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# A Generalization of the Pascal and Stirling Matrices

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**Abstract**

The purpose of this article is threefold. First, we introduce a natural generalization of the Pascal and Stirling matrices and investigate its LU decomposition and determinant. Second, we study several well-known integer sequences arising from Pascal's triangle and present a systematic method for recovering these sequences using the Pascal matrix and the exponential function. Third, we obtain determinant representations for linear recurrence relations with constant coefficients.

# 1 Introduction, background and definitions

We begin by recalling the standard definitions and notation for the Pascal and Stirling matrices. We then introduce shifted versions of these matrices and a natural generalization that will be studied throughout the article.

## 1.1 The classical Pascal matrix

One of the most well-known numerical triangles in mathematics is Pascal's triangle. It is named after the 17th-century French mathematician Blaise Pascal (1623–1662), although Pascal himself referred to it as the "Arithmetical Triangle." It should be noted, however, that Pascal was not the first to discover this triangle. It had been known and used by mathematicians in Persia, China, India, and elsewhere for centuries before Pascal [2]. Pascal's triangle is shown in Fig. 1. It is constructed as follows: the top entry and the two edges of the triangle are filled with 1's, and each remaining entry is obtained by adding the two entries directly above it in the preceding row.

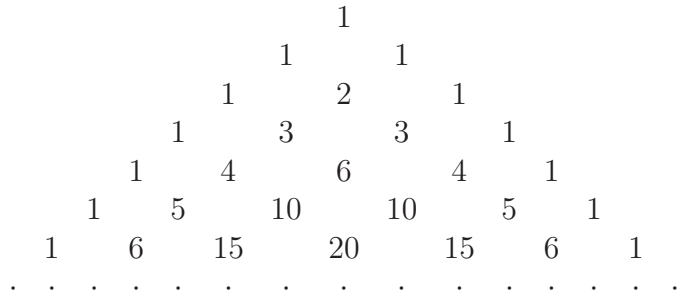


Figure 1: Pascal's triangle.

Here, we are interested in square arrays of numbers derived from Pascal's triangle, which can be constructed in various ways. We will mention some of these methods subsequently.

Given a positive integer $n$, the $n \times n$ *classical Pascal matrix* $P(n) = [P_{i,j}]_{1 \le i,j \le n}$ is defined by

$$P_{i,j} = \binom{i-1}{j-1}, \qquad 1 \le i, j \le n,$$

where $\binom{i-1}{j-1} = 0$ whenever $j > i$. Equivalently, the entries of $P(n)$ can be defined recursively by

$$P_{i,j} = \begin{cases} 1, & \text{if } i \geq 1,\ j = 1; \\ 0, & \text{if } i = 1,\ j > 1; \\ P_{i-1,j-1} + P_{i-1,j}, & \text{if } i, j \geq 2. \end{cases}$$

For example, when $n = 5$, we have

$$P(5) = \begin{pmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 2 & 1 & . & . \\ 1 & 3 & 3 & 1 & . \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

## 1.2 Stirling numbers and matrices

Stirling numbers are divided into two classes: Stirling numbers of the first kind and Stirling numbers of the second kind. In this article, we consider only Stirling numbers of the second kind. For positive integers $n$ and $k$ with $1 \leq k \leq n$, the *Stirling number of the second kind*, denoted by $S(n, k)$, is defined by

$$x^n = \sum_{k=1}^{n} S(n, k)x(x-1)(x-2)\cdots(x-k+1).$$

In particular, $S(n, n) = 1$. We use the convention that $S(n, k) = 0$ whenever $k > n$. The Stirling numbers of the second kind satisfy the recurrence relation

$$S(n, k) = S(n-1, k-1) + kS(n-1, k), \qquad n, k \geq 2. \tag{1}$$

Similarly, we define the $n \times n$ (lower) triangular matrix $S(n) = [S_{i,j}]_{1 \leq i,j \leq n}$ with $S_{i,j} = S(i, j)$, which is called the *Stirling matrix of the second kind*. The recurrence relation (1) yields the following recursive description of $S(n)$.

$$S_{i,j} = \begin{cases} 1, & \text{if } i \geq 1, j = 1; \\ 0, & \text{if } i = 1, j > 1; \\ S_{i-1,j-1} + jS_{i-1,j}, & \text{if } i, j \geq 2. \end{cases}$$

For instance, when $n = 5$, the Stirling matrix $S(5)$ has the following form:

$$S(5) = \begin{pmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 3 & 1 & . & . \\ 1 & 7 & 6 & 1 & . \\ 1 & 15 & 25 & 10 & 1 \end{pmatrix}.$$

## 1.3 Shifted Pascal and Stirling matrices

The $n \times n$ *shifted Pascal and Stirling matrices*, $\mathsf{P}(n) = [\mathsf{P}_{i,j}]_{1\leq i,j\leq n}$ and $\mathsf{S}(n) = [\mathsf{S}_{i,j}]_{1\leq i,j\leq n}$, respectively, are defined by

$$\mathsf{P}_{i,j} = \begin{cases} 1, & \text{if } i \geq 1, j = 1; \\ 1, & \text{if } i = 1, j > 1; \\ \mathsf{P}_{i,j-1} + \mathsf{P}_{i-1,j}, & \text{if } i, j \geq 2, \end{cases}$$

and

$$\mathsf{S}_{i,j} = \begin{cases} 1, & \text{if } i \geq 1, j = 1; \\ 1, & \text{if } i = 1, j > 1; \\ \mathsf{S}_{i,j-1} + j\mathsf{S}_{i-1,j}, & \text{if } i, j \geq 2. \end{cases}$$

For instance, for $n = 5$, these matrices have the following form:

$$\mathsf{P}(5) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix} \quad \text{and} \quad \mathsf{S}(5) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 7 & 25 & 65 & 140 \\ 1 & 15 & 90 & 350 & 1050 \\ 1 & 31 & 301 & 1701 & 6951 \end{pmatrix}.$$

It is well known that

$$\mathsf{P}(n) = P(n)P(n)^t,$$

for every positive integer $n$; see, for example, [1]. The relationship between the shifted Pascal and Stirling matrices has also been investigated in several works; see, for example, [6, 11]. Motivated by these constructions, we introduce the following generalizations.

## 1.4 Generalizations

Let $\alpha = (\alpha_j)_{j\geq 1}$ be an arbitrary sequence, and define the matrices $S_\alpha(n) = [\widehat{S}_{i,j}]_{1\leq i,j\leq n}$ and $\mathsf{S}_\alpha(n) = [\widehat{\mathsf{S}}_{i,j}]_{1\leq i,j\leq n}$ by

$$\widehat{S}_{i,j} = \begin{cases} \alpha_1, & \text{if } i \geq 1, j = 1; \\ 0, & \text{if } i = 1, j > 1; \\ \widehat{S}_{i-1,j-1} + \alpha_j \widehat{S}_{i-1,j}, & \text{if } i, j \geq 2, \end{cases} \tag{2}$$

and

$$\widehat{\mathsf{S}}_{i,j} = \begin{cases} \alpha_1, & \text{if } i \geq 1, j = 1; \\ \alpha_1, & \text{if } i = 1, j > 1; \\ \widehat{\mathsf{S}}_{i,j-1} + \alpha_j \widehat{\mathsf{S}}_{i-1,j}, & \text{if } i, j \geq 2. \end{cases}$$

For example, the matrices $S_\alpha(n)$ and $\mathsf{S}_\alpha(n)$ corresponding to $n = 3$ are

$$S_\alpha(3) = \begin{pmatrix} \alpha_1 & . & . \\ \alpha_1 & \alpha_1 & . \\ \alpha_1 & \alpha_1 + \alpha_2\alpha_1 & \alpha_1 \end{pmatrix},$$

and

$$\mathsf{S}_\alpha(3) = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_1 & \alpha_1 + \alpha_2\alpha_1 & \alpha_1 + \alpha_2\alpha_1 + \alpha_3\alpha_1 \\ \alpha_1 & \alpha_1 + \alpha_2\alpha_1 + \alpha_2^2\alpha_1 & \alpha_1 + \alpha_2\alpha_1 + \alpha_2^2\alpha_1 + \alpha_3\alpha_1 + \alpha_3\alpha_2\alpha_1 + \alpha_3^2\alpha_1 \end{pmatrix}.$$

Since

$$S_{(\alpha_1,\alpha_2,\alpha_3,\ldots)}(n) = \alpha_1 S_{(1,\alpha_2,\alpha_3,\ldots)}(n),$$

and

$$\mathsf{S}_{(\alpha_1,\alpha_2,\alpha_3,\ldots)}(n) = \alpha_1 \mathsf{S}_{(1,\alpha_2,\alpha_3,\ldots)}(n),$$

we may, without loss of generality, restrict our attention to sequences $\alpha = (\alpha_j)_{j\geq 1}$ satisfying $\alpha_1 = 1$. Observe that when $\alpha = (1, 1, 1, \ldots)$, we have

$$S_\alpha(n) = P(n) \qquad \text{and} \qquad \mathsf{S}_\alpha(n) = \mathsf{P}(n).$$

On the other hand, when $\alpha = (1, 2, 3, \ldots)$, we have

$$S_\alpha(n) = S(n) \qquad \text{and} \qquad \mathsf{S}_\alpha(n) = \mathsf{S}(n).$$

Thus, $S_\alpha(n)$ provides a natural common generalization of the Pascal and Stirling matrices, while $\mathsf{S}_\alpha(n)$ provides a corresponding generalization of their shifted versions.

In this article, for an arbitrary sequence $\alpha = (\alpha_j)_{j\geq 1}$ with $\alpha_1 = 1$, we investigate the $n \times n$ matrix $\mathsf{S}_\alpha(n)$. In particular, we obtain an LU decomposition of it as $\mathsf{S}_\alpha(n) = S_\alpha(n) \cdot \mathsf{U}_\alpha(n)$, where $\mathsf{U}_\alpha(n)$ is an upper triangular matrix associated with $\alpha$ (defined by a recurrence relation in Section 2), and find its determinant. This result generalizes the corresponding results in [1, 3, 8].

Since $S_\alpha(n)$ is a lower triangular matrix with 1's on the diagonal, to find the determinant of the matrix $\mathsf{S}_\alpha(n)$, we must explicitly specify the diagonal entries of $\mathsf{U}_\alpha(n)$. For this purpose, we use complete homogeneous symmetric polynomials, which we will introduce below.

## 1.5 The complete homogeneous symmetric polynomials

*The complete homogeneous symmetric polynomials* $h_k(x_1, \ldots, x_n)$, $k \geq 0$, are defined by $h_0(x_1, x_2, \ldots, x_n) = 1$, and

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

or equivalently

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{\substack{l_1+l_2+\cdots+l_n=k \\ l_1,l_2,\ldots,l_n\geq 0}} x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n},$$

for all $k \geq 1$. Thus, we have

$$\begin{aligned}
h_1(x_1, x_2, \ldots, x_n) &= x_1 + x_2 + \cdots + x_n, \\
h_2(x_1, x_2, \ldots, x_n) &= x_1^2 + x_1x_2 + \cdots + x_1x_n + x_2^2 + x_2x_3 + \cdots + x_{n-1}x_n + x_n^2, \\
&\vdots \\
h_n(x_1, x_2, \ldots, x_n) &= \sum_{\substack{l_1+l_2+\cdots+l_n=n \\ l_1,l_2,\ldots,l_n\geq 0}} x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}.
\end{aligned}$$

We now define [11] the infinite lower triangular matrix $G(x_1, x_2, \ldots) = [G_{i,j}]_{i,j\geq 1}$, where

$$G_{i,j} = \begin{cases} 0, & \text{if } i < j; \\ h_{i-j}(x_1, x_2, \ldots, x_j), & \text{if } i \geq j. \end{cases}$$

Let $G_n(x_1, x_2, \ldots)$ denote the leading $n \times n$ principal submatrix of $G(x_1, x_2, \ldots)$. Then

$$G_4(x_1, x_2, \ldots) = \begin{pmatrix} 1 & . & . & . \\ x_1 & 1 & . & . \\ x_1^2 & x_1 + x_2 & 1 & . \\ x_1^3 & x_1^2 + x_1x_2 + x_2^2 & x_1 + x_2 + x_3 & 1 \end{pmatrix}.$$

Note that $G_n(1, 1, 1, \ldots)$ is the $n \times n$ Pascal matrix $P(n)$, while $G_n(1, 2, 3, \ldots)$ is the $n \times n$ Stirling matrix of the second kind [11]. For instance,

$$G_4(1, 1, 1, \ldots) = \begin{pmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 2 & 1 & . \\ 1 & 3 & 3 & 1 \end{pmatrix} \quad \text{and} \quad G_4(1, 2, 3, \ldots) = \begin{pmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 3 & 1 & . \\ 1 & 7 & 6 & 1 \end{pmatrix}.$$

Similarly, if $x_1 = 1$, then we have

$$G_n(1, x_2, x_3, \ldots) = S_{(1,x_2,x_3,\ldots)}(n),$$

and, in particular, for $n = 4$, we have

$$G_4(1, x_2, x_3, \ldots) = S_{(1,x_2,x_3,\ldots)}(4) = \begin{pmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 1 + x_2 & 1 & . \\ 1 & 1 + x_2 + x_2^2 & 1 + x_2 + x_3 & 1 \end{pmatrix}.$$

Let us introduce some basic notation and conventions to be utilized in the subsequent discussion. Let $M_n(\mathbb{F})$ denote the set of $n \times n$ matrices over a field $\mathbb{F}$. For any matrix $A \in M_n(\mathbb{F})$, we employ $A_{i,j}$ to denote its $(i,j)$th entry, and $A^t$ to represent its transpose. Additionally, we use the notation $\mathrm{R}_i(A)$ (resp. $\mathrm{C}_j(A)$) to denote the $i$th row (resp. the $j$th column) of $A$. A square matrix is said to be *diagonal* if its nondiagonal elements are all zero. A diagonal matrix can be specified by the diag$(\cdot)$ constructor function which takes a vector as input and produces the diagonal matrix having the entries of the vector on its main diagonal:

$$\mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \begin{pmatrix} \alpha_1 & . & . & . \\ . & \alpha_2 & . & . \\ . & . & \ddots & . \\ . & . & . & \alpha_n \end{pmatrix}.$$

Recall that we write $A(\infty) = [A_{i,j}]_{i,j \geq 1}$ to denote an infinite matrix. For an infinite lower triangular matrix $L(\infty) = [L_{i,j}]_{i,j \geq 1}$, we denote by $\widehat{L}(\infty) = [\widehat{L}_{i,j}]_{i,j \geq 1}$ the *shifted matrix* associated with $L(\infty)$, where

$$\widehat{L}_{i,j} = L_{i+j-1,j} \quad \text{for all} \quad i,j \geq 1.$$

Given two $m \times n$ matrices $A = [A_{i,j}]$ and $B = [B_{i,j}]$, the *Hadamard product* $A \odot B$ of $A$ and $B$ is the $m \times n$ matrix of entry-wise products:

$$A \odot B = [A_{i,j} B_{i,j}] \quad \text{for all} \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

The *matrix exponential* of an $n \times n$ matrix $A$ over $\mathbb{R}$ or $\mathbb{C}$, denoted by $\exp(A)$, is given by

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots,$$

where $I_n = A^0$ is the $n \times n$ identity matrix. A matrix $T(n) = [T_{i,j}]_{1 \leq i,j \leq n}$ of the form

$$T(n) = \begin{pmatrix} t_0 & t_1 & & t_{n-1} \\ t_{-1} & t_0 & \ddots & \\ & \ddots & \ddots & t_1 \\ t_{-(n-1)} & & t_{-1} & t_0 \end{pmatrix}$$

is called a *Toeplitz matrix*. Equivalently, a Toeplitz matrix is one for which the entries along each diagonal are constant: $T_{i,j} = t_{j-i}$. Obviously, a Toeplitz matrix is determined by its first row and first column; henceforth, we will use $T_{\alpha,\beta}(n)$ to describe a Toeplitz matrix $T(n)$, where $\alpha = (T_{i,1})_{1 \leq i \leq n}$ and $\beta = (T_{1,j})_{1 \leq j \leq n}$.

## 2 The LU decomposition of $\mathsf{S}_\alpha(n)$ and its determinant

Given a sequence $\alpha = (\alpha_i)_{i\geq 1}$, we define the matrix $\mathsf{U}_\alpha(n) = [\mathsf{U}_{i,j}]_{1\leq i,j\leq n}$ by

$$\mathsf{U}_{i,j} = \mathsf{U}_{i,j-1} + \alpha_{i-1}\mathsf{U}_{i-1,j-1} + (\alpha_j - \alpha_{i-1})\mathsf{U}_{i-1,j}, \quad 2 \leq i,j \leq n, \tag{3}$$

with initial conditions

$$\mathsf{U}_{1,1} = 1, \qquad \mathsf{U}_{i,1} = 0 \quad (2 \leq i \leq n), \qquad \mathsf{U}_{1,j} = 1 \quad (2 \leq j \leq n).$$

It follows immediately from these initial conditions and (3) that $\mathsf{U}_\alpha(n)$ is upper triangular. For example, when $n = 4$, we have

$$\mathsf{U}_\alpha(4) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \cdot & \alpha_2 & \alpha_2+\alpha_3 & \alpha_2+\alpha_3+\alpha_4 \\ \cdot & \cdot & \alpha_3^2 & \alpha_3^2+\alpha_3\alpha_4+\alpha_4^2 \\ \cdot & \cdot & \cdot & \alpha_4^3 \end{pmatrix}.$$

We next determine the entries of $\mathsf{U}_\alpha(n)$ explicitly.

**Lemma 1.** *With the above notation, for every $j \geq i \geq 1$, we have $\mathsf{U}_{i,j} = h_{i-1}(\alpha_i, \ldots, \alpha_j)$. In particular, $\mathsf{U}_{i,i} = h_{i-1}(\alpha_i) = \alpha_i^{i-1}$, and hence*

$$\det \mathsf{U}_\alpha(n) = \prod_{i=1}^{n} \alpha_i^{i-1}.$$

*Proof.* We proceed by induction on $i + j$. If $i + j = 2$, then $i = j = 1$, and hence

$$\mathsf{U}_{1,1} = 1 = h_0(\alpha_1),$$

as required.

Suppose now that $i + j > 2$. We distinguish several cases. First, if $i = 1$, then

$$\mathsf{U}_{1,j} = 1 = h_0(\alpha_1, \ldots, \alpha_j),$$

for every $j > 1$. Next, suppose that $i = 2$ and $j > 2$. It follows from (3) that

$$\begin{aligned} \mathsf{U}_{2,j} &= \mathsf{U}_{2,j-1} + \alpha_1\mathsf{U}_{1,j-1} + (\alpha_j - \alpha_1)\mathsf{U}_{1,j} \\ &= \mathsf{U}_{2,j-1} + \alpha_j. \end{aligned}$$

By the inductive hypothesis,

$$\mathsf{U}_{2,j-1} = h_1(\alpha_2, \ldots, \alpha_{j-1}) = \alpha_2 + \cdots + \alpha_{j-1},$$

and therefore

$$\mathsf{U}_{2,j} = \alpha_2 + \cdots + \alpha_{j-1} + \alpha_j = h_1(\alpha_2, \ldots, \alpha_j).$$

We next consider the diagonal case $i = j \geq 2$. Once again, it follows from (3) that

$$\mathsf{U}_{i,i} = \mathsf{U}_{i,i-1} + \alpha_{i-1}\mathsf{U}_{i-1,i-1} + (\alpha_i - \alpha_{i-1})\mathsf{U}_{i-1,i}. \tag{4}$$

Since $\mathsf{U}_\alpha(n)$ is upper triangular, $\mathsf{U}_{i,i-1} = 0$. Also, since $(i-1)+(i-1) < 2i$ and $(i-1)+i < 2i$, the inductive hypothesis gives

$$\mathsf{U}_{i-1,i-1} = \alpha_{i-1}^{i-2} \quad \text{and} \quad \mathsf{U}_{i-1,i} = h_{i-2}(\alpha_{i-1}, \alpha_i).$$

Substituting these into (4) implies that

$$\mathsf{U}_{i,i} = \alpha_{i-1}^{i-1} + (\alpha_i - \alpha_{i-1})h_{i-2}(\alpha_{i-1}, \alpha_i).$$

On the other hand, using the standard identity

$$(y - x)h_r(x, y) = y^{r+1} - x^{r+1},$$

we obtain

$$(\alpha_i - \alpha_{i-1})h_{i-2}(\alpha_{i-1}, \alpha_i) = \alpha_i^{i-1} - \alpha_{i-1}^{i-1}.$$

Hence

$$\mathsf{U}_{i,i} = \alpha_i^{i-1} = h_{i-1}(\alpha_i),$$

as required.

It remains to consider the case $j > i \geq 3$. By the inductive hypothesis, we have

$$\begin{aligned}
\mathsf{U}_{i,j-1} &= h_{i-1}(\alpha_i, \ldots, \alpha_{j-1}), \\
\mathsf{U}_{i-1,j-1} &= h_{i-2}(\alpha_{i-1}, \ldots, \alpha_{j-1}), \\
\mathsf{U}_{i-1,j} &= h_{i-2}(\alpha_{i-1}, \ldots, \alpha_j).
\end{aligned}$$

We also use the following standard identities for complete homogeneous symmetric polynomials:

$$h_k(x_1, \ldots, x_m) = h_k(x_1, \ldots, x_{m-1}) + x_m h_{k-1}(x_1, \ldots, x_m),$$

and

$$h_k(x_1, \ldots, x_m) = x_1 h_{k-1}(x_1, \ldots, x_m) + h_k(x_2, \ldots, x_m).$$

Using (3) and the two identities above, we obtain

$$\begin{aligned}
\mathsf{U}_{i,j} =& h_{i-1}(\alpha_i, \ldots, \alpha_{j-1}) + \alpha_{i-1}h_{i-2}(\alpha_{i-1}, \ldots, \alpha_{j-1}) + (\alpha_j - \alpha_{i-1})h_{i-2}(\alpha_{i-1}, \ldots, \alpha_j) \\
=& h_{i-1}(\alpha_i, \ldots, \alpha_{j-1}) + \alpha_{i-1}h_{i-2}(\alpha_{i-1}, \ldots, \alpha_{j-1}) \\
&+ \alpha_j \left[\alpha_{i-1}h_{i-3}(\alpha_{i-1}, \ldots, \alpha_j) + h_{i-2}(\alpha_i, \ldots, \alpha_j)\right] \\
&- \alpha_{i-1} \left[h_{i-2}(\alpha_{i-1}, \ldots, \alpha_{j-1}) + \alpha_j h_{i-3}(\alpha_{i-1}, \ldots, \alpha_j)\right] \\
=& h_{i-1}(\alpha_i, \ldots, \alpha_{j-1}) + \alpha_j h_{i-2}(\alpha_i, \ldots, \alpha_j) \\
=& h_{i-1}(\alpha_i, \ldots, \alpha_j).
\end{aligned}$$

This completes the induction.

Finally, since $\mathsf{U}_\alpha(n)$ is upper triangular, its determinant is the product of its diagonal entries. Thus

$$\det \mathsf{U}_\alpha(n) = \prod_{i=1}^{n} \mathsf{U}_{i,i} = \prod_{i=1}^{n} \alpha_i^{i-1}.$$

The proof is now complete. □

We are now ready to state the main result of this section.

**Theorem 2.** *Let $\alpha = (\alpha_j)_{j\geq 1}$ be an arbitrary sequence with $\alpha_1 = 1$, and let $n$ be a positive integer. Then, we have the following matrix decomposition:*

$$\mathsf{S}_\alpha(n) = S_\alpha(n) \cdot \mathsf{U}_\alpha(n). \tag{5}$$

*Additionally, we obtain*

$$\det \mathsf{S}_\alpha(n) = \prod_{j=1}^{n} \alpha_j^{j-1}.$$

Before proving this theorem, it is useful to highlight some special cases:

(a) If $\alpha = (1, 1, 1, \ldots)$, then $\mathsf{S}_\alpha(n) = \mathsf{P}(n)$, $S_\alpha(n) = P(n)$ and $\mathsf{U}_\alpha(n) = P(n)^t$, and we get $\mathsf{P}(n) = P(n) \cdot P(n)^t$ and $\det \mathsf{P}(n) = 1$ [1].

(b) If $\alpha = (1, 2, 3, \ldots)$, then $\mathsf{S}_\alpha(n) = \mathsf{S}(n)$ and $S_\alpha(n) = S(n)$, and we get $\mathsf{S}(n) = S(n) \cdot \mathsf{U}_\alpha(n)$ and $\det \mathsf{S}(n) = \prod_{j=1}^{n} j^{j-1}$ [3, 8].

*Proof.* As a convenient notation, let us write $\mathsf{S} = \mathsf{S}_\alpha(n)$, $S = S_\alpha(n)$, and $\mathsf{U} = \mathsf{U}_\alpha(n)$. In order to prove (5), namely $\mathsf{S} = S \cdot \mathsf{U}$, therefore, it suffices to show that their entries correspond to the same recurrence relations satisfying the same initial conditions, that is

$$(S \cdot \mathsf{U})_{1,j} = \widehat{\mathsf{S}}_{1,j} \quad \text{and} \quad (S \cdot \mathsf{U})_{i,1} = \widehat{\mathsf{S}}_{i,1},$$

for every $1 \leq i, j \leq n$, and

$$(S \cdot \mathsf{U})_{i,j} = (S \cdot \mathsf{U})_{i,j-1} + \alpha_j (S \cdot \mathsf{U})_{i-1,j}, \tag{6}$$

for every $2 \leq i, j \leq n$. First of all, using simple calculations, we have

$$(S \cdot \mathsf{U})_{1,j} = \sum_{k=1}^{n} \widehat{S}_{1,k} \mathsf{U}_{k,j} = \widehat{S}_{1,1} \mathsf{U}_{1,j} = 1 \cdot 1 = 1 = \widehat{\mathsf{S}}_{1,j},$$

and

$$(S \cdot \mathsf{U})_{i,1} = \sum_{k=1}^{n} \widehat{S}_{i,k} \mathsf{U}_{k,1} = \widehat{S}_{i,1} \mathsf{U}_{1,1} = 1 \cdot 1 = 1 = \widehat{\mathsf{S}}_{i,1},$$

as required. Now, to verify (6), we proceed as follows. Let us, for the moment, assume that $2 \le i, j \le n$. Since

$$\begin{aligned}(S \cdot \mathsf{U})_{i,j} &= \sum_{k=1}^{n} \widehat{S}_{i,k}\mathsf{U}_{k,j} = \widehat{S}_{i,1}\mathsf{U}_{1,j} + \sum_{k=2}^{n} \widehat{S}_{i,k}\mathsf{U}_{k,j} \\ &= \widehat{S}_{i,1}\mathsf{U}_{1,j-1} + \sum_{k=2}^{n} \widehat{S}_{i,k}\big[\mathsf{U}_{k,j-1} + \alpha_{k-1}\mathsf{U}_{k-1,j-1} + (\alpha_j - \alpha_{k-1})\mathsf{U}_{k-1,j}\big] \\ &= (S \cdot \mathsf{U})_{i,j-1} + \sum_{k=2}^{n} \widehat{S}_{i,k}\big[\alpha_{k-1}\mathsf{U}_{k-1,j-1} + (\alpha_j - \alpha_{k-1})\mathsf{U}_{k-1,j}\big],\end{aligned}$$

it suffices to show that

$$\sum_{k=2}^{n} \widehat{S}_{i,k}\big[\alpha_{k-1}\mathsf{U}_{k-1,j-1} + (\alpha_j - \alpha_{k-1})\mathsf{U}_{k-1,j}\big] = \alpha_j (S \cdot \mathsf{U})_{i-1,j}. \tag{7}$$

To do this, we note that using (2), the sum on the left-hand side of (7) is equal to

$$\begin{aligned}&\sum_{k=2}^{n} \big[\widehat{S}_{i-1,k-1} + \alpha_k \widehat{S}_{i-1,k}\big]\big[\alpha_{k-1}\mathsf{U}_{k-1,j-1} + (\alpha_j - \alpha_{k-1})\mathsf{U}_{k-1,j}\big] \\ &= \alpha_1 \widehat{S}_{i-1,1}\mathsf{U}_{1,j-1} + \sum_{k=3}^{n} \alpha_{k-1}\widehat{S}_{i-1,k-1}\mathsf{U}_{k-1,j-1} \\ &\quad + (\alpha_j - \alpha_1)\widehat{S}_{i-1,1}\mathsf{U}_{1,j} + \sum_{k=3}^{n} (\alpha_j - \alpha_{k-1})\widehat{S}_{i-1,k-1}\mathsf{U}_{k-1,j} \\ &\quad + \sum_{k=2}^{n} \alpha_k\alpha_{k-1}\widehat{S}_{i-1,k}\mathsf{U}_{k-1,j-1} + \sum_{k=2}^{n} \alpha_k(\alpha_j - \alpha_{k-1})\widehat{S}_{i-1,k}\mathsf{U}_{k-1,j} \\ &= \alpha_j \widehat{S}_{i-1,1}\mathsf{U}_{1,j} + \sum_{k=2}^{n-1} \alpha_k \widehat{S}_{i-1,k}\mathsf{U}_{k,j-1} + \sum_{k=2}^{n-1} (\alpha_j - \alpha_k)\widehat{S}_{i-1,k}\mathsf{U}_{k,j} \\ &\quad + \sum_{k=2}^{n-1} \alpha_k\alpha_{k-1}\widehat{S}_{i-1,k}\mathsf{U}_{k-1,j-1} + \sum_{k=2}^{n-1} \alpha_k(\alpha_j - \alpha_{k-1})\widehat{S}_{i-1,k}\mathsf{U}_{k-1,j} \\ &\quad \text{(note that} \quad \mathsf{U}_{1,j-1} = \mathsf{U}_{1,j} \quad \text{and} \quad \widehat{S}_{i-1,n} = 0)\end{aligned}$$

$$
\begin{aligned}
&= \alpha_j \widehat{S}_{i-1,1}\mathsf{U}_{1,j} + \sum_{k=2}^{n-1} \widehat{S}_{i-1,k}\Big(\alpha_k\big[\mathsf{U}_{k,j-1} + \alpha_{k-1}\mathsf{U}_{k-1,j-1} + (\alpha_j - \alpha_{k-1})\mathsf{U}_{k-1,j}\big] \\
&\quad + (\alpha_j - \alpha_k)\mathsf{U}_{k,j}\Big) \\
&= \alpha_j \widehat{S}_{i-1,1}\mathsf{U}_{1,j} + \sum_{k=2}^{n-1} \widehat{S}_{i-1,k}\big[\alpha_k\mathsf{U}_{k,j} + (\alpha_j - \alpha_k)\mathsf{U}_{k,j}\big] \quad \text{(by (3))} \\
&= \alpha_j \sum_{k=1}^{n-1} \widehat{S}_{i-1,k}\mathsf{U}_{k,j} = \alpha_j \sum_{k=1}^{n} \widehat{S}_{i-1,k}\mathsf{U}_{k,j} = \alpha_j (S \cdot \mathsf{U})_{i-1,j}, \quad \text{(note that } \widehat{S}_{i-1,n} = 0)
\end{aligned}
$$

as required. The first part of the theorem is now complete.

To prove the second statement, using the matrix decomposition (5), Lemma 1 and the fact that $\det S_\alpha(n) = 1$, we observe that

$$
\det \mathsf{S}_\alpha(n) = \det(S_\alpha(n) \cdot \mathsf{U}_\alpha(n)) = \det S_\alpha(n) \det \mathsf{U}_\alpha(n) = \prod_{j=1}^{n} \alpha_j^{j-1},
$$

and the proof is complete. □

# 3 Sequences associated with Pascal's triangle

In this section, let $d^{(k)}$ denote the $k$th diagonal of Pascal's triangle (see Fig. 1). Hence,

$$
d^{(k)} = (d_i^{(k)})_{i\geq 0} = \left(\binom{k+i}{i}\right)_{i\geq 0} \quad (k = 0, 1, 2, \ldots).
$$

For instance, we have

| $k$ | $d^{(k)}$ | OEIS# |
|---|---|---|
| 0 | $(1, 1, 1, 1, 1, \ldots)$ | A000012 |
| 1 | $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots)$ | A000027 |
| 2 | $(1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots)$ | A000217 |
| 3 | $(1, 4, 10, 20, 35, 56, 84, 120, 165, 220, \ldots)$ | A000292 |
| 4 | $(1, 5, 15, 35, 70, 126, 210, 330, 495, 715, \ldots)$ | A000332 |

Table 1: The $k$th diagonal of Pascal's triangle for $k = 0, 1, 2, 3, 4$.

In the sequel, we introduce some lower triangular matrices that are constructed using $d^{(k)}$:

$$
\mathsf{L}_{d^{(k)}}(\infty) = \begin{pmatrix} \mathbf{0}_{k\times\infty} \\ \operatorname{diag} d^{(k)} \end{pmatrix}, \quad k = 0, 1, 2, \ldots,
$$

for example, for $k = 0, 1, 2, 3, 4$, we have $\mathsf{L}_{d^{(0)}}(\infty) = I_\infty$,

$$\mathsf{L}_{d^{(1)}}(\infty) = \begin{pmatrix} . & . & . & . & . & . & \cdots \\ 1 & . & . & . & . & . & \cdots \\ . & 2 & . & . & . & . & \cdots \\ . & . & 3 & . & . & . & \cdots \\ . & . & . & 4 & . & . & \cdots \\ . & . & . & . & 5 & . & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathsf{L}_{d^{(2)}}(\infty) = \begin{pmatrix} . & . & . & . & . & . & \cdots \\ . & . & . & . & . & . & \cdots \\ 1 & . & . & . & . & . & \cdots \\ . & 3 & . & . & . & . & \cdots \\ . & . & 6 & . & . & . & \cdots \\ . & . & . & 10 & . & . & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\mathsf{L}_{d^{(3)}}(\infty) = \begin{pmatrix} . & . & . & . & . & . & \cdots \\ . & . & . & . & . & . & \cdots \\ . & . & . & . & . & . & \cdots \\ 1 & . & . & . & . & . & \cdots \\ . & 4 & . & . & . & . & \cdots \\ . & . & 10 & . & . & . & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathsf{L}_{d^{(4)}}(\infty) = \begin{pmatrix} . & . & . & . & . & . & \cdots \\ . & . & . & . & . & . & \cdots \\ . & . & . & . & . & . & \cdots \\ . & . & . & . & . & . & \cdots \\ 1 & . & . & . & . & . & \cdots \\ . & 5 & . & . & . & . & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

**Theorem 3.** *Let $k \geq 1$ be an integer. With the notation above, we have the following*

$$\exp(\mathsf{L}_{d^{(k)}}(n)) = T_{\alpha^{(k)},\beta}(n) \odot \sum_{i=0}^{\infty} \mathsf{L}_{d^{(ki)}}(n), \tag{8}$$

*where $\alpha^{(k)} = (\alpha_i^{(k)})_{i\geq 1}$ is the sequence given by the recurrence*

$$\alpha_i^{(k)} = \binom{i-2}{k-1} \alpha_{i-k}^{(k)}, \quad i > k,$$

*and the initial conditions $\alpha_1^{(k)} = 1$, $\alpha_i^{(k)} = 0$, $i = 2, 3, \ldots, k$, and $\beta = (1, 0, 0, 0, \ldots)$.*

Before we proceed with the proof of Theorem 3, it seems appropriate to mention a routine observation.

Given an arbitrary sequence $\gamma = (\gamma_i)_{i\geq 1}$ and a positive integer $k$, the infinite matrix $\mathsf{L}_{k,\gamma}(\infty) = [L_{i,j}]_{i,j\geq 1}$ is defined so that

$$L_{i,j} = \begin{cases} \gamma_j, & \text{if } i - j = k; \\ 0, & \text{otherwise,} \end{cases}$$

or equivalently

$$\mathsf{L}_{k,\gamma}(\infty) = \begin{pmatrix} \mathbf{0}_{k\times\infty} \\ \operatorname{diag}\gamma \end{pmatrix}, \quad k \geq 1.$$

Thus, for example, we have

$$\mathsf{L}_{2,\gamma}(6) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \gamma_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \gamma_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \gamma_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \gamma_4 & \cdot & \cdot \end{pmatrix}.$$

Let $\exp(\mathsf{L}_{k,\gamma}(\infty)) = [E_{i,j}]_{i,j\geq 1}$. It is easy to check that

$$E_{i,j} = \begin{cases} 0, & \text{if } i < j; \\ 1, & \text{if } i = j; \\ 0, & \text{if } i > j, \quad i-j \not\equiv 0 \pmod{k}; \\ \frac{\gamma_j \gamma_{j+k} \gamma_{j+2k} \cdots \gamma_{i-k}}{\left(\frac{i-j}{k}\right)!}, & \text{if } i > j, \quad i-j \equiv 0 \pmod{k}. \end{cases} \tag{9}$$

*Proof.* Fix an integer $k \geq 1$, and let $\gamma_i = d_{i-1}^{(k)} = \binom{i-1+k}{i-1}$. We first claim that the sequence $(E_{i,1})_{i\geq 1}$ (the first column of $\exp(\mathsf{L}_{d^{(k)}}(\infty))$) is equal to the sequence $\alpha^{(k)}$, and for this purpose, it is enough to check by induction on $i$ that

$$E_{i,1} = \binom{i-2}{k-1} E_{i-k,1} \quad (i > k),$$

and $E_{1,1} = 1$, $E_{2,1} = E_{3,1} = \ldots = E_{k,1} = 0$.

Indeed, for $i > k$ and $i - 1 \equiv 0 \pmod{k}$, we observe by the inductive hypothesis that

$$\begin{aligned} E_{i,1} &= \frac{\gamma_1 \gamma_{1+k} \gamma_{1+2k} \cdots \gamma_{i-2k} \gamma_{i-k}}{\left(\frac{i-1}{k}\right)!} \\ &= \frac{\gamma_1 \gamma_{1+k} \gamma_{1+2k} \cdots \gamma_{i-2k}}{\left(\frac{(i-k)-1}{k}\right)!} \cdot \frac{\gamma_{i-k}}{\frac{i-1}{k}} \\ &= E_{i-k,1} \cdot \frac{\binom{i-1}{k}}{\frac{i-1}{k}} \\ &= \binom{i-2}{k-1} E_{i-k,1}. \end{aligned}$$

Next, we show that if $r - s = i - 1$ and $r - s \equiv 0 \pmod{k}$, then

$$E_{r,s} = E_{i,1} \cdot \binom{r-1}{i-1}. \tag{10}$$

In fact, we have

$$\begin{aligned}
E_{r,s} &= \frac{\gamma_s\gamma_{s+k}\cdots\gamma_{r-k}}{\left(\frac{r-s}{k}\right)!} = \frac{\gamma_1\gamma_{1+k}\cdots\gamma_{i-k}}{\left(\frac{i-1}{k}\right)!}\cdot\frac{\gamma_s\gamma_{s+k}\cdots\gamma_{r-k}}{\gamma_1\gamma_{1+k}\cdots\gamma_{i-k}} \\
&= E_{i,1}\cdot\frac{\gamma_s\gamma_{s+k}\cdots\gamma_{r-k}}{\gamma_1\gamma_{1+k}\cdots\gamma_{i-k}} \\
&= E_{i,1}\cdot\frac{\binom{s-1+k}{s-1}\binom{s-1+2k}{s-1+k}\binom{s-1+3k}{s-1+2k}\cdots\binom{r-1}{r-1-k}}{\binom{0+k}{0}\binom{2k}{k}\binom{3k}{2k}\cdots\binom{i-1}{i-1-k}} \\
&= E_{i,1}\cdot\frac{(r-1)!}{(s-1)!(i-1)!} \\
&= E_{i,1}\cdot\frac{(r-1)!}{(r-i)!(i-1)!} \\
&= E_{i,1}\cdot\binom{r-1}{i-1}.
\end{aligned}$$

Finally, it follows from the fact that $(E_{i,1})_{i\geq 1} = \alpha^{(k)}$, and (9), (10) that

$$\exp(\mathsf{L}_{d^{(k)}}(n)) = T_{\alpha^{(k)},\beta}(n)\odot\sum_{i=0}^{\infty}\mathsf{L}_{d^{(ki)}}(n),$$

and the proof is complete. □

**Example 4.** We have

$$\exp\left(\begin{pmatrix}
. & . & . & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & .\\
1 & . & . & . & . & . & . & . & . & .\\
. & 3 & . & . & . & . & . & . & . & .\\
. & . & 6 & . & . & . & . & . & . & .\\
. & . & . & 10 & . & . & . & . & . & .\\
. & . & . & . & 15 & . & . & . & . & .\\
. & . & . & . & . & 21 & . & . & . & .\\
. & . & . & . & . & . & 28 & . & . & .\\
. & . & . & . & . & . & . & 36 & . & .
\end{pmatrix}\right) = \begin{pmatrix}
1 & . & . & . & . & . & . & . & . & .\\
. & 1 & . & . & . & . & . & . & . & .\\
1 & . & 1 & . & . & . & . & . & . & .\\
. & 3 & . & 1 & . & . & . & . & . & .\\
3 & . & 6 & . & 1 & . & . & . & . & .\\
. & 15 & . & 10 & . & 1 & . & . & . & .\\
15 & . & 45 & . & 15 & . & 1 & . & . & .\\
. & 105 & . & 105 & . & 21 & . & 1 & . & .\\
105 & . & 420 & . & 210 & . & 28 & . & 1 & .\\
. & 945 & . & 1260 & . & 378 & . & 36 & . & 1
\end{pmatrix}$$

$$= \begin{pmatrix}
1 & . & . & . & . & . & . & . & . & .\\
. & 1 & . & . & . & . & . & . & . & .\\
1 & . & 1 & . & . & . & . & . & . & .\\
. & 1 & . & 1 & . & . & . & . & . & .\\
3 & . & 1 & . & 1 & . & . & . & . & .\\
. & 3 & . & 1 & . & 1 & . & . & . & .\\
15 & . & 3 & . & 1 & . & 1 & . & . & .\\
. & 15 & . & 3 & . & 1 & . & 1 & . & .\\
105 & . & 15 & . & 3 & . & 1 & . & 1 & .\\
. & 105 & . & 15 & . & 3 & . & 1 & . & 1
\end{pmatrix}\odot\begin{pmatrix}
1 & . & . & . & . & . & . & . & . & .\\
. & 1 & . & . & . & . & . & . & . & .\\
1 & . & 1 & . & . & . & . & . & . & .\\
. & 3 & . & 1 & . & . & . & . & . & .\\
1 & . & 6 & . & 1 & . & . & . & . & .\\
. & 5 & . & 10 & . & 1 & . & . & . & .\\
1 & . & 15 & . & 15 & . & 1 & . & . & .\\
. & 7 & . & 35 & . & 21 & . & 1 & . & .\\
1 & . & 28 & . & 70 & . & 28 & . & 1 & .\\
. & 9 & . & 84 & . & 126 & . & 36 & . & 1
\end{pmatrix}.$$

Given a formal power series

$$f(t) = \sum_{r=0}^{\infty} a_r t^r,$$

over a field of characteristic zero, define the lower triangular matrix

$$M(f) = [M(f)_{i,j}]_{i,j\geq 1}$$

by

$$M(f)_{i,j} = \begin{cases} \dfrac{(i-1)!}{(j-1)!}[t^{i-j}]f(t), & i \geq j, \\ 0, & i < j, \end{cases}$$

where $[t^r]f(t)$ denotes the coefficient of $t^r$ in $f(t)$.

The following elementary observation will be useful.

**Lemma 5.** *Let $f(t)$ and $g(t)$ be formal power series. Then*

$$M(f)M(g) = M(fg).$$

*Proof.* For $i \geq j$, we have

$$\begin{aligned} (M(f)M(g))_{i,j} &= \sum_{r=j}^{i} \frac{(i-1)!}{(r-1)!}[t^{i-r}]f(t)\frac{(r-1)!}{(j-1)!}[t^{r-j}]g(t) \\ &= \frac{(i-1)!}{(j-1)!}\sum_{r=j}^{i}[t^{i-r}]f(t)[t^{r-j}]g(t) \\ &= \frac{(i-1)!}{(j-1)!}[t^{i-j}]f(t)g(t) \\ &= M(fg)_{i,j}. \end{aligned}$$

For $i < j$, both sides are zero. Hence the result follows. □

*Remark* 6. (1) Since formal power series $f$ and $g$ commute under multiplication, we also have

$$M(f)M(g) = M(g)M(f).$$

(2) Note that

$$\exp(M(f)) = \sum_{r\geq 0} \frac{M(f)^r}{r!} = \sum_{r\geq 0} M\left(\frac{f^r}{r!}\right) = M(e^f).$$

By Lemma 5, for every $k \geq 1$ we have

$$\mathsf{L}_{d^{(k)}}(\infty) = M\left(\frac{t^k}{k!}\right).$$

Indeed, if $i = j + k$, then

$$M\left(\frac{t^k}{k!}\right)_{i,j} = \frac{(j+k-1)!}{(j-1)!k!} = \binom{j+k-1}{k} = \binom{j+k-1}{j-1} = d^{(k)}_{j-1},$$

and all other entries are zero. Consequently, for $r, s \geq 1$,

$$\mathsf{L}_{d^{(r)}}(\infty)\mathsf{L}_{d^{(s)}}(\infty) = \binom{r+s}{r}\mathsf{L}_{d^{(r+s)}}(\infty).$$

We now define

$$\mathbb{L}_m(\infty) = \sum_{k=1}^{m} \mathsf{L}_{d^{(k)}}(\infty), \quad m = 1, 2, 3, \ldots,$$

thus, for example,

$$\mathbb{L}_1(\infty) = \begin{pmatrix} . & . & . & . & . & . & \cdots \\ 1 & . & . & . & . & . & \cdots \\ . & 2 & . & . & . & . & \cdots \\ . & . & 3 & . & . & . & \cdots \\ . & . & . & 4 & . & . & \cdots \\ . & . & . & . & 5 & . & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \mathbb{L}_2(\infty) = \begin{pmatrix} . & . & . & . & . & . & \cdots \\ 1 & . & . & . & . & . & \cdots \\ 1 & 2 & . & . & . & . & \cdots \\ . & 3 & 3 & . & . & . & \cdots \\ . & . & 6 & 4 & . & . & \cdots \\ . & . & . & 10 & 5 & . & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\mathbb{L}_3(\infty) = \begin{pmatrix} . & . & . & . & . & . & \cdots \\ 1 & . & . & . & . & . & \cdots \\ 1 & 2 & . & . & . & . & \cdots \\ 1 & 3 & 3 & . & . & . & \cdots \\ . & 4 & 6 & 4 & . & . & \cdots \\ . & . & 10 & 10 & 5 & . & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathbb{L}_4(\infty) = \begin{pmatrix} . & . & . & . & . & . & \cdots \\ 1 & . & . & . & . & . & \cdots \\ 1 & 2 & . & . & . & . & \cdots \\ 1 & 3 & 3 & . & . & . & \cdots \\ 1 & 4 & 6 & 4 & . & . & \cdots \\ . & 5 & 10 & 10 & 5 & . & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since the matrices $\mathsf{L}_{d^{(1)}}(\infty), \ldots, \mathsf{L}_{d^{(m)}}(\infty)$ commute, we obtain

$$\begin{aligned} \exp(\mathbb{L}_m(\infty)) &= \exp\left(\sum_{k=1}^{m} \mathsf{L}_{d^{(k)}}(\infty)\right) \\ &= \prod_{k=1}^{m} \exp\left(\mathsf{L}_{d^{(k)}}(\infty)\right) \\ &= \prod_{k=1}^{m} \left(T_{\alpha^{(k)},\beta}(\infty) \odot \sum_{i=0}^{\infty} \mathsf{L}_{d^{(ki)}}(\infty)\right), \end{aligned}$$

where the last equality follows from Theorem 3.

When $m \leq 4$, our computations show that

$$\exp(\mathbb{L}_1(\infty)) = \begin{pmatrix} \mathbf{1} & . & . & . & . & . & \cdots \\ \mathbf{1} & 1 & . & . & . & . & \cdots \\ \mathbf{1} & 2 & 1 & . & . & . & \cdots \\ \mathbf{1} & 3 & 3 & 1 & . & . & \cdots \\ \mathbf{1} & 4 & 6 & 4 & 1 & . & \cdots \\ \mathbf{1} & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \exp(\mathbb{L}_2(\infty)) = \begin{pmatrix} \mathbf{1} & . & . & . & . & . & \cdots \\ \mathbf{1} & 1 & . & . & . & . & \cdots \\ \mathbf{2} & 2 & 1 & . & . & . & \cdots \\ \mathbf{4} & 6 & 3 & 1 & . & . & \cdots \\ \mathbf{10} & 16 & 12 & 4 & 1 & . & \cdots \\ \mathbf{26} & 50 & 40 & 20 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\exp(\mathbb{L}_3(\infty)) = \begin{pmatrix} \mathbf{1} & . & . & . & . & . & \cdots \\ \mathbf{1} & 1 & . & . & . & . & \cdots \\ \mathbf{2} & 2 & 1 & . & . & . & \cdots \\ \mathbf{5} & 6 & 3 & 1 & . & . & \cdots \\ \mathbf{14} & 20 & 12 & 4 & 1 & . & \cdots \\ \mathbf{46} & 70 & 50 & 20 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \exp(\mathbb{L}_4(\infty)) = \begin{pmatrix} \mathbf{1} & . & . & . & . & . & \cdots \\ \mathbf{1} & 1 & . & . & . & . & \cdots \\ \mathbf{2} & 2 & 1 & . & . & . & \cdots \\ \mathbf{5} & 6 & 3 & 1 & . & . & \cdots \\ \mathbf{15} & 20 & 12 & 4 & 1 & . & \cdots \\ \mathbf{51} & 75 & 50 & 20 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In particular, we observe that the first column of each of these matrices is a known integer sequence, as listed below:

| $\mathbb{L}(\infty)$ | $\mathrm{C}_1(\exp(\mathbb{L}(\infty)))^t$ | OEIS# |
|---|---|---|
| $\mathbb{L}_1(\infty)$ | $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \ldots)$ | A000012 |
| $\mathbb{L}_2(\infty)$ | $(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{10}, \mathbf{26}, \ldots)$ | A000085 |
| $\mathbb{L}_3(\infty)$ | $(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{14}, \mathbf{46}, \ldots)$ | A001680 |
| $\mathbb{L}_4(\infty)$ | $(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{15}, \mathbf{51}, \ldots)$ | A001681 |

Moreover, if $\mathbb{E}_m(\infty) = \exp(\mathbb{L}_m(\infty))$, then we obtain the following matrix decomposition:

$$\widehat{\mathbb{E}}_m(\infty) = \mathrm{diag}(\mathrm{C}_1(\mathbb{E}_m(\infty))^t) \cdot \mathsf{P}(\infty) = \mathrm{diag}(\mathrm{C}_1(\mathbb{E}_m(\infty))^t) \cdot P(\infty) \cdot P(\infty)^t,$$

which implies that

$$\det \widehat{\mathbb{E}}_m(n) = \prod_{i=1}^{n} (\mathbb{E}_m(n))_{i,1}.$$

For instance, if $n = 6$, then we have

$$\det\begin{pmatrix} \mathbf{1} & 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & 2 & 3 & 4 & 5 & 6 \\ \mathbf{1} & 3 & 6 & 10 & 15 & 21 \\ \mathbf{1} & 4 & 10 & 20 & 35 & 56 \\ \mathbf{1} & 5 & 15 & 35 & 70 & 126 \\ \mathbf{1} & 6 & 21 & 56 & 126 & 252 \end{pmatrix} = \mathbf{1}\times\mathbf{1}\times\mathbf{1}\times\mathbf{1}\times\mathbf{1}\times\mathbf{1},$$

$$\det\begin{pmatrix} \mathbf{1} & 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & 2 & 3 & 4 & 5 & 6 \\ \mathbf{2} & 6 & 12 & 20 & 30 & 42 \\ \mathbf{4} & 16 & 40 & 80 & 140 & 224 \\ \mathbf{10} & 50 & 150 & 350 & 700 & 1260 \\ \mathbf{26} & 156 & 546 & 1456 & 3276 & 6552 \end{pmatrix} = \mathbf{1}\times\mathbf{1}\times\mathbf{2}\times\mathbf{4}\times\mathbf{10}\times\mathbf{26},$$

$$\det\begin{pmatrix} \mathbf{1} & 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & 2 & 3 & 4 & 5 & 6 \\ \mathbf{2} & 6 & 12 & 20 & 30 & 42 \\ \mathbf{5} & 20 & 50 & 100 & 175 & 280 \\ \mathbf{14} & 70 & 210 & 490 & 980 & 1764 \\ \mathbf{46} & 276 & 966 & 2576 & 5796 & 11592 \end{pmatrix} = \mathbf{1}\times\mathbf{1}\times\mathbf{2}\times\mathbf{5}\times\mathbf{14}\times\mathbf{46},$$

$$\det\begin{pmatrix} \mathbf{1} & 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & 2 & 3 & 4 & 5 & 6 \\ \mathbf{2} & 6 & 12 & 20 & 30 & 42 \\ \mathbf{5} & 20 & 50 & 100 & 175 & 280 \\ \mathbf{15} & 75 & 225 & 525 & 1050 & 1890 \\ \mathbf{51} & 306 & 1071 & 2856 & 6426 & 12852 \end{pmatrix} = \mathbf{1}\times\mathbf{1}\times\mathbf{2}\times\mathbf{5}\times\mathbf{15}\times\mathbf{51}.$$

# 4 Exponentials of the classical Pascal matrix

The constructions in the preceding section motivate a different iteration of the matrix exponential, now starting from the classical Pascal matrix itself. We define a sequence of infinite matrices $(P_k(\infty))_{k\geq 1}$ recursively by

$$P_1(\infty) = P(\infty), \qquad P_k(\infty) = \exp\big(P_{k-1}(\infty) - I\big), \quad k \geq 2.$$

Let $P_k(n)$ denote the $n \times n$ submatrix of $P_k(\infty)$. For instance, the matrix $P_2(6)$ is given by:

$$P_2(6) = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 2 & 2 & 1 & . & . & . \\ 5 & 6 & 3 & 1 & . & . \\ 15 & 20 & 12 & 4 & 1 & . \\ 52 & 75 & 50 & 20 & 5 & 1 \end{bmatrix}.$$

The first column of $P_2(\infty)$ is the Bell-number sequence $(B_{i-1})_{i\geq 1}$, namely $1, 1, 2, 5, 15, 52, \ldots$ (A000110 in [10]). The next three columns of $P_2(\infty)$ appear in the OEIS [10] as A033306, A105479, A105480, respectively. It is also noteworthy that $P_2(6)$ admits the following Hadamard factorization:

$$P_2(6) = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 2 & 1 & 1 & . & . & . \\ 5 & 2 & 1 & 1 & . & . \\ 15 & 5 & 2 & 1 & 1 & . \\ 52 & 15 & 5 & 2 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}.$$

More generally, $P_k(n)$ admits a Hadamard factorization of the form

$$P_k(n) = T_{\alpha,\beta}(n) \odot P_1(n),$$

where

$$\alpha = \mathrm{C}_1(P_k(n))^t \quad \text{and} \quad \beta = (1, 0, \ldots, 0).$$

Indeed, the Toeplitz matrix $T_{\alpha,\beta}(n)$ has first column equal to the first column of $P_k(n)$ and first row equal to the first row of $P_k(n)$.

The following observation gives a proof of the Hadamard factorization in the general case. Let

$$f_1(t) = e^t, \qquad f_{k+1}(t) = \exp(f_k(t) - 1), \quad k \geq 1,$$

and write

$$f_k(t) = \sum_{r\geq 0} a_r^{(k)} t^r.$$

By Lemma 5, we have

$$P_1(\infty) = P(\infty) = M(e^t) = M(f_1).$$

Moreover, if $P_k(\infty) = M(f_k)$, then

$$P_{k+1}(\infty) = \frac{\exp(P_k(\infty))}{\exp(1)} = \frac{M(e^{f_k})}{e} = M\left(\exp(f_k - 1)\right) = M(f_{k+1}).$$

Thus, by induction,

$$P_k(\infty) = M(f_k), \qquad k \geq 1.$$

Now write

$$a_r^{(k)} = \frac{b_r^{(k)}}{r!}, \qquad b_r^{(k)} = r![t^r] f_k(t).$$

For $i \geq j$, it follows that

$$(P_k(\infty))_{i,j} = \frac{(i-1)!}{(j-1)!} a_{i-j}^{(k)} = \frac{(i-1)!}{(j-1)!(i-j)!} b_{i-j}^{(k)} = \binom{i-1}{j-1} b_{i-j}^{(k)}.$$

On the other hand, the first column of $P_k(\infty)$ is

$$\left(b_0^{(k)}, b_1^{(k)}, b_2^{(k)}, \ldots\right)^t.$$

Since $f_k(0) = 1$, we have $b_0^{(k)} = 1$. Therefore, if

$$\alpha = \mathrm{C}_1(P_k(n))^t \quad \text{and} \quad \beta = (1, 0, \ldots, 0),$$

then

$$P_k(n) = T_{\alpha,\beta}(n) \odot P_1(n).$$

Indeed, for $i \geq j$,

$$(T_{\alpha,\beta}(n))_{i,j} = b_{i-j}^{(k)}, \qquad (P_1(n))_{i,j} = \binom{i-1}{j-1},$$

and both matrices are zero above the main diagonal. Hence the asserted Hadamard factorization follows coefficient by coefficient.

The first column of $P_k$, as well as the corresponding sequence in the OEIS, for $k = 2, 3, 4, \ldots, 11$, are as follows:

| $k$ | $\mathrm{C}_1(P_k)^t$ | OEIS# |
|---|---|---|
| 2 | $1, 1, 2, 5, 15, 52, 203, 877, \ldots$ | A000110 |
| 3 | $1, 1, 3, 12, 60, 358, 2471, 19302, \ldots$ | A000258 |
| 4 | $1, 1, 4, 22, 154, 1304, 12915, 146115, \ldots$ | A000307 |
| 5 | $1, 1, 5, 35, 315, 3455, 44590, 660665, \ldots$ | A000357 |
| 6 | $1, 1, 6, 51, 561, 7556, 120196, 2201856, \ldots$ | A000405 |
| 7 | $1, 1, 7, 70, 910, 14532, 274778, 5995892, \ldots$ | A001669 |
| 8 | $1, 1, 8, 92, 1380, 25488, 558426, 14140722, \ldots$ | A081624 |
| 9 | $1, 1, 9, 117, 1989, 41709, 1038975, 29947185, \ldots$ | A081629 |
| 10 | $1, 1, 10, 145, 2755, 64660, 1804705, 58336855, \ldots$ | A081697 |
| 11 | $1, 1, 11, 176, 3696, 95986, 2967041, 106296586, \ldots$ | A081740 |

# 5 Determinant representations of some recurrence relations

An infinite matrix $A(\infty) = [A_{i,j}]_{i,j\geq 1}$ is said to be a *determinant representation* of a sequence $\omega = (\omega_i)_{i\geq 1}$ if $\omega_n = \det[A_{i,j}]_{1\leq i,j\leq n}$, the $n$th leading principal minor of $A(\infty)$ [9]. In what follows, we derive a determinant representation for a class of recursive sequences. The idea of the construction of this matrix is taken from [5].

Recall that a *linear homogeneous recurrence relation of degree $k$ with constant coefficients* is a recurrence relation of the form:

$$\alpha_n = c_1\alpha_{n-1} + c_2\alpha_{n-2} + \cdots + c_k\alpha_{n-k} \quad (n > k), \tag{11}$$

where $c_1, c_2, \ldots, c_k$ are constants, and $c_k \neq 0$. It follows from the general recursion theorem that for any $k$ initial conditions $\alpha_1, \alpha_2, \ldots, \alpha_k$, there exists a unique sequence $(\alpha_n)_{n\geq 1}$ that satisfies (11) with the same initial conditions.

Let $(\beta_n)_{n\geq 1}$ be a sequence defined as $\beta_1 = \beta_2 = \ldots = \beta_{k-1} = 0$, $\beta_k = 1$, and

$$\beta_n - c_1\beta_{n-1} - c_2\beta_{n-2} - \cdots - c_k\beta_{n-k} = 0 \quad (n > k). \tag{12}$$

Consider the $n \times n$ upper Hessenberg-Toeplitz matrix

$$H_n(-1; c_1, c_2, \ldots, c_k) = [H_{i,j}]_{1\leq i,j\leq n},$$

where

$$H_{i,j} = \begin{cases} c_{j-i+1}, & \text{if } 0 \leq j-i \leq k-1; \\ -1, & \text{if } j-i = -1; \\ 0, & \text{otherwise.} \end{cases}$$

For instance, for $n = 5$ and $k = 3$, the corresponding matrix is

$$H_5(-1; c_1, c_2, c_3) = \begin{pmatrix} c_1 & c_2 & c_3 & . & . \\ -1 & c_1 & c_2 & c_3 & . \\ . & -1 & c_1 & c_2 & c_3 \\ . & . & -1 & c_1 & c_2 \\ . & . & . & -1 & c_1 \end{pmatrix}.$$

Note that the rows of $H_n(-1; c_1, c_2, \ldots, c_k)$ represent the negatives of the coefficients of the difference equation (12) satisfied by the sequence $(\beta_n)_{n\geq 1}$.

Let $D_{n,k} = \det H_n(-1; c_1, c_2, \ldots, c_k)$. For $n > k$, using expansion along the last column, we obtain

$$D_{n,k} = \sum_{j=1}^{k} c_j D_{n-j,k}. \tag{13}$$

If we define

$$D_{1-k,k} = D_{2-k,k} = \ldots = D_{-2,k} = D_{-1,k} = 0; \quad D_{0,k} = 1, \tag{14}$$

then (13) remains valid for $n \geq 1$. Hence, for fixed $k$, it is evident that both $D_{n,k}$ and $(\beta_n)_{n\geq 1}$ obey the identical recurrence relation of order $k$. Consequently, $D_{n,k} = \beta_{n+k}$.

Therefore, the subsequent theorem has been established.

**Theorem 7.** *With the above notation, we have $D_{n,k} = \beta_{n+k}$ for every $n \geq 1$.*

Now, let us give two examples to illustrate the result of Theorem 7:

(a) *The Berstel sequence* (A007420 in [10]). Suppose $(b_n)_{n\geq 1}$ is Berstel's ternary recurrence sequence defined by

$$b_1 = b_2 = 0, \quad b_3 = 1, \quad b_n = 2b_{n-1} - 4b_{n-2} + 4b_{n-3}, \quad n > 3.$$

Then, the following determinant representation of $(b_{n+3})_{n\geq 1}$ follows from Theorem 7,

$$b_{n+3} = \begin{vmatrix} 2 & -4 & 4 & . & & . \\ -1 & 2 & -4 & 4 & \ddots & . \\ . & -1 & 2 & -4 & \ddots & . \\ . & . & -1 & 2 & \ddots & 4 \\ & \ddots & \ddots & \ddots & \ddots & -4 \\ . & . & . & & -1 & 2 \end{vmatrix}_{n\times n}.$$

(b) The *k-generalized Fibonacci sequence*. The $k$-generalized Fibonacci sequence $(F_n^{(k)})_{n\geq 1}$ satisfies the $k$-th order recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \quad n > k,$$

with initial conditions

$$F_1^{(k)} = F_2^{(k)} = \ldots = F_{k-1}^{(k)} = 0, \quad F_k^{(k)} = 1.$$

We recall that these numbers are also called Fibonacci $k$-sequences, generalized Fibonacci numbers of order $k$, Fibonacci $k$-step numbers, and $k$-bonacci numbers (see also [4, 7]). The original Fibonacci sequence $(F_{i-1})_{i\geq 1}$ (A000045 in [10]) can be obtained by putting $k = 2$:

$$F_1^{(2)} = 0, \quad F_2^{(2)} = 1, \quad F_n^{(2)} = F_{n-1}^{(2)} + F_{n-2}^{(2)}, \quad n > 2.$$

Using Theorem 7, for instance, $(F_{n+5}^{(5)})_{n\geq 1}$ has the following determinant representation:

$$F_{n+5}^{(5)} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & . & . & . \\ -1 & 1 & 1 & 1 & 1 & 1 & \ddots & . \\ . & -1 & 1 & 1 & 1 & 1 & \ddots & . \\ . & . & -1 & 1 & 1 & 1 & \ddots & 1 \\ . & . & . & -1 & 1 & 1 & \ddots & 1 \\ . & . & . & . & -1 & 1 & \ddots & 1 \\ . & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ . & . & . & . & . & . & -1 & 1 \end{vmatrix}_{n\times n}.$$

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# References


[1] A. Edelman and G. Strang, Pascal matrices, *Amer. Math. Monthly* **111** (2004), 189–197.

[2] A. W. F. Edwards, *Pascal's Arithmetical Triangle: The Story of a Mathematical Idea*, Charles Griffin, 1987 and Johns Hopkins University Press, Baltimore, MD, 2002.

[3] C. Krattenthaler, Advanced determinant calculus: A complement, *Linear Algebra Appl.* **411** (2005) 68–166.

[4] G. Y. Lee and S. G. Lee, A note on generalized Fibonacci numbers, *Fibonacci Quart.* **33** (1995), 273–278.

[5] D. A. Lind, A determinant involving generalized binomial coefficients, *Fibonacci Quart.* **9** (1971), 113–119.

[6] P. Maltais and T. A. Gulliver, Pascal matrices and Stirling numbers, *Appl. Math. Lett.* **11** (1998), 7–11.

[7] E. P. Miles, Jr., Generalized Fibonacci numbers and associated matrices, *Amer. Math. Monthly* **67** (1960), 745–752.

[8] A. R. Moghaddamfar, S. Navid Salehy, and S. Nima Salehy, The determinants of matrices with recursive entries, *Linear Algebra Appl.* **428** (2008), 2468–2481.

[9] A. R. Moghaddamfar, S. Navid Salehy, and S. Nima Salehy, Determinant representations of sequences: a survey, *Spec. Matrices* **1** (2013) 46–60.

[10] N. J. A. Sloane et al., The On-Line Encyclopedia of Integer Sequences, 2023. Available at https://oeis.org.

[11] M. Z. Spivey and A. M. Zimmer, Symmetric polynomials, Pascal matrices, and Stirling matrices, *Linear Algebra Appl.* **428** (2008), 1127–1134.


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