficient of $u^k v^\ell$ is zero. Therefore,

$$\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)(\mathrm{exc}(\sigma))_k(\mathrm{fix}(\sigma))_\ell = \begin{cases} \dfrac{(-1)^k n!}{k+1}, & \text{if } k+\ell = n-1, \\ 0, & \text{if } k+\ell \neq n-1. \end{cases}$$

This proves the result. □

Corollary 11 (Identity at $(x, y) = (0, b)$)**.** *Let $n \geq 1$, $k \geq 0$, and $0 \leq \ell \leq n$, and let $b \in \mathbb{R}$. Then we have*

$$\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma)=k \\ \mathrm{fix}(\sigma)\geq \ell}} \varepsilon(\sigma)\,(\mathrm{fix}(\sigma))_\ell\, b^{\mathrm{fix}(\sigma)-\ell} = -\mathbf{1}_{\{k\geq 1\}}(n)_\ell (b-1)^{n-\ell} + \sum_{r=0}^{\min(k,n-\ell)} \frac{(-1)^r (n)_{r+\ell}}{r!} b^{n-r-\ell},$$

where $\mathbf{1}_{\{P\}}$ denotes the indicator function equal to 1 if the proposition P is true and 0 otherwise.

Proof. By Proposition 6, setting $x = 0$ isolates the terms with $\mathrm{exc}(\sigma) = k$, since $x^{\mathrm{exc}(\sigma)-k}$ vanishes for $\mathrm{exc}(\sigma) \neq k$ (with the standard convention $0^0 = 1$). On the left-hand side, this yields $k! \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma)=k \\ \mathrm{fix}(\sigma)\geq \ell}} \varepsilon(\sigma)(\mathrm{fix}(\sigma))_\ell b^{\mathrm{fix}(\sigma)-\ell}$. Let us now evaluate the right-hand side of Proposition 7, setting $x = 0$ and $y = b$. In the summation over $A^{(n)}_{k-j,\ell}(0, b)$, the term $A^{(n)}_{r,\ell}(0, b)$ is non-zero only for $r = 1$ and $\ell \leq n$. Thus, $k - j = 1$ (i.e., $j = k - 1$), which requires $k \geq 1$. This contributes $-k!(n)_\ell (b-1)^{n-\ell} \mathbf{1}_{\{k\geq 1\}}$. For the $B^{(n)}_{k-j,\ell}(0, b)$ terms, the condition $r + \ell \leq n$ (where $r = k - j$) restricts the summation indices such that $0 \leq k - j \leq n - \ell$, i.e., j ranges from $\max(0, k - n + \ell)$ to k. Substituting $r = k - j$ gives

$$\sum_{j=\max(0,k-n+\ell)}^{k} \binom{k}{j} (-1)^j \frac{j!}{(0-1)^{j+1}} \left(-(-1)^r (n)_{r+\ell} b^{n-r-\ell}\right) = k! \sum_{r=0}^{\min(k,n-\ell)} \frac{(-1)^r (n)_{r+\ell}}{r!} b^{n-r-\ell}.$$

Dividing the entire identity by $k!$ completes the proof. □

As a numerical verification, for $n = 2$, $k = 0$, $\ell = 0$, and $b = 2$, the restriction $\mathrm{exc}(\sigma) = 0$ selects only the identity permutation, yielding a left-hand side equal to $1 \cdot 2^2 = 4$. The formula yields $-\mathbf{1}_{\{0\geq 1\}} \cdot 1 \cdot (2-1)^2 + \sum_{r=0}^{0} \frac{(-1)^0 (2)_0}{0!} 2^{2-0} = 0 + 4 = 4$, validating the edge case.

Corollary 12 (Identity at $(x, y) = (a, 0)$)**.** *Let $n \geq 1$, $k \geq 0$ and $0 \leq \ell \leq n$, and let $a \in \mathbb{R}$. The factorial moments satisfy:*

$$\ell! \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{fix}(\sigma)=\ell \\ \mathrm{exc}(\sigma)\geq k}} \varepsilon(\sigma)\,(\mathrm{exc}(\sigma))_k\, a^{\mathrm{exc}(\sigma)-k} = (n)_\ell \frac{d^k}{da^k} D_{n-\ell}(a, 0).$$

Proof. By Proposition 6, setting $y = 0$ restricts the summation to permutations with $\mathrm{fix}(\sigma) = \ell$, as $y^{\mathrm{fix}(\sigma)-\ell}$ vanishes for $\mathrm{fix}(\sigma) > \ell$. The factor $(\mathrm{fix}(\sigma))_\ell$ evaluates to $\ell!$. The left-hand side therefore becomes

$$\ell! \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{fix}(\sigma)=\ell \\ \mathrm{exc}(\sigma)\geq k}} \varepsilon(\sigma)(\mathrm{exc}(\sigma))_k a^{\mathrm{exc}(\sigma)-k} = \left.\frac{\partial^{k+\ell} D_n(x, y)}{\partial x^k \partial y^\ell}\right|_{(x,y)=(a,0)}.$$

By conditioning on the subset of m fixed points (chosen in $\binom{n}{m}$ ways, each contributing a weight y^m, while the remaining $n-m$ elements are evaluated with zero fixed points via $D_{n-m}(x,0)$), we obtain the expansion

$$D_n(x,y) = \sum_{m=0}^{n} \binom{n}{m} D_{n-m}(x,0) y^m, \tag{4}$$

taking the ℓ-th partial derivative with respect to y and evaluating at $y=0$ yields

$$\left.\frac{\partial^\ell D_n(x,y)}{\partial y^\ell}\right|_{y=0} = (n)_\ell D_{n-\ell}(x,0).$$

Differentiating k times with respect to x at $x=a$ gives $(n)_\ell \frac{d^k}{da^k} D_{n-\ell}(a,0)$, which proves the assertion. □

As a numerical verification, for $n=3$, $k=2$, $\ell=0$, and $a=2$, the derangement polynomial is $D_3(x,0) = x^2 + x$. The left-hand side yields

$$\sum_{\substack{\mathrm{fix}(\sigma)=0 \\ \mathrm{exc}(\sigma)\geq 2}} \varepsilon(\sigma)(\mathrm{exc}(\sigma))_2 2^{\mathrm{exc}(\sigma)-2} = 2,$$

while the right-hand side gives $\frac{d^2}{da^2}(a^2+a) = 2$, confirming the relation.

5 Conclusion

We have introduced a determinantal framework to study the exceedance and fixed-point statistics of permutations. While connecting with known polynomial evaluations via cycle-weight specializations [5, 6], the matrix provides a direct algebraic encoding of these statistics, leading to a recurrence relation for the associated determinant. From this recurrence, we derived an exponential generating function satisfying a first-order linear differential equation, which yields an explicit expression for the determinant and enables a systematic study of its specializations. In particular, several classical signed enumerations arise as direct consequences, including alternating exceedance sums and fixed-point-weighted identities.

Beyond explicit formulas, the determinantal structure provides a unified treatment of mixed factorial moments via derivatives of the determinant. These derivatives admit a combinatorial interpretation in terms of signed factorial-moment sums of the joint distribution of exceedances and fixed points. Evaluations at special parameter values yield sum rules involving falling factorials of these statistics.

The present approach suggests several directions for further investigation. An interesting avenue for future work is to investigate whether the present determinantal reduction can be adapted to classical Weyl groups (types B and D), extending the linear character evaluations

(i.e., enumerations weighted by one-dimensional group representations or linear characters) of Sivasubramanian [12] via direct recurrences. Another promising direction is the asymptotic analysis of the determinant through its explicit analytic form, which may shed light on limiting distributions of signed permutation statistics.

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