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Explicit Congruences for Modular Forms of Higher Weights

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Abstract

We give explicit formulas for congruences between Eisenstein series and certain cusp forms of even weight at least twelve. These formulas generalize several classical congruences of Ramanujan type and express the relevant Fourier coefficients through additive convolutions involving the Ramanujan tau function and divisor functions. As a consequence, we prove that every irregular prime occurs as the modulus of at least one Ramanujan-type congruence.

1 Introduction

Ramanujan [14, 15] observed the following famous congruence:

$$\sigma_{11}(n) \equiv \tau(n) \pmod{691}, \tag{1}$$

for all $n \geq 1$, where σ_k is the divisor function

$$\sigma_k(n) = \sum_{d|n, d>0} d^k,$$

for integers $k \geq 0$ and $n \geq 1$. Throughout, let z lie in the complex upper half-plane and put $q = e^{2\pi i z}$, so that $|q| < 1$. The Ramanujan tau function $\tau(n)$ is defined by the expansion

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \cdots .$$

From a modern perspective, this congruence is the manifestation of a congruence between the Fourier coefficients of two modular forms of weight 12: the unique normalized cusp form

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}, \quad q = e^{2\pi iz},$$

and the Eisenstein series

$$G_{12}(z) = \frac{691}{65520} + \sum_{n=1}^{\infty}\sigma_{11}(n)q^n, \quad q = e^{2\pi iz}.$$

This congruence has fueled numerous profound investigations. Important contributions include those of Serre [18, 19], Deligne [5], Swinnerton-Dyer [21, 22], Katz [10], Ribet [17], Manin [12], and many others. Diamond [6] and Ciolan, Languasco, and Moree [3] give further results.

In this paper, we prove explicit congruences between Eisenstein series G_k and certain cusp forms for all even weights $k \geq 12$. Datskovsky and Guerzhoy [4] noticed the existence of congruences of this kind, but to the best of our knowledge, the explicit formulas we obtain for all $k \geq 28$ are new. We also show that for every irregular prime p, there is a Ramanujan-type congruence modulo p in at least one weight.

2 Main theorem

To state our theorem, we first introduce the following notation.

For a nonzero integer n, let

$$\mathbb{Z}_{(n)} = \big\{a/b \in \mathbb{Q} : a, b \in \mathbb{Z},\ b \neq 0,\ \gcd(b, n) = 1\big\}.$$

We extend congruence modulo n to rational numbers by writing

$$a \equiv b \pmod{n} \quad \text{if } a - b \in n\mathbb{Z}_{(n)}.$$

For two modular forms f, g with rational Fourier coefficients, we write $f \equiv g \pmod{n}$ if their corresponding Fourier coefficients are congruent modulo n.

Recall that the Bernoulli numbers B_k are defined by the standard generating function

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Thus $B_0 = 1$, $B_1 = -\frac{1}{2}$, and $B_k = 0$ for every odd $k > 1$. For each even integer $k \geq 4$, let

$$G_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \qquad q = e^{2\pi iz}.$$

Then $G_k(z)$ is the Eisenstein series of weight k. We define $G_0(z) = 1$. We extend the definition of $\sigma_{k-1}(n)$ by setting

$$\sigma_{k-1}(0) = -\frac{B_k}{2k} \qquad (k \geq 4 \text{ even}),$$

and

$$\sigma_{-1}(n) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n \geq 1. \end{cases}$$

Thus, for every even $k \geq 4$,

$$G_k(z) = \sum_{n=0}^{\infty} \sigma_{k-1}(n) q^n,$$

with $G_0 = 1$. For even $k \geq 4$, let

$$E_k(z) = 1 + \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{\sigma_{k-1}(0)} q^n,$$

and set $E_0 = 1$. Then E_k is the normalized Eisenstein series of weight k.

For even $k \geq 4$, let $\sigma_{k-1}(0) = N_k/D_k$ where $N_k, D_k \in \mathbb{Z}$, $N_k > 0$, and $\gcd(N_k, D_k) = 1$. We also set $N_0 = N_2 = 1$. For $j \geq 1$, the value N_{2j} is the absolute value of the j-th term of A001067, which lists the numerators of $B_{2j}/(2j)$.

For an even integer $k \geq 4$, let $\mathcal{S}_k$ denote the complex vector space of cusp forms of $\mathrm{SL}_2(\mathbb{Z})$ of weight k, and let m be its dimension. By the dimension formula of Diamond and Shurman [7, Sec. 3.5],

$$m = \begin{cases} \lfloor k/12 \rfloor - 1, & \text{if } k \equiv 2 \pmod{12}; \\ \lfloor k/12 \rfloor, & \text{otherwise.} \end{cases} \tag{2}$$

For two functions $f, g : \mathbb{Z}_{\geq 0} \to \mathbb{Q}$, define their *additive convolution* by

$$\big(f * g\big)(n) = \sum_{i=0}^{n} f(i) g(n-i).$$

Here $n \geq 0$, and $*$ is not to be confused with Dirichlet convolution. For formal power series, the coefficients of a product are exactly the additive convolution of the coefficient sequences, i.e.,

$$\left(\sum_{n=0}^{\infty} f(n) q^n\right)\left(\sum_{n=0}^{\infty} g(n) q^n\right) = \sum_{n=0}^{\infty} \big(f * g\big)(n) q^n. \tag{3}$$

We extend the definition of Ramanujan's tau function by $\tau(0) = 0$, and write $\tau^{*1} = \tau$ and $\tau^{*(r+1)} = \tau * (\tau^{*r})$ for integers $r \geq 1$. The sequence $(\tau^{*2}(n))_{n \geq 2}$ is listed in A010839, and $(\tau^{*3}(n))_{n \geq 3}$ is listed in A035118.

Theorem 1. *Let k be an even integer, $k \geq 12$, $k \neq 14$, and let m be defined as in (2). Let C be the $m \times m$ matrix (c_{ij}), where*

$$c_{ij} = \frac{1}{\sigma_{k-12i-1}(0)}\big(\tau^{*i} * \sigma_{k-12i-1}\big)(j), \qquad 1 \leq i, j \leq m.$$

Let

$$\begin{pmatrix}\gamma_1 & \gamma_2 & \cdots & \gamma_m\end{pmatrix} = \begin{pmatrix}\sigma_{k-1}(1) & \sigma_{k-1}(2) & \cdots & \sigma_{k-1}(m)\end{pmatrix} C^{-1}.$$

Then

$$G_k \equiv \sum_{i=1}^{m} \gamma_i \Delta^i E_{k-12i} \pmod{N_k'}, \tag{4}$$

where

$$N_k' = \max\left(d : d \mid N_k,\ \gcd(d, N_{k-12i}) = 1 \text{ for every integer } i \text{ with } 1 \leq i \leq \lfloor k/12 \rfloor\right).$$

Remark 2. When $k = 14$, the modulus N_k' defined above equals 1. In this case, the corresponding congruence trivially holds.

Corollary 3. *With the notation above, we have*

$$\sigma_{k-1}(n) \equiv \sum_{i=1}^{m} \frac{\gamma_i}{\sigma_{k-12i-1}(0)}\big(\tau^{*i} * \sigma_{k-12i-1}\big)(n) \pmod{N_k'}, \tag{5}$$

for all $n \geq 1$.

Proof of Theorem 1 and Corollary 3. Choose $r, s \in \mathbb{Z}_{\geq 0}$ such that

$$4r + 6s = k,$$

and let $h = E_4^r E_6^s$. It is clear that

$$G_k - \sigma_{k-1}(0)h \in \mathcal{S}_k.$$

Since the Eisenstein series

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

and

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \quad (q = e^{2\pi i z})$$

have integral Fourier coefficients (listed in A004009 and A013973, respectively), we have

$$G_k \equiv G_k - \sigma_{k-1}(0)h \pmod{N_k}. \tag{6}$$

Datskovsky and Guerzhoy [4] already obtained this congruence.

For $1 \le i \le m$, extend the notation c_{ij} to all integers $j \ge 1$ by the same formula. By (3),

$$c_{ij} = \frac{1}{\sigma_{k-12i-1}(0)}\big(\tau^{*i} * \sigma_{k-12i-1}\big)(j)$$

is defined so that it is the j-th Fourier coefficient of $\Delta^i E_{k-12i}$, i.e.,

$$\Delta^i E_{k-12i} = \sum_{j=1}^{\infty} c_{ij} q^j.$$

Each $\Delta^i E_{k-12i}$ is a cusp form of weight k with leading term q^i. Thus $c_{ij} = 0$ for $i > j$ and $c_{ii} = 1$, so C is upper triangular with every diagonal entry equal to 1. The distinct leading powers imply linear independence. Since there are $m = \dim \mathcal{S}_k$ such forms, they form a basis of $\mathcal{S}_k$. Hence there exist $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_m \in \mathbb{C}$ such that

$$G_k - \sigma_{k-1}(0)h = \sum_{i=1}^{m} \tilde{\gamma}_i \Delta^i E_{k-12i}. \tag{7}$$

Comparing the Fourier expansion of both sides of (7) up to the q^m-term gives a linear system with rational coefficients. For each $i = 1, 2, \ldots, m$, the diagonal entry $c_{ii} = 1$ as it is the leading Fourier coefficient of $\Delta^i E_{k-12i}$. Therefore, the upper triangular matrix C is invertible over $\mathbb{Q}$. The first m Fourier coefficients of $G_k - \sigma_{k-1}(0)h$ are rational, and therefore

$$\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_m \in \mathbb{Q}.$$

Combining (7) with the congruence

$$G_k \equiv G_k - \sigma_{k-1}(0)h \pmod{N_k}$$

now yields

$$G_k \equiv \sum_{i=1}^{m} \tilde{\gamma}_i \Delta^i E_{k-12i} \pmod{N_k}.$$

Comparing the first m Fourier coefficients in this congruence, we obtain

$$\begin{pmatrix}\tilde{\gamma}_1 & \tilde{\gamma}_2 & \cdots & \tilde{\gamma}_m\end{pmatrix} C \equiv \begin{pmatrix}\sigma_{k-1}(1) & \sigma_{k-1}(2) & \cdots & \sigma_{k-1}(m)\end{pmatrix} \pmod{N_k}.$$

We may write $C = I + N$, where I is the identity matrix and N a nilpotent matrix satisfying $N^m = 0$. Hence

$$C^{-1} = (I+N)^{-1} = I - N + N^2 - N^3 + \cdots + (-1)^{m-1} N^{m-1}.$$

For each i, the Fourier coefficients of E_{k-12i} have denominators whose prime factors divide N_{k-12i}. Hence the same is true for the coefficients c_{ij} of $\Delta^i E_{k-12i}$. By the definition of N'_k, all

these coefficients lie in the localization $\mathbb{Z}_{(N_k')}$. The displayed finite expansion of C^{-1} therefore shows that every entry of C^{-1} also lies in $\mathbb{Z}_{(N_k')}$.

Reducing the preceding vector congruence modulo N_k' and multiplying by C^{-1} inside $\mathbb{Z}_{(N_k')}$ gives

$$\begin{pmatrix}\tilde{\gamma}_1 & \tilde{\gamma}_2 & \cdots & \tilde{\gamma}_m\end{pmatrix} \equiv \begin{pmatrix}\sigma_{k-1}(1) & \sigma_{k-1}(2) & \ldots & \sigma_{k-1}(m)\end{pmatrix} C^{-1} \pmod{N_k'}.$$

Since all Fourier coefficients of the basis elements $\Delta^i E_{k-12i}$ lie in $\mathbb{Z}_{(N_k')}$, substitution back into (6) yields

$$G_k \equiv \sum_{i=1}^{m} \gamma_i \Delta^i E_{k-12i} \pmod{N_k'}.$$

Corollary 3 follows by taking Fourier coefficients of both sides.

□

Remark 4. If we use the Miller basis described by Stein [20, Sec. 2.3] instead of $\Delta^i E_{k-12i}$, we obtain congruences modulo N_k. However, the present basis $\Delta^i E_{k-12i}$ yields the particularly simple formula (5) in Corollary 3, which involves only the additive convolution of τ and σ.

3 Explicit formulas in the cases of low dimensions

In this section, we apply the main theorem to produce congruences of Ramanujan type in the cases $m = \dim \mathcal{S}_k \leq 3$. For $k \leq 26$, we recover Ramanujan's congruence (1) and other congruences in the literature. For $k \geq 28$, our results are new to the best of our knowledge.

Corollary 5. *For $k = 12, 16, 18, 20, 22, 26$, we have*

$$G_k \equiv \Delta E_{k-12} \pmod{N_k'} \tag{8}$$

and

$$\sigma_{k-1}(n) \equiv \frac{1}{\sigma_{k-13}(0)}(\tau * \sigma_{k-13})(n) \pmod{N_k'}. \tag{9}$$

Proof. For $k = 12, 16, 18, 20, 22, 26$, we have $\dim \mathcal{S}_k = 1$. Therefore the matrix $C = C^{-1} = (1)$ is the 1×1 identity matrix. Hence $\gamma_1 = \sigma_{k-1}(1) = 1$ and Theorem 1 directly implies (8) and (9). □

Substituting in $k = 12, 16, 18, 20, 22, 26$, we obtain congruences of Ramanujan type:

$$\sigma_{11}(n) \equiv \tau(n) \pmod{691},$$

$$\sigma_{15}(n) \equiv \tau(n) + 240 \sum_{m=1}^{n-1} \sigma_3(m)\tau(n-m) \pmod{3617},$$

$$\sigma_{17}(n) \equiv \tau(n) - 504 \sum_{m=1}^{n-1} \sigma_5(m)\tau(n-m) \pmod{43867},$$

$$\sigma_{19}(n) \equiv \tau(n) + 480 \sum_{m=1}^{n-1} \sigma_7(m)\tau(n-m) \pmod{283 \cdot 617},$$
$$\sigma_{21}(n) \equiv \tau(n) - 264 \sum_{m=1}^{n-1} \sigma_9(m)\tau(n-m) \pmod{131 \cdot 593},$$
$$\sigma_{25}(n) \equiv \tau(n) - 24 \sum_{m=1}^{n-1} \sigma_{13}(m)\tau(n-m) \pmod{657931}.$$

Congruence (8) can be deduced from formulae given by Ramanujan [14, Table 1]. The congruence modulo 691 appears in Ramanujan's manuscript [1], in which he also studied the Fourier coefficients of ΔE_{k-12} for the above k. In fact, he used the notation τ_t for our $\tau * \sigma_{2t-1}$.

Wilton [23] published proofs of the congruences modulo 691 and 3617. Swinnerton-Dyer [21] and Manin [12] gave modern proofs of all six congruences in this corollary. Rankin [16] and Berndt and Ono [1] discuss the history of these congruences.

Corollary 6. *For $k = 24, 28, 30, 32, 34, 38$, we have*

$$G_k \equiv \Delta E_{k-12} + \left(2^{k-1} + 25 - \frac{1}{\sigma_{k-13}(0)}\right)\Delta^2 E_{k-24} \pmod{N'_k} \tag{10}$$

and

$$\begin{aligned}\sigma_{k-1}(n) \equiv{}& \frac{1}{\sigma_{k-13}(0)}(\tau * \sigma_{k-13})(n) \\ &+ \frac{1}{\sigma_{k-25}(0)}\left(2^{k-1} + 25 - \frac{1}{\sigma_{k-13}(0)}\right)(\tau * \tau * \sigma_{k-25})(n) \pmod{N'_k}.\end{aligned} \tag{11}$$

Proof. For $k = 24, 28, 30, 32, 34, 38$, we have $\dim \mathcal{S}_k = 2$. The space $\mathcal{S}_k$ is spanned by ΔE_{k-12} and $\Delta^2 E_{k-24}$.

From

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n = q - 24q^2 + (\text{higher order terms}),$$

we calculate the first few terms of the Fourier expansion of ΔE_{k-12} and $\Delta^2 E_{k-24}$:

$$\begin{aligned}\Delta E_{k-12}(z) &= q + \left(-24 + \frac{1}{\sigma_{k-13}(0)}\right)q^2 + \cdots, \\ \Delta^2 E_{k-24}(z) &= q^2 + \cdots.\end{aligned}$$

Therefore the matrix C and its inverse in Theorem 1 are

$$C = \begin{pmatrix} 1 & -24 + \frac{1}{\sigma_{k-13}(0)} \\ 0 & 1 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 1 & 24 - \frac{1}{\sigma_{k-13}(0)} \\ 0 & 1 \end{pmatrix},$$

and thus

$$\begin{pmatrix}\gamma_1 & \gamma_2\end{pmatrix} = \begin{pmatrix}1 & \sigma_{k-1}(2)\end{pmatrix} C^{-1} = \begin{pmatrix}1 & \sigma_{k-1}(2) + 24 - \frac{1}{\sigma_{k-13}(0)}\end{pmatrix}.$$

Theorem 1 gives congruence (10), namely,

$$G_k \equiv \Delta E_{k-12} + \left(2^{k-1} + 25 - \frac{1}{\sigma_{k-13}(0)}\right)\Delta^2 E_{k-24} \pmod{N'_k}$$

and (11) follows. □

For example, for $k = 24$ we have $B_{24} = -(103 \times 2294797)/2730$. Therefore $N'_{24} = 103 \times 2294797$. As $\sigma_{11}(0) = -B_{12}/24 = 691/65520$, we have

$$G_{24} \equiv \Delta E_{12} + \left(2^{23} + 25 - \frac{65520}{691}\right)\Delta^2 \pmod{103 \times 2294797},$$

and

$$\begin{aligned}\sigma_{23}(n) \equiv{}& \frac{65520}{691}\sum_{i=0}^{n}\tau(i)\sigma_{11}(n-i) \\ &+ \left(2^{23} + 25 - \frac{65520}{691}\right)\sum_{i=0}^{n}\tau(i)\tau(n-i) \pmod{103 \cdot 2294797}.\end{aligned}$$

Datskovsky and Guerzhoy [4] obtained the same explicit formula for $k = 24$. Their proof involved congruence of Hecke eigenforms modulo prime ideals.

For $k = 24, 28, 30, 32, 34, 38$, we list N'_k and the residues of γ_1, γ_2 modulo N'_k in Table 1:

k	N'_k	γ_1	$\gamma_2 \pmod{N'_k}$
24	103×2294797	1	107586203
28	9349×362903	1	2008360611
30	1721×1001259881	1	419043597883
32	$37 \times 683 \times 305065927$	1	2426412849149
34	151628697551	1	144283399404
38	154210205991661	1	80609153788200

Table 1: Values of N'_k and residues of γ_1 and γ_2 modulo N'_k.

Corollary 7. *When $k = 36, 40, 42, 44, 46, 50$, take*

$$\gamma_2 = 2^{k-1} + 25 - \frac{1}{\sigma_{k-13}(0)},$$

$$\gamma_3 = 3^{k-1} + 2^{k-1} \cdot 48 + 949 - \frac{2^{k-13}+25}{\sigma_{k-13}(0)} - \frac{2^{k-1}+25}{\sigma_{k-25}(0)} + \frac{1}{\sigma_{k-13}(0)\sigma_{k-25}(0)}.$$

Then we have

$$G_k \equiv \Delta E_{k-12} + \gamma_2 \Delta^2 E_{k-24} + \gamma_3 \Delta^3 E_{k-36} \pmod{N'_k},$$

and

$$\begin{aligned}\sigma_{k-1}(n) \equiv{}& \frac{1}{\sigma_{k-13}(0)}\big(\tau * \sigma_{k-13}\big)(n) \\ &+ \frac{\gamma_2}{\sigma_{k-25}(0)}\big(\tau^{*2} * \sigma_{k-25}\big)(n) \\ &+ \frac{\gamma_3}{\sigma_{k-37}(0)}\big(\tau^{*3} * \sigma_{k-37}\big)(n) \pmod{N'_k}.\end{aligned}$$

Proof. When $k = 36, 40, 42, 44, 46, 50$, we have $\dim \mathcal{S}_k = 3$, and the space $\mathcal{S}_k$ is spanned by ΔE_{k-12}, $\Delta^2 E_{k-24}$, and $\Delta^3 E_{k-36}$. From the calculation of the Fourier coefficients of the three modular forms, we see

$$C = \begin{pmatrix} 1 & -24 + \frac{1}{\sigma_{k-13}(0)} & 252 + \frac{2^{k-13}-23}{\sigma_{k-13}(0)} \\ 0 & 1 & -48 + \frac{1}{\sigma_{k-25}(0)} \\ 0 & 0 & 1 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 1 & -c_{12} & c_{12}c_{23} - c_{13} \\ 0 & 1 & -c_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix}\gamma_1 & \gamma_2 & \gamma_3\end{pmatrix} = \begin{pmatrix}\sigma_{k-1}(1) & \sigma_{k-1}(2) & \sigma_{k-1}(3)\end{pmatrix} C^{-1}.$$

So

$$\begin{aligned}\gamma_1 &= 1, \\ \gamma_2 &= 2^{k-1} + 25 - \frac{1}{\sigma_{k-13}(0)}, \\ \gamma_3 &= 3^{k-1} + 2^{k-1} \cdot 48 + 949 - \frac{2^{k-13}+25}{\sigma_{k-13}(0)} - \frac{2^{k-1}+25}{\sigma_{k-25}(0)} + \frac{1}{\sigma_{k-13}(0)\sigma_{k-25}(0)}.\end{aligned}$$

Theorem 1 gives the claimed congruences. □

Table 2 lists N'_k and the residues of γ_1, γ_2, and γ_3 modulo N'_k for each k.

k	N'_k	γ_1	$\gamma_2 \pmod{N'_k}$	$\gamma_3 \pmod{N'_k}$
36	26315271553053477373	1	8390735099767322508	23753298026670817518
40	137616929 × 1897170067619	1	143325441834485554993	249170693679242863157
42	1520097643918070802691	1	1268541799120853738102	213658362938250302834
44	59 × 8089 × 2947939 × 1798482437	1	2310744223299992745919	1741035903638575027145
46	383799511 × 67568238839737	1	25393878538540355611918	6784674507998026661076
50	417202699 × 47464429777438199	1	13556968179673327086390836	15511858160808671948447160

Table 2: Values of N'_k and residues of γ_1, γ_2, and γ_3 modulo N'_k.

In all the above cases, N_k does not share common prime factors with any of the N_{k-12i} ($1 \le i \le \frac{k}{12}$), and therefore N'_k coincides with N_k. In fact, the smallest k such that $N'_k \neq N_k$ is $k = 68$.

4 Remarks on irregular primes

Recall that an odd prime p is irregular if and only if p divides the numerator of B_k for some even integer k with $2 \le k \le p-3$ (see entry A000928 in [13]). By the Clausen–von Staudt and Kummer congruences, N_k is a product of irregular primes. Conversely, every irregular prime p appears as a prime factor of N_k for some k. Consequently, it is also a factor of N'_k for some k. (To see this, it suffices to choose the least even $k > 0$ such that $p \mid N_k$.)

Corollary 3 thereby provides at least one Ramanujan-type congruence for each irregular prime. Jensen [9] proved that there are infinitely many irregular primes. Carlitz [2], Ireland and Rosen [8, Chap. 15], and Luca, Pizarro-Madariaga, and Pomerance [11] give further discussion.

The smallest irregular prime is 37. Since it is a prime factor of N'_{32}, we have $\gamma_1 = 1$ and $\gamma_2 \equiv 25 \pmod{37}$,

$$G_{32} \equiv \Delta E_{20} + 25\Delta^2 E_8 \pmod{37}$$

and

$$\sigma_{31}(n) \equiv 22(\tau * \sigma_{19})(n) + 12(\tau * \tau * \sigma_7)(n) \pmod{37}.$$

The second smallest irregular prime is 59, which is a prime factor of N'_{44}. We have $\gamma_1 = 1$, $\gamma_2 \equiv 16 \pmod{59}$, $\gamma_3 \equiv 4 \pmod{59}$. Thus we obtain

$$G_{44} \equiv \Delta E_{32} + 16\Delta^2 E_{20} + 4\Delta^3 E_8 \pmod{59}$$

and

$$\sigma_{43}(n) \equiv 27(\tau * \sigma_{31})(n) + 19(\tau^{*2} * \sigma_{19})(n) + 32(\tau^{*3} * \sigma_7)(n) \pmod{59}.$$

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