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The Log-Balancedness of the Sequences for the Product of Fibonacci and Some Combinatorial Numbers

Krzysztof Kowitz
Institute of Mathematics
University of Gdańsk
Gdańsk, 80-952
Poland
Krzysztof.Kowitz@ug.edu.pl

Feng-Zhen Zhao
Department of Mathematics
Shanghai University
Shanghai, 200444
China
fengzhenzhao@shu.edu.cn

Abstract

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence and $(z_n)_{n\geq 0}$ be a positive sequence. We mainly discuss the log-behavior of the sequence $(z_nF_{n+1})_{n\geq 0}$. We give some sufficient conditions for the log-balancedness of $(z_nF_{n+1})_{n\geq 0}$. For instance, we prove that the sequence $(C_nF_{n+1})_{n\geq 2}$ is log-balanced, where C_n is the n-th Catalan number.

1 Introduction

Let $N \in \mathbb{N}$. A positive sequence $(z_n)_{n\geq N}$ is called *log-convex* (*log-concave*) if $z_n^2 \leq z_{n-1}z_{n+1}$ ($z_n^2 \geq z_{n-1}z_{n+1}$) for each $n \geq N+1$ (for example, see [6]). A log-convex sequence $(z_n)_{n\geq N}$ is

called *log-balanced* if $(\frac{z_n}{n!})_{n\geq N}$ is log-concave (Došlić [4] gave this definition for $N=0$). It is well known that a sequence $(z_n)_{n\geq N}$ is log-convex (log-concave) if and only if its quotient sequence $(\frac{z_{n+1}}{z_n})_{n\geq N}$ is nondecreasing (nonincreasing). A positive sequence $(z_n)_{n\geq N}$ is *tempered* (Došlić and Kolarec [6] gave this definition for $N=0$) if its quotient sequence satisfies

$$\frac{nz_n}{(n+1)z_{n-1}} \leq \frac{z_{n+1}}{z_n} \leq \frac{(n+1)z_n}{nz_{n-1}}.$$

Log-balancedness is a special case of log-convexity and a log-convex sequence $(z_n)_{n\geq N}$ is log-balanced if and only if $(\frac{z_{n+1}}{(n+1)z_n})_{n\geq N}$ is nonincreasing. A log-balanced sequence is also tempered. It is clear that a sequence $(z_n)_{n\geq N}$ is tempered if and only if $(n!z_n)_{n\geq N}$ is log-convex and $(\frac{z_n}{n!})_{n\geq N}$ is log-concave. Log-convexity (Log-concavity) is one of the main sources of combinatorial inequalities. Since log-balancedness involves log-convexity and log-concavity, it can help us find more inequalities. Many combinatorial sequences, including the Catalan numbers, the Motzkin numbers, the large Schröder numbers, the central Delannoy numbers, and the Bell numbers are log-balanced. One can find more log-balanced sequences in Došlić [4], Gu and Zhao [8], Liu and Zhao [12], Zhang and Zhao [14], Zhao [15], Zhao [17], and Zhao [18]. In addition, log-balanced sequences have applications in other subjects. For instance, Bell numbers provide important examples in white noise distribution theory (one can find more details in Asai et al. [1]). Hence the log-balancedness of sequences deserves to be studied.

In this paper, we are interested in the sequences for the product of Fibonacci and some combinatorial numbers (product of Catalan and some combinatorial numbers). Let $(F_n)_{n\geq 0}$ denote the Fibonacci sequence. The explicit expression of F_n is

$$F_n = \frac{\alpha^n - (-1)^n\alpha^{-n}}{\sqrt{5}},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$. The Fibonacci sequence $(F_n)_{n\geq 0}$ is sequence A000045 in the OEIS [13] and satisfies the recurrence

$$F_{n+1} = F_n + F_{n-1},$$

where $F_0 = 0$ and $F_1 = 1$. For various properties of $(F_n)_{n\geq 0}$, see sequence A000045 in the OEIS [13]. The Fibonacci sequence $(F_n)_{n\geq 0}$ is neither log-convex nor log-concave, $(F_{2n+2})_{n\geq 0}$ is log-concave, and $(F_{2n+1})_{n\geq 0}$ is log-convex.

The Catalan number C_n is the number of lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis. The explicit expression of C_n is $C_n = \frac{1}{n+1}\binom{2n}{n}$. The sequence of Catalan numbers $(C_n)_{n\geq 0}$ is sequence A000108 in the OEIS [13] and satisfies the recurrence relation $(n+2)C_{n+1} = 2(2n+1)C_n$. In [7, Theorems 3.3(d) and 3.7(b)], the authors present a combinatorial interpretation of the product of Fibonacci numbers and Catalan numbers (in this paper, the authors define $F_0 = 1$, meaning that the indices of the Fibonacci numbers are shifted by 1 compared to our definition). That was the main motivation for writing the final part of the article.

In Sections 2–3, we discuss the log-balancedness of the sequences for the product of Fibonacci and some combinatorial numbers (product of Catalan and some combinatorial numbers).

2 The log-balancedness of the sequences for the product of Fibonacci numbers and other numbers

Let $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ be positive sequences. Zhao [15, Theorem 3.1] gave a criterion for the log-balancedness of $(u_nv_n)_{n\geq 0}$ when $(u_n)_{n\geq 0}$ is log-balanced and $(v_n)_{n\geq 0}$ is log-concave. Zhang and Zhao [14, Theorem 3] derived a sufficient condition for the log-balancedness of $(u_nv_n)_{n\geq 0}$ when $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ are log-convex. It is well known that the Fibonacci sequence $(F_n)_{n\geq 0}$ is neither log-concave nor log-convex. Now we discuss the log-balancedness of the sequences for the product of Fibonacci and some combinatorial numbers.

Theorem 1. *Suppose that $(F_n)_{n\geq 0}$ is the Fibonacci sequence and $(z_n)_{n\geq 0}$ is a positive sequence satisfying the recurrence*

$$z_{n+1} = R_n z_n + S_n z_{n-1} \tag{1}$$

for $n \geq 1$, where $R_n > 0$ and $S_n \geq 0$ for each $n \geq 1$. For $n \geq 0$, let $q_n = \frac{z_{n+1}}{z_n}$. There exist a nonnegative integer N_0 and $\lambda_n, \theta_n > 0$ such that $\theta_n \leq q_n \leq \lambda_n$ for $n \geq N_0$. For $n \geq N_0$, set

$$\Gamma_n = \left(R_{n+1} + \frac{S_{n+1}}{\lambda_n} - \lambda_n\right)F_{n+2}^2 - \lambda_{n+1}$$

and

$$\Omega_n = \left((n+2)\theta_n - (n+1)R_{n+1} - \frac{(n+1)S_{n+1}}{\theta_n}\right)F_{n+2}^2 - (n+1)\lambda_{n+1}.$$

If there exists a nonnegative integer $N \geq N_0$ such that $\Gamma_n \geq 0$ and $\Omega_n \geq 0$ for $n \geq N$, the sequence $(z_nF_{n+1})_{n\geq N}$ is log-balanced.

Proof. Let $a_n = z_nF_{n+1}$ and $q_n = \frac{z_{n+1}}{z_n}$. By applying

$$F_nF_{n+2} - F_{n+1}^2 = (-1)^{n+1} \tag{2}$$

(see [11, p. 19]), we get

$$\frac{a_na_{n+2}}{a_{n+1}^2} = \frac{z_nz_{n+2}F_{n+1}F_{n+3}}{z_{n+1}^2F_{n+2}^2} = \frac{q_{n+1}}{q_n}\Big(1 + \frac{(-1)^n}{F_{n+2}^2}\Big).$$

For $(a_n)_{n\geq N}$ to be log-balanced, we must have

$$1 \leq \frac{a_na_{n+2}}{a_{n+1}^2} \leq \frac{n+2}{n+1},$$

for $n \geq N$, so

$$1 \leq \frac{q_{n+1}}{q_n}\Big(1 + \frac{(-1)^n}{F_{n+2}^2}\Big) \leq \frac{n+2}{n+1}$$

for $n \geq N$. Therefore, we must have

$$(q_{n+1} - q_n)F_{n+2}^2 + (-1)^n q_{n+1} \geq 0$$

for $n \geq N$ and

$$((n+2)q_n - (n+1)q_{n+1})F_{n+2}^2 - (-1)^n(n+1)q_{n+1} \geq 0$$

for $n \geq N$. We have

$$(q_{n+1} - q_n)F_{n+2}^2 + (-1)^n q_{n+1} \geq (q_{n+1} - q_n)F_{n+2}^2 - q_{n+1}$$

and

$$\begin{aligned}&((n+2)q_n - (n+1)q_{n+1})F_{n+2}^2 - (-1)^n(n+1)q_{n+1}\\ &\quad \geq ((n+2)q_n - (n+1)q_{n+1})F_{n+2}^2 - (n+1)q_{n+1}.\end{aligned}$$

It follows from (1) that

$$q_n = R_n + \frac{S_n}{q_{n-1}}. \tag{3}$$

By (3), we obtain

$$(q_{n+1} - q_n)F_{n+2}^2 + (-1)^n q_{n+1} \geq \left(R_{n+1} + \frac{S_{n+1}}{q_n} - q_n\right)F_{n+2}^2 - q_{n+1}$$

and

$$\begin{aligned}&((n+2)q_n - (n+1)q_{n+1})F_{n+2}^2 - (-1)^n(n+1)q_{n+1}\\ &\quad \geq \left((n+2)q_n - (n+1)R_{n+1} - \frac{(n+1)S_{n+1}}{q_n}\right)F_{n+2}^2 - (n+1)q_{n+1}.\end{aligned}$$

For $n \geq N_0$, noting that $\theta_n \leq q_n \leq \lambda_n$ and we have

$$(q_{n+1} - q_n)F_{n+2}^2 + (-1)^n q_{n+1} \geq \left(R_{n+1} + \frac{S_{n+1}}{\lambda_n} - \lambda_n\right)F_{n+2}^2 - \lambda_{n+1} = \Gamma_n$$

and

$$\begin{aligned}&((n+2)q_n - (n+1)q_{n+1})F_{n+2}^2 - (-1)^n(n+1)q_{n+1}\\ &\quad \geq \left((n+2)\theta_n - (n+1)R_{n+1} - \frac{(n+1)S_{n+1}}{\theta_n}\right)F_{n+2}^2 - (n+1)\lambda_{n+1} = \Omega_n.\end{aligned}$$

Because $\Gamma_n \geq 0$ and $\Omega_n \geq 0$ for $n \geq N$, the sequence $(z_n F_{n+1})_{n \geq N}$ is log-balanced. □

Remark 2. Note that when $(z_n)_{n\geq 0}$ is log-convex, then inequality $1 \leq \frac{q_{n+1}}{q_n}\left(1+\frac{(-1)^n}{F_{n+2}^2}\right)$ is automatic satisfied at even n; when $(z_n)_{n\geq 0}$, then inequality $\frac{q_{n+1}}{q_n}\left(1+\frac{(-1)^n}{F_{n+2}^2}\right) \leq \frac{n+2}{n+1}$ is automatic satisfied at odd n.

Remark 3. In the proof of Theorem 1 for $q_n = \frac{z_{n+1}}{z_n}$, we obtained

$$\frac{z_n z_{n+2} F_{n+1} F_{n+3}}{z_{n+1}^2 F_{n+2}^2} = \frac{q_{n+1}}{q_n}\left(1+\frac{(-1)^n}{F_{n+2}^2}\right).$$

It is well known that $\frac{1}{F_{n+2}^2} \sim \frac{5}{(\frac{\sqrt{5}+1}{2})^{2n+4}}$ when $n \to \infty$. If the positive sequence $(z_n)_{n\geq 0}$ has the controlled ratio expansion

$$\frac{q_{n+1}}{q_n} = 1 + \frac{c}{n^2} + O(n^{-3}),$$

where $c > 0$ is a constant, there necessarily exists a positive integer N such that $1 < \frac{z_n z_{n+2} F_{n+1} F_{n+3}}{z_{n+1}^2 F_{n+2}^2} < \frac{n+2}{n+1}$ for $n \geq N$. Thus, the sequence $(z_n F_{n+1})_{n\geq N}$ is log-balanced.

Theorem 4. *Suppose that $(F_n)_{n\geq 0}$ is the Fibonacci sequence and $(z_n)_{n\geq 0}$ is a positive sequence satisfying the recurrence*

$$z_{n+1} = R_n z_n - S_n z_{n-1}$$

for $n \geq 1$, where $R_n > 0$ and $S_n \geq 0$ for each $n \geq 1$. For $n \geq 0$, let $q_n = \frac{z_{n+1}}{z_n}$. There exist a nonnegative integer N_1 and $\lambda_n, \theta_n > 0$ such that $\theta_n \leq q_n \leq \lambda_n$ for $n \geq N_1$. For $n \geq N_1$, set

$$\Phi_n = \left(R_{n+1} - \frac{S_{n+1}}{\theta_n} - \lambda_n\right) F_{n+2}^2 - \lambda_{n+1}$$

and

$$\Psi_n = \left((n+2)\theta_n - (n+1)R_{n+1} + \frac{(n+1)S_{n+1}}{\lambda_n}\right) F_{n+2}^2 - (n+1)\lambda_{n+1}.$$

If there exists a nonnegative integer $N \geq N_1$ such that $\Phi_n \geq 0$ and $\Psi_n \geq 0$ for $n \geq N$, the sequence $(z_n F_{n+1})_{n\geq N}$ is log-balanced.

The proof of Theorem 4 is similar to that of Theorem 1 and is omitted here.

Now we apply the result of Theorem 1 and Theorem 4 to some sequences.

Example 5. For the Catalan sequence $(C_n)_{n\geq 0}$ and the Fibonacci sequence $(F_n)_{n\geq 0}$, $(C_n F_{n+1})_{n\geq 0}$ is sequence A098614 in the OEIS [13]. In paper [2, pp. 21–22], the authors analyzed the sequence $(C_n F_{n+1})_{n\geq 0}$ and derived its generating function. For the values of $(C_n F_{n+1})_{n\geq 0}$, see Table 1.

n	0	1	2	3	4	5	6	7	8	9	10
C_nF_{n+1}	1	1	4	15	70	336	1716	9009	48620	267410	1494844

Table 1: Some values of $(C_nF_{n+1})_{n\geq 0}$.

Now we discuss the log-behavior of the sequence $(C_nF_{n+1})_{n\geq 0}$ by using Theorem 1.

Corollary 6. *For the Catalan sequence $(C_n)_{n\geq 0}$ and the Fibonacci sequence $(F_n)_{n\geq 0}$, $(C_nF_{n+1})_{n\geq 2}$ is log-balanced and $(C_nF_{n+1})_{n\geq 1}$ is tempered.*

Proof. The Catalan sequence $(C_n)_{n\geq 0}$ satisfies the recurrence $C_{n+1}=\frac{2(2n+1)}{n+2}C_n$. It is clear that $R_n=\frac{2(2n+1)}{n+2}$ and $S_n=0$. For $n\geq 0$, let $q_n=\frac{C_{n+1}}{C_n}$. It is evident that $q_n=\frac{2(2n+1)}{n+2}$.

$$\begin{aligned}\Gamma_n &= \left(R_{n+1}+\frac{S_{n+1}}{q_n}-q_n\right)F_{n+2}^2-q_{n+1}\\ &= \frac{6F_{n+2}^2}{(n+2)(n+3)}-\frac{2(2n+3)}{n+3} > \frac{6F_{n+2}^2}{(n+2)(n+3)}-4\end{aligned}$$

and

$$\begin{aligned}\Omega_n &= \left((n+2)q_n-(n+1)R_{n+1}-\frac{(n+1)S_{n+1}}{q_n}\right)F_{n+2}^2-(n+1)q_{n+1}\\ &= \frac{4nF_{n+2}^2}{n+2}-\frac{2(n+1)(2n+3)}{n+3} > \frac{4nF_{n+2}^2}{n+2}-4(n+1).\end{aligned}$$

One can prove by induction that

$$F_n \geq n \tag{4}$$

for $n\geq 5$. For $n\geq 3$, we have

$$\Gamma_n > \frac{6(n+2)}{n+3}-4>0$$

and

$$\Omega_n > 4n(n+2)-4(n+1)>0.$$

It follows from Theorem 1 that the sequence $(C_nF_{n+1})_{n\geq 3}$ is log-balanced. We find that $(C_nF_{n+1})_{2\leq n\leq 4}$ is log-balanced. Thus, the sequence $(C_nF_{n+1})_{n\geq 2}$ is log-balanced. Clearly, the sequence $(C_nF_{n+1})_{n\geq 2}$ is tempered. On the other hand, we observe that $(C_nF_{n+1})_{1\leq n\leq 3}$ is tempered. Thus, the sequence $(C_nF_{n+1})_{n\geq 1}$ is tempered. □

Example 7. The Motzkin number M_n is the number of lattice paths, from $(0,0)$ to $(n,0)$ with steps $(1,0)$, $(1,1)$, and $(1,-1)$, never going below the x-axis. The Motzkin sequence $(M_n)_{n\geq 0}$ is sequence A001006 in the OEIS [13] and satisfies the recurrence relation

$$M_{n+1}=\frac{2n+3}{n+3}M_n+\frac{3n}{n+3}M_{n-1},$$

where $M_0 = M_1 = 1$. The sequence $(F_{n+1}M_n)_{n\geq 0}$ is sequence A200539 in the OEIS [13]. Some values of $(F_{n+1}M_n)_{n\geq 0}$ are listed in Table 2.

n	0	1	2	3	4	5	6	7	8	9
$F_{n+1}M_n$	1	1	4	12	45	168	663	2667	10982	45925

Table 2: Some values of $(F_{n+1}M_n)_{n\geq 0}$.

Došlić [4, Corollary 1] proved that the Motzkin sequence $(M_n)_{n\geq 0}$ is log-balanced. Now we discuss the log-behavior of the sequence $(F_{n+1}M_n)_{n\geq 0}$.

Corollary 8. *For the Motzkin sequence $(M_n)_{n\geq 0}$ and the Fibonacci sequence $(F_n)_{n\geq 0}$, $(F_{n+1}M_n)_{n\geq 4}$ is log-balanced and $(F_{n+1}M_n)_{n\geq 1}$ is tempered.*

Proof. It is clear that

$$R_n = \frac{2n+3}{n+3} \quad \text{and} \quad S_n = \frac{3n}{n+3}.$$

For $n \geq 0$, let $q_n = \frac{M_{n+1}}{M_n}$. For $n \geq 2$, Došlić and Veljan [5, Proof of Theorem 3.4] proved that $b_n \leq q_n \leq b_{n+1}$, where $b_n = \frac{6(n+1)}{2n+5}$. For $n \geq 2$, we have

$$\begin{aligned}\Gamma_n &= \left(R_{n+1} + \frac{S_{n+1}}{b_{n+1}} - b_{n+1}\right)F_{n+2}^2 - b_{n+2} \\ &= \frac{3(n-1)F_{n+2}^2}{2(n+2)(n+4)(2n+7)} - \frac{6(n+3)}{2n+9}\end{aligned}$$

and

$$\begin{aligned}\Omega_n &= \left((n+2)b_n - (n+1)R_{n+1} - \frac{(n+1)S_{n+1}}{b_n}\right)F_{n+2}^2 - (n+1)b_{n+2} \\ &= \frac{3(n+1)(4n+7)F_{n+2}^2}{2(n+4)(2n+5)} - \frac{6(n+1)(n+3)}{2n+9}.\end{aligned}$$

One can prove by induction that

$$F_n^2 > 6n(n+4) \quad \text{for} \quad n \geq 9. \tag{5}$$

For $n \geq 7$, it follows from (5) that

$$\Gamma_n \geq \frac{9(n-1)(n+6)}{(n+4)(2n+7)} - \frac{6(n+3)}{2n+9} = \frac{3(2n^3+15n^2-47n-330)}{(n+4)(2n+7)(2n+9)} > 0$$

and

$$\Omega_n \geq \frac{9(n+1)(n+2)(n+6)(4n+7)}{(n+4)(2n+5)} - \frac{6(n+1)(n+3)}{2n+9}$$
$$\geq 18(n+1)(n+2) - \frac{6(n+1)(n+3)}{2n+9} = \frac{6(n+1)(6n^2+38n+51)}{2n+9} > 0.$$

It follows from Theorem 1 that the sequence $(F_{n+1}M_n)_{n\geq 7}$ is log-balanced. On the other hand, we find that $(F_{n+1}M_n)_{4\leq n\leq 8}$ is log-balanced. Thus, the sequence $(F_{n+1}M_n)_{n\geq 4}$ is log-balanced. It is clear that the sequence $(F_{n+1}M_n)_{n\geq 4}$ is tempered. We observe that $(F_{n+1}M_n)_{1\leq n\leq 5}$ is tempered. Thus, $(F_{n+1}M_n)_{n\geq 1}$ is tempered. □

Example 9. Dyck paths are paths starting and ending on the horizontal axis using steps $(1,1)$ and $(1,-1)$, and never going below the horizontal axis. The Fine number B_n is the number of Dyck paths of length $2n$ with no hills (a hill is a pair of consecutive steps giving a peak of height 1). Let $(a_n)_{n\geq 0}$ be sequence A000957 in the OEIS [13]. For $n \geq 0$, $B_n = a_{n+1}$. The Fine sequence $(B_n)_{n\geq 0}$ satisfies the recurrence

$$B_{n+1} = \frac{7n+2}{2n+4}B_n + \frac{2n+1}{n+2}B_{n-1}, \tag{6}$$

where $B_0 = 1$ and $B_1 = 0$. Some values of $(F_nB_{n+1})_{n\geq 0}$ are given in Table 3.

n	0	1	2	3	4	5	6	7	8	9
F_nB_{n+1}	0	1	2	12	54	285	1488	8086	44520	249492

Table 3: Some values of $(F_nB_{n+1})_{n\geq 0}$.

Došlić [4, Corollary 2] proved that the sequence of Fine numbers $(B_n)_{n\geq 2}$ is log-balanced (one can find more information on these numbers in paper [3]). Now we discuss the log-behavior of the sequence $(F_nB_{n+1})_{n\geq 0}$. We first give a lemma.

Lemma 10. *Let $(B_n)_{n\geq 0}$ be the Fine sequence. For $n \geq 2$, let $x_n = \frac{B_{n+1}}{B_n}$. For $n \geq 4$, we have*

$$x_n \leq \varphi_n, \tag{7}$$

where $\varphi_n = \frac{2(2n+3)}{n+3} - \frac{1}{n^2}$.

Proof. It follows from (6) that

$$x_n = \frac{7n+2}{2(n+2)} + \frac{2n+1}{(n+2)x_{n-1}} \tag{8}$$

for $n \geq 3$. Liu and Wang [10, proof of Corollary 3.2] proved that

$$x_n \geq \frac{2(2n+1)}{n+2} \tag{9}$$

Using (8), we have

$$x_{n+1}-\varphi_{n+1}=\frac{7n+9}{2(n+3)}+\frac{2n+3}{(n+3)x_n}-\frac{2(2n+5)}{n+4}+\frac{1}{(n+1)^2}$$

for $n\geq 4$. By (9), we derive that

$$\begin{aligned}x_{n+1}-\varphi_{n+1}&\leq \frac{7n+9}{2(n+3)}+\frac{(2n+3)(n+2)}{2(n+3)(2n+1)}-\frac{2(2n+5)}{n+4}+\frac{1}{(n+1)^2}\\&=\frac{8n+11}{2(n+3)}+\frac{n+2}{(n+3)(2n+1)}-\frac{2(2n+5)}{n+4}+\frac{1}{(n+1)^2}\\&=\frac{-17n^3-12n^2+41n+24}{2(n+3)(n+4)(2n+1)(n+1)^2}<0\quad\text{for}\quad n\geq 4.\end{aligned}$$

For $n=4$ we have

$$x_4=\frac{B_5}{B_4}=\frac{6}{2}=3\leq\frac{345}{112}=\varphi_4.$$

Thus, (7) holds. □

Corollary 11. *For the Fine sequence $(B_n)_{n\geq 0}$ and the Fibonacci sequence $(F_n)_{n\geq 0}$, $(F_nB_{n+1})_{n\geq 5}$ is log-balanced and $(F_nB_{n+1})_{n\geq 2}$ is tempered.*

Proof. For $n\geq 0$, let $z_n=B_{n+2}$ and $q_n=\frac{z_{n+1}}{z_n}$. Then the sequence $(z_n)_{n\geq 0}$ satisfies the recurrence

$$z_{n+1}=\frac{7n+16}{2(n+4)}z_n+\frac{2n+5}{n+4}z_{n-1}.$$

It is evident that

$$R_n=\frac{7n+16}{2(n+4)}\quad\text{and}\quad S_n=\frac{2n+5}{n+4}.$$

It follows from (7) and (9) that

$$\theta_n\leq q_n\leq\lambda_n$$

for $n\geq 2$, where $\theta_n=\frac{2(2n+5)}{n+4}$ and $\lambda_n=\frac{2(2n+7)}{n+5}-\frac{1}{(n+2)^2}$. For $n\geq 2$, we have

$$\begin{aligned}\Gamma_n&=\left(R_{n+1}+\frac{S_{n+1}}{\lambda_n}-\lambda_n\right)F_{n+2}^2-\lambda_{n+1}\\&=\left(\frac{7n+23}{2(n+5)}+\frac{(2n+7)(n+2)^2}{2(2n+7)(n+2)^2-(n+5)}-\frac{2(2n+7)}{n+5}+\frac{1}{(n+2)^2}\right)F_{n+2}^2\\&\quad-\frac{2(2n+9)}{n+6}+\frac{1}{(n+3)^2}\\&=\left(\frac{n+5}{4(2n+7)(n+2)^2-2(n+5)}+\frac{1}{(n+2)^2}\right)F_{n+2}^2-\frac{2(2n+9)}{n+6}+\frac{1}{(n+3)^2}\\&>\frac{F_{n+2}^2}{(n+2)^2}-4\end{aligned}$$

and

$$\begin{aligned}\Omega_n &= \left((n+2)\theta_n - (n+1)R_{n+1} - \frac{(n+1)S_{n+1}}{\theta_n}\right)F_{n+2}^2 - (n+1)\lambda_{n+1}\\ &= \left(\frac{2(n+2)(2n+5)}{n+4} - \frac{(n+1)(7n+23)}{2(n+5)} - \frac{(n+1)(n+4)(2n+7)}{2(n+5)(2n+5)}\right)F_{n+2}^2\\ &- (n+1)\left(\frac{2(2n+9)}{n+6} - \frac{1}{(n+3)^2}\right)\\ &= \frac{16n^3+133n^2+401n+428}{2(n+4)(n+5)(2n+5)}F_{n+2}^2 - (n+1)\left(\frac{2(2n+9)}{n+6} - \frac{1}{(n+3)^2}\right)\\ &> 2F_{n+2}^2 - 4(n+1).\end{aligned}$$

For $n \geq 7$, we have that $\Gamma_n > 0$ and $\Omega_n > 0$ from (5). It follows from Theorem 1 that the sequence $(F_{n+1}B_{n+2})_{n\geq 7}$ is log-balanced. To check whether a sequence $(F_nB_{n+1})_{n\geq 8}$ is log-balanced, we need to modify the condition for Ω_n and verify the condition

$$\Omega_n' = \left((n+3)\theta_n - (n+2)R_{n+1} - \frac{(n+2)S_{n+1}}{\theta_n}\right)F_{n+2}^2 - (n+2)\lambda_{n+1} > 0$$

for $n \geq 7$.

Note that

$$\begin{aligned}\Omega_n' &= \left((n+3)\theta_n - (n+2)R_{n+1} - \frac{(n+2)S_{n+1}}{\theta_n}\right)F_{n+2}^2 - (n+2)\lambda_{n+1}\\ &= \left(\frac{2(n+3)(2n+5)}{n+4} - \frac{(n+2)(7n+23)}{2(n+5)} - \frac{(n+2)(n+4)(2n+7)}{2(n+5)(2n+5)}\right)F_{n+2}^2\\ &- (n+2)\left(\frac{2(2n+9)}{n+6} - \frac{1}{(n+3)^2}\right)\\ &= \left(2 + \frac{8n^3+41n^2+34n-44}{2(n+4)(n+5)(2n+5)}\right)F_{n+2}^2 - (n+2)\left(\frac{2(2n+9)}{n+6} - \frac{1}{(n+3)^2}\right)\\ &> 2F_{n+2}^2 - 4(n+2).\end{aligned}$$

Hence, $\Omega_n' > 0$ for $n \geq 2$. Thus, $(F_nB_{n+1})_{n\geq 8}$ is log-balanced. On the other hand, we observe that $(F_nB_{n+1})_{5\leq n\leq 9}$ is log-balanced. Thus, the sequence $(F_nB_{n+1})_{n\geq 5}$ is log-balanced. Naturally, the sequence $(F_nB_{n+1})_{n\geq 5}$ is also tempered. We observe that $(F_nB_{n+1})_{2\leq n\leq 6}$ is tempered. Therefore, the sequence $(F_nB_{n+1})_{n\geq 2}$ is tempered. □

Example 12. Let $(w_n)_{n\geq 0}$ be the sequence counting walks on cubic lattice. The sequence $(w_n)_{n\geq 0}$ is sequence A005572 in the OEIS [13] and satisfies the recurrence

$$(n+3)w_{n+1} = 4(2n+3)w_n - 12nw_{n-1} \tag{10}$$

for $n \geq 1$, where $w_0 = 1$ and $w_1 = 4$. Some values of $(w_nF_{n+1})_{n\geq 0}$ are listed in Table 4.

n	0	1	2	3	4	5	6	7
w_nF_{n+1}	1	4	34	228	1770	13632	109473	892668

Table 4: Some values of $(w_nF_{n+1})_{n\geq 0}$.

Liu and Wang [10, Corollary 3.15] proved that the sequence $(w_n)_{n\geq 0}$ defined by (10) is log-convex (one can find more information on these numbers in paper [12, Section 3.3]). Now we consider the log-behavior of the sequence $(w_nF_{n+1})_{n\geq 0}$.

Corollary 13. *For the sequence $(w_n)_{n\geq 0}$ defined by (10) and the Fibonacci sequence $(F_n)_{n\geq 0}$, $(w_nF_{n+1})_{n\geq 4}$ is log-balanced and $(w_nF_{n+1})_{n\geq 1}$ is tempered.*

Proof. It is obvious that

$$R_n = \frac{4(2n+3)}{n+3} \quad \text{and} \quad S_n = \frac{12n}{n+3}.$$

For $n \geq 0$, put $q_n = \frac{w_{n+1}}{w_n}$. Zhao [16, proof of Theorem 2.10] showed that $\theta_n \leq q_n \leq \theta_{n+1}$ for $n \geq 3$, where $\theta_n = \frac{6n+9}{n+3}$. For $n \geq 3$, we get

$$\begin{aligned}
\Phi_n &= \left(R_{n+1} - \frac{S_{n+1}}{\theta_n} - \theta_{n+1}\right)F_{n+2}^2 - \theta_{n+2} \\
&= \left(\frac{4(2n+5)}{n+4} - \frac{12(n+1)(n+3)}{(n+4)(6n+9)} - \frac{6n+15}{n+4}\right)F_{n+2}^2 - \frac{6n+21}{n+5} \\
&= \frac{3}{(n+4)(2n+3)}F_{n+2}^2 - \frac{6n+21}{n+5} > \frac{3}{(n+4)(2n+3)}F_{n+2}^2 - 6.
\end{aligned}$$

For $n \geq 7$, we get

$$\begin{aligned}
\Psi_n &= \left((n+2)\theta_n - (n+1)R_{n+1} + \frac{(n+1)S_{n+1}}{\theta_{n+1}}\right)F_{n+2}^2 - (n+1)\theta_{n+2} \\
&= \left(\frac{(n+2)(6n+9)}{n+3} - \frac{4(n+1)(2n+5)}{n+4} + \frac{4(n+1)^2}{2n+5}\right)F_{n+2}^2 - \frac{(n+1)(6n+21)}{n+5} \\
&> \left(3 + \frac{9}{n+3}\right)F_{n+2}^2 - 6(n+1) > 3F_{n+2}^2 - 6(n+1).
\end{aligned}$$

By means of (5), we have $\Phi_n > 0$ and $\Psi_n > 0$ for $n \geq 7$. It follows from Theorem 4 that the sequence $(w_nF_{n+1})_{n\geq 7}$ is log-balanced. On the other hand, we observe that $(w_nF_{n+1})_{4\leq n\leq 8}$ is log-balanced. Thus, the sequence $(w_nF_{n+1})_{n\geq 4}$ is log-balanced. Naturally, the sequence $(w_nF_{n+1})_{n\geq 4}$ is also tempered. We find that $(w_nF_{n+1})_{1\leq n\leq 5}$ is tempered. Thus, the sequence $(w_nF_{n+1})_{n\geq 1}$ is tempered. □

Example 14. The large Schröder number r_n is the number of lattice paths, Schröder paths, from $(0,0)$ to (n,n) using steps $(1,0)$, $(1,1)$ and $(0,1)$, and never rising above the line $y=x$. The large Schröder sequence $(r_n)_{n\geq 0}$ is sequence A006318 in the OEIS [13] and satisfies the recurrence relation

$$r_{n+1} = \frac{3(2n+1)}{n+2} r_n - \frac{n-1}{n+2} r_{n-1}, \tag{11}$$

where $r_0 = 1$ and $r_1 = 2$.

Some values of $(r_n F_{n+1})_{n\geq 0}$ are given in Table 5.

n	$r_n F_{n+1}$
0	1
1	2
2	12
3	66
4	450
5	3152
6	23478
7	179718
8	1413924
9	11335390
10	92356902
11	762256224

Table 5: Some values of $(r_n F_{n+1})_{n\geq 0}$.

Došlić [4, Corollary 5] proved that the sequence of large Schröder numbers $(r_n)_{n\geq 0}$ is log-balanced. Now we discuss the log-behavior of the sequence $(r_n F_{n+1})_{n\geq 0}$. We first give a lemma.

Lemma 15. *Let $(r_n)_{n\geq 0}$ be the large Schröder sequence. For $n \geq 0$, let $x_n = \frac{r_{n+1}}{r_n}$. For $n \geq 0$, we have*

$$\varphi_n \leq x_n \leq \varphi_{n+1}, \tag{12}$$

where $\varphi_n = \frac{(3+2\sqrt{2})(2n+1)}{2n+4}$.

Proof. It follows from (11) that

$$x_n = \frac{6n+3}{n+2} - \frac{n-1}{(n+2)x_{n-1}} \tag{13}$$

for $n \geq 1$. We prove by induction that (12) holds. We observe that $\varphi_j \leq x_j \leq \varphi_{j+1}$ for $0 \leq j \leq 2$. Assume that $\varphi_j \leq x_j \leq \varphi_{j+1}$ for $j \geq 2$. Using (13), we have

$$x_{j+1} - \varphi_{j+2} = \frac{6j+9}{j+3} - \frac{j}{(j+3)x_j} - \frac{(6+4\sqrt{2})j + 15 + 10\sqrt{2}}{2j+8}$$

and

$$\varphi_{j+1} - x_{j+1} = \frac{(6+4\sqrt{2})j + 9 + 6\sqrt{2}}{2j+6} - \frac{6j+9}{j+3} + \frac{j}{(j+3)x_j}.$$

For $j \geq 2$, we get that

$$\begin{aligned} x_{j+1} - \varphi_{j+2} &\leq \frac{6j+9}{j+3} - \frac{2j}{(6+4\sqrt{2})j + 9 + 6\sqrt{2}} - \frac{(6+4\sqrt{2})j + 15 + 10\sqrt{2}}{2j+8} \\ &= \frac{-(93+72\sqrt{2})j - (117+108\sqrt{2})}{(2j+8)(j+3)((6+4\sqrt{2})j + 9 + 6\sqrt{2})} < 0 \end{aligned}$$

and

$$\begin{aligned} \varphi_{j+1} - x_{j+1} &\leq \frac{(6+4\sqrt{2})j + 9 + 6\sqrt{2}}{2j+6} - \frac{6j+9}{j+3} + \frac{j(2j+4)}{(j+3)((6+4\sqrt{2})j + 3 + 2\sqrt{2})} \\ &= \frac{-3}{(2j+6)((6+4\sqrt{2})j + 3 + 2\sqrt{2})} < 0. \end{aligned}$$

Thus, (12) holds. □

Corollary 16. *For the large Schröder sequence $(r_n)_{n\geq 0}$ defined by (10) and the Fibonacci sequence $(F_n)_{n\geq 0}$, $(r_n F_{n+1})_{n\geq 2}$ is log-balanced (hence, $(r_n F_{n+1})_{n\geq 2}$ is tempered).*

Proof. It is obvious that

$$R_n = \frac{6n+3}{n+2} \quad \text{and} \quad S_n = \frac{n-1}{n+2}.$$

For $n \geq 0$, put $q_n = \frac{r_{n+1}}{r_n}$. It follows from (12) that

$$\theta_n \leq q_n \leq \theta_{n+1}$$

for $n \geq 0$, where $\theta_n = \frac{(6+4\sqrt{2})n+3+2\sqrt{2}}{2n+4}$. One can prove by induction that

$$F_n^2 > 64(n+1)^2 \quad \text{for} \quad n \geq 12. \tag{14}$$

By means of (14), for $n \geq 10$, we get

$$\begin{aligned} \Phi_n &= \left(R_{n+1} - \frac{S_{n+1}}{\theta_n} - \theta_{n+1}\right) F_{n+2}^2 - \theta_{n+2} \\ &= \left(\frac{6n+9}{n+3} - \frac{n(2n+4)}{(n+3)((6+4\sqrt{2})n + 3 + 2\sqrt{2})} - \frac{(6+4\sqrt{2})n + 9 + 6\sqrt{2}}{2n+6}\right) F_{n+2}^2 \\ &\quad - \frac{(6+4\sqrt{2})n + 15 + 10\sqrt{2}}{2n+8} \\ &= \frac{3}{(2n+6)((6+4\sqrt{2})n + 3 + 2\sqrt{2})} F_{n+2}^2 - \frac{(6+4\sqrt{2})n + 15 + 10\sqrt{2}}{2n+8} \\ &\geq \frac{1}{4(2n+1)(n+3)} F_{n+2}^2 - 6 > \frac{16(n+3)}{2n+1} - 6 = \frac{4n+42}{2n+1} > 0 \end{aligned}$$

and

$$\begin{aligned}\Psi_n &= \left((n+2)\theta_n - (n+1)R_{n+1} + \frac{(n+1)S_{n+1}}{\theta_{n+1}}\right)F_{n+2}^2 - (n+1)\theta_{n+2} \\ &= \left(\frac{(6+4\sqrt{2})n+3+2\sqrt{2}}{2} - \frac{(n+1)(6n+9)}{n+3} + \frac{2n(n+1)}{(6+4\sqrt{2})n+9+6\sqrt{2}}\right)F_{n+2}^2 \\ &- \frac{(n+1)((6+4\sqrt{2})n+15+10\sqrt{2})}{2n+8} \\ &\geq \left(\frac{(68+48\sqrt{2})n^2+(93+72\sqrt{2})n-9}{(2n+6)((6+4\sqrt{2})n+9+6\sqrt{2})}\right)F_{n+2}^2 - 6(n+1) \\ &\geq \frac{n}{n+3}F_{n+2}^2 - 6(n+1) > 64n(n+3) - 6(n+1) = 64n^2+186n-6 > 0.\end{aligned}$$

It follows from Theorem 4 that the sequence $(r_nF_{n+1})_{n\geq 10}$ is log-balanced. On the other hand, we observe that $(r_nF_{n+1})_{2\leq n\leq 11}$ is log-balanced. Thus, the sequence $(r_nF_{n+1})_{n\geq 2}$ is log-balanced. Hence, $(r_nF_{n+1})_{n\geq 2}$ is tempered. □

Example 17. The central Delannoy number D_n is the number of lattice paths from $(0,0)$ to (n,n) using only the steps $(1,0)$, $(0,1)$, and $(1,1)$. Equivalently, it is number of lattice paths from the lower left to the upper right corner of an $(n+1)\times(n+1)$ chess board. The central Delannoy sequence $(D_n)_{n\geq 0}$ is sequence A001850 in the OEIS [13] and satisfies the recurrence relation

$$D_{n+1} = \frac{3(2n+1)}{n+1}D_n - \frac{n}{n+1}D_{n-1}, \tag{15}$$

where $D_0 = 1$ and $D_1 = 3$.

Some values of $(D_nF_{n+1})_{n\geq 0}$ are given in Table 6.

n	0	1	2	3	4	5	6	7
D_nF_{n+1}	1	3	26	189	1605	13464	116857	1021419

Table 6: Some values of $(D_nF_{n+1})_{n\geq 0}$.

Došlić [4, Corollary 7] proved that the sequence of central Delannoy numbers $(D_n)_{n\geq 0}$ is log-balanced. Now we discuss the log-behavior of the sequence $(D_nF_{n+1})_{n\geq 0}$. We first give a lemma.

Lemma 18. *Let $(D_n)_{n\geq 0}$ be the central Delannoy sequence. For $n\geq 0$, let $x_n = \frac{D_{n+1}}{D_n}$. For $n\geq 1$, we have*

$$\varphi_n \leq x_n \leq \varphi_{n+1}, \tag{16}$$

where $\varphi_n = \frac{(3+2\sqrt{2})(2n-1)}{2n}$.

Proof. It follows from (15) that

$$x_n = \frac{6n+3}{n+1} - \frac{n}{(n+1)x_{n-1}} \tag{17}$$

for $n \geq 1$. We prove by induction that (16) holds. We observe that $\varphi_j \leq x_j \leq \varphi_{j+1}$ for $1 \leq j \leq 3$. Assume that $\varphi_j \leq x_j \leq \varphi_{j+1}$ for $j \geq 3$. Using (17), we have

$$x_{j+1} - \varphi_{j+2} = \frac{6j+9}{j+2} - \frac{j+1}{(j+2)x_j} - \frac{(6+4\sqrt{2})j + 9 + 6\sqrt{2}}{2j+4}$$

and

$$\varphi_{j+1} - x_{j+1} = \frac{(6+4\sqrt{2})j + 3 + 2\sqrt{2}}{2j+2} - \frac{6j+9}{j+2} + \frac{j+1}{(j+2)x_j}.$$

For $j \geq 3$, we get that

$$\begin{aligned} x_{j+1} - \varphi_{j+2} &\leq \frac{6j+9}{j+2} - \frac{(2j+2)(j+1)}{(j+2)((6+4\sqrt{2})j + 3 + 2\sqrt{2})} - \frac{(6+4\sqrt{2})j + 9 + 6\sqrt{2}}{2j+4} \\ &= \frac{-1}{(2j+4)((6+4\sqrt{2})j + 3 + 2\sqrt{2})} < 0 \end{aligned}$$

and

$$\begin{aligned} \varphi_{j+1} - x_{j+1} &\leq \frac{(6+4\sqrt{2})j + 3 + 2\sqrt{2}}{2j+2} - \frac{6j+9}{j+2} + \frac{2j(j+1)}{(j+2)((6+4\sqrt{2})j - (3+2\sqrt{2}))} \\ &= \frac{-(31+24\sqrt{2})j + 20 + 12\sqrt{2}}{(2j+2)(j+2)((6+4\sqrt{2})j - (3+2\sqrt{2}))} < 0. \end{aligned}$$

Thus, (16) holds. □

Corollary 19. *For the central Delannoy sequence $(D_n)_{n\geq 0}$ defined by (15) and the Fibonacci sequence $(F_n)_{n\geq 0}$, $(D_n F_{n+1})_{n\geq 4}$ is log-balanced and $(D_n F_{n+1})_{n\geq 1}$ is tempered.*

Proof. It is obvious that

$$R_n = \frac{6n+3}{n+1} \quad \text{and} \quad S_n = \frac{n}{n+1}.$$

For $n \geq 0$, put $q_n = \frac{D_{n+1}}{D_n}$. It follows from (16) that

$$\theta_n \leq q_n \leq \theta_{n+1}$$

for $n \geq 1$, where $\theta_n = \frac{(6+4\sqrt{2})n - (3+2\sqrt{2})}{2n}$. One can prove by induction that

$$F_n^2 > 4n^2 \quad \text{for} \quad n \geq 8. \tag{18}$$

By means of (18), for $n \geq 6$, we get

$$\begin{aligned}
\Phi_n &= \left(R_{n+1} - \frac{S_{n+1}}{\theta_n} - \theta_{n+1}\right)F_{n+2}^2 - \theta_{n+2} \\
&= \left(\frac{6n+9}{n+2} - \frac{2n(n+1)}{(n+2)((6+4\sqrt{2})n-(3+2\sqrt{2}))} - \frac{(6+4\sqrt{2})n+3+2\sqrt{2}}{2n+2}\right)F_{n+2}^2 \\
&- \frac{(6+4\sqrt{2})n+9+6\sqrt{2}}{2n+4} \\
&= \frac{(31+24\sqrt{2})n-(20+12\sqrt{2})}{(2n+2)(n+2)((6+4\sqrt{2})n-(3+2\sqrt{2}))}F_{n+2}^2 - \frac{(6+4\sqrt{2})n+9+6\sqrt{2}}{2n+4} \\
&\geq \frac{8n-5}{3n(n+1)(n+2)}F_{n+2}^2 - 6 > \frac{4(n+2)(8n-5)}{3n(n+1)} - 6 = \frac{14n^2+26n-40}{3n(n+1)} > 0
\end{aligned}$$

and

$$\begin{aligned}
\Psi_n &= \left((n+2)\theta_n - (n+1)R_{n+1} + \frac{(n+1)S_{n+1}}{\theta_{n+1}}\right)F_{n+2}^2 - (n+1)\theta_{n+2} \\
&= \left(\frac{(n+2)((6+4\sqrt{2})n-(3+2\sqrt{2}))}{2n} - \frac{(n+1)(6n+9)}{n+2}\right. \\
&\left. + \frac{2(n+1)^3}{(n+2)((6+4\sqrt{2})n+3+2\sqrt{2})}\right)F_{n+2}^2 - \frac{(n+1)((6+4\sqrt{2})n+9+6\sqrt{2})}{2n+4} \\
&\geq \left(\frac{(68+48\sqrt{2})n^3+(69+48\sqrt{2})n^2-(118+84\sqrt{2})n-68-48\sqrt{2}}{2n(n+2)((6+4\sqrt{2})n+3+2\sqrt{2})}\right)F_{n+2}^2 - 6(n+1) \\
&\geq \frac{3n}{(n+2)(2n+1)}F_{n+2}^2 - 6(n+1) > \frac{12n(n+2)}{2n+1} - 6(n+1) = \frac{6n-6}{2n+1} > 0.
\end{aligned}$$

It follows from Theorem 4 that the sequence $(D_nF_{n+1})_{n\geq 6}$ is log-balanced. On the other hand, we observe that $(D_nF_{n+1})_{4\leq n\leq 7}$ is log-balanced. Thus, the sequence $(D_nF_{n+1})_{n\geq 4}$ is log-balanced. Therefore, the sequence $(D_nF_{n+1})_{n\geq 4}$ is tempered. We observe that $(D_nF_{n+1})_{1\leq n\leq 5}$ is tempered. Thus, the sequence $(D_nF_{n+1})_{n\geq 1}$ is tempered. □

3 The log-balancedness of the sequences for the product of Catalan numbers and other numbers

For the Catalan sequence $(C_n)_{n\geq 0}$ and the Fibonacci sequence $(F_n)_{n\geq 0}$, we have proved that $(C_nF_{n+1})_{n\geq 2}$ is log-balanced. In this section of the article, we consider the log-balancedness of the sequences for the products of Catalan numbers and some other combinatorial numbers such as Pell, Pell-Lucas, and Lucas numbers. We introduce a definition. Let $N \in \mathbb{N}$. A positive sequence $(z_n)_{n\geq N}$ is called *log-alternating* if $z_n^2 - z_{n-1}z_{n+1} = (-1)^n\beta$ for each

$n \geq N+1$, where $\beta > 0$ is a constant. It is clear that the Fibonacci, Pell, Pell-Lucas, and Lucas sequences are log-alternating after a certain term. The common logarithmic structure of these sequences allows us to formulate and apply general criteria for the log-balancedness of the products under consideration.

Theorem 20. *Let $(C_n)_{n\geq 0}$ be the sequence of Catalan numbers. If there exist a positive integer $N \geq 2$ and a positive number β such that $(z_n)_{n\geq N}$ is a positive sequence, $3z_nz_{n+2} \geq \beta(n+3)(2n+1)$, and $|z_nz_{n+2} - z_{n+1}^2| \leq \beta$ for $n \geq N$, the sequence $(C_nz_n)_{n\geq N}$ is log-balanced.*

Proof. For $n \geq N$, let $a_n = C_nz_n$. We know that

$$\frac{C_nC_{n+2}}{C_{n+1}^2} = \frac{\frac{1}{n+1}\binom{2n}{n}\frac{1}{n+3}\binom{2n+4}{n+2}}{(\frac{1}{n+2}\binom{2n+2}{n+1})^2} = \frac{(n+2)(2n+3)}{(2n+1)(n+3)} = 1 + \frac{3}{(2n+1)(n+3)}.$$

Hence sequence $(a_n)_{n\geq N}$ is log-balanced if

$$\begin{aligned} a_na_{n+2} - a_{n+1}^2 &= C_nC_{n+2}z_nz_{n+2} - C_{n+1}^2z_{n+1}^2 \\ &= C_{n+1}^2\Big(\Big(1 + \frac{3}{(2n+1)(n+3)}\Big)z_nz_{n+2} - z_{n+1}^2\Big) \\ &= C_{n+1}^2\Big(\frac{3z_nz_{n+2}}{(2n+1)(n+3)} + z_nz_{n+2} - z_{n+1}^2\Big) \geq 0 \end{aligned}$$

and

$$\begin{aligned} (n+2)a_{n+1}^2 - (n+1)a_na_{n+2} &= (n+2)C_{n+1}^2z_{n+1}^2 - (n+1)C_nC_{n+2}z_nz_{n+2} \\ &= (n+2)C_{n+1}^2\Big(z_{n+1}^2 - \frac{n+1}{n+2}\Big(1 + \frac{3}{(2n+1)(n+3)}\Big)z_nz_{n+2}\Big) \\ &= (n+2)C_{n+1}^2\Big(\frac{2nz_nz_{n+2}}{(2n+1)(n+3)} - (z_nz_{n+2} - z_{n+1}^2)\Big) \geq 0 \end{aligned}$$

for $n \geq N$. For $n \geq N$, noting that $3z_nz_{n+2} \geq \beta(n+3)(2n+1)$ and $|z_nz_{n+2} - z_{n+1}^2| \leq \beta$, we have that

$$\frac{3z_nz_{n+2}}{(2n+1)(n+3)} + z_nz_{n+2} - z_{n+1}^2 \geq \frac{3z_nz_{n+2}}{(2n+1)(n+3)} - \beta \geq 0$$

and

$$\frac{2nz_nz_{n+2}}{(2n+1)(n+3)} - (z_nz_{n+2} - z_{n+1}^2) \geq \frac{3z_nz_{n+2}}{(2n+1)(n+3)} - \beta \geq 0.$$

This implies that

$$a_na_{n+2} - a_{n+1}^2 \geq 0 \quad \text{and} \quad (n+2)a_{n+1}^2 - (n+1)a_na_{n+2} \geq 0 \quad \text{for} \quad n \geq N.$$

Thus, the sequence $(C_nz_n)_{n\geq N}$ is log-balanced. □

Lemma 21. [15, Theorem 3.1] *For two sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$, suppose that $(x_n)_{n\geq 0}$ is log-balanced and $(y_n)_{n\geq 0}$ is log-concave. For $n \geq 0$, let $s_n = \frac{x_{n+1}y_{n+1}}{x_n y_n}$. If the sequence $(s_n)_{n\geq 0}$ is increasing, then $(x_n y_n)_{n\geq 0}$ is log-balanced.*

Theorem 22. *Let $(C_n)_{n\geq 0}$ be the sequence of Catalan numbers. Suppose that there exists a nonnegative integer N such that $(z_n)_{n\geq N}$ is a positive sequence and $z_n z_{n+2} - z_{n+1}^2 = -1$ for $n \geq N$. The sequence $(C_n z_n)_{n\geq N}$ is log-balanced if one of the following conditions holds.*

(i) $3z_n z_{n+2} \geq 2n^2 + 7n + 3$ *for* $n \geq N$.

(ii) $3z_{n+1}^2 \geq 2n^2 + 7n + 6$ *for* $n \geq N$.

Proof. We note that the sequence of Catalan numbers $(C_n)_{n\geq 0}$ is log-balanced and $(z_n)_{n\geq N}$ is log-concave. For $n \geq N$, let $s_n = \frac{C_{n+1}z_{n+1}}{C_n z_n}$. In order to prove that the sequence $(C_n z_n)_{n\geq N}$ is log-balanced, we only need to show that $(s_n)_{n\geq N}$ is increasing. For $n \geq N$, it follows from $z_n z_{n+2} - z_{n+1}^2 = -1$ that

$$\begin{aligned} s_{n+1} - s_n &= \frac{2(2n^2+7n+6)z_n z_{n+2} - 2(2n^2+7n+3)z_{n+1}^2}{(n+2)(n+3)z_n z_{n+1}} \\ &= \frac{6z_n z_{n+2} - 2(2n^2+7n+3)}{(n+2)(n+3)z_n z_{n+1}} \end{aligned}$$

or

$$s_{n+1} - s_n = \frac{6z_{n+1}^2 - 2(2n^2+7n+6)}{(n+2)(n+3)z_n z_{n+1}}.$$

For the case (i) or (ii), the sequence $(s_n)_{n\geq N}$ is increasing. Therefore, the theorem follows from Lemma 21. □

Now we discuss the log-balancedness of some sequences by Theorems 20 and 22.

The Pell sequence $(P_n)_{n\geq 0}$ is sequence A000129 in the OEIS [13] and satisfies the recurrence $P_{n+1} = 2P_n + P_{n-1}$ with $P_0 = 0$, $P_1 = 1$. In paper [9], the authors consider, among other things, Hadamard product of Fibonacci and Catalan numbers (Fibonacci and Catalan numbers). The Pell-Lucas sequence $(Q_n)_{n\geq 0}$ is sequence A001333 in the OEIS [13] and satisfies the recurrence $Q_{n+1} = 2Q_n + Q_{n-1}$ with $Q_0 = 1$, $Q_1 = 1$.

The Lucas sequence $(L_n)_{n\geq 0}$ is sequence A000032 in the OEIS [13] and satisfies the recurrence $L_{n+1} = L_n + L_{n-1}$ with $L_0 = 2$, $L_1 = 1$. The sequence $(C_n L_n)_{n\geq 0}$ is sequence A371986 in the OEIS [13].

For the values of $(C_n F_{2n+2})_{n\geq 0}$, $(C_n P_n)_{n\geq 0}$, $(C_n Q_n)_{n\geq 0}$, and $(C_n L_n)_{n\geq 0}$, see Table 7, Table 8, Table 9, and Table 10.

n	0	1	2	3	4	5	6	7	8	9
$C_n F_{2n+2}$	1	3	16	105	770	6048	49764	423423	3695120	32891430

Table 7: Some values of $(C_n F_{2n+2})_{n\geq 0}$.

n	0	1	2	3	4	5	6	7	8	9
C_nP_n	0	1	4	25	168	1218	9240	72501	583440	4789070

Table 8: Some values of $(C_nP_n)_{n\geq 0}$.

n	0	1	2	3	4	5	6	7	8	9
C_nQ_n	1	1	6	35	238	1722	13068	102531	825110	6772766

Table 9: Some values of $(C_nQ_n)_{n\geq 0}$.

n	0	1	2	3	4	5	6	7	8	9
C_nL_n	2	1	6	20	98	462	2376	12441	67210	369512

Table 10: Some values of $(C_nL_n)_{n\geq 0}$.

In the final of this section, we discuss the log-behavior of the sequences $(C_nP_n)_{n\geq 0}$, $(C_nQ_n)_{n\geq 0}$, $(C_nL_n)_{n\geq 0}$, and $(C_nF_{2n+2})_{n\geq 0}$.

Corollary 23. *The sequences $(C_nP_n)_{n\geq 2}$, $(C_nQ_n)_{n\geq 2}$, and $(C_nL_n)_{n\geq 4}$ are log-balanced and the sequences $(C_nP_n)_{n\geq 2}$, $(C_nQ_n)_{n\geq 1}$, and $(C_nL_n)_{n\geq 3}$ are tempered.*

Proof. Let $z_n = P_n$. For $n \geq 0$, we have $P_nP_{n+2} - P_{n+1}^2 = (-1)^{n+1}$ (see [11, p. 154]). One can prove by induction that $P_n \geq n+1$ for $n \geq 3$. Hence, $3P_nP_{n+2} \geq (n+3)(2n+1)$ for $n \geq 3$. For $n = 2$, we have $3P_2P_4 = 72 \geq 5 \cdot 5 = 25$. Therefore, from Theorem 20 we have that the sequence $(C_nP_n)_{n\geq 2}$ is log-balanced. Hence, $(C_nP_n)_{n\geq 2}$ is tempered.

Let $z_n = Q_n$. For $n \geq 0$, we have $Q_nQ_{n+2} - Q_{n+1}^2 = 2(-1)^n$ (see [11, p. 155]). One can prove by induction that $Q_n \geq 2n+1$ for $n \geq 3$. Hence, $3Q_nQ_{n+2} \geq 2(n+3)(2n+1)$ for $n \geq 3$. For $n = 2$, we have $3Q_2Q_4 = 153 \geq 2 \cdot 5 \cdot 5 = 50$. Therefore, from Theorem 20 we have that the sequence $(C_nQ_n)_{n\geq 2}$ is log-balanced. Hence, $(C_nQ_n)_{n\geq 2}$ is tempered.

Let $z_n = L_n$. For $n \geq 0$, we have $L_nL_{n+2} - L_{n+1}^2 = 5(-1)^n$ (see [11, p. 19]). One can prove by induction that $L_n \geq 3n$ for $n \geq 6$. Hence, $3L_nL_{n+2} \geq 5(n+3)(2n+1)$ for $n \geq 6$. We find that $3L_nL_{n+2} \geq 5(n+3)(2n+1)$ for $4 \leq n \leq 5$. Therefore, from Theorem 20 we have that the sequence $(C_nL_n)_{n\geq 4}$ is log-balanced. Hence, $(C_nL_n)_{n\geq 4}$ is tempered.

We observe that $(C_nQ_n)_{1\leq n\leq 3}$ and $(C_nL_n)_{3\leq n\leq 5}$ are tempered. Thus, the sequences $(C_nQ_n)_{n\geq 1}$ and $(C_nL_n)_{n\geq 3}$ are tempered. □

Remark 24. For the Catalan sequence $(C_n)_{n\geq 0}$ and the Fibonacci sequence $(F_n)_{n\geq 0}$, we can prove that $(C_nF_{n+1})_{n\geq 2}$ is log-balanced by Theorem 20.

Corollary 25. *The sequence $(C_nF_{2n+2})_{n\geq 0}$ is log-balanced (hence, $(C_nF_{2n+2})_{n\geq 0}$ is tempered).*

Proof. For $n \geq 0$, let $z_n = F_{2n+2}$. It follows from (4) that $z_n z_{n+2} \geq 4(n+1)(n+3) > 2n^2 + 7n + 3$ for $n \geq 2$. Note that $F_{2n+2}F_{2n+6} - F_{2n+4}^2 = -1$ for $n \geq 0$. For Theorem 22 we have that the sequence $(C_n F_{2n+2})_{n\geq 2}$ is log-balanced. On the other hand, we observe that $(C_n F_{2n+2})_{0\leq n\leq 3}$ is log-balanced. Thus, the sequence $(C_n F_{2n+2})_{n\geq 0}$ is log-balanced. Hence, $(C_n F_{2n+2})_{n\geq 0}$ is tempered.

□

4 Concluding remarks

For the Fibonacci sequence $(F_n)_{n\geq 0}$ and a positive sequence $(z_n)_{n\geq 0}$, we mainly discussed the log-behavior of $(z_n F_{n+1})_{n\geq 0}$. We derived some sufficient conditions for the log-balancedness of $(z_n F_{n+1})_{n\geq 0}$. For instance, for the Catalan sequence $(C_n)_{n\geq 0}$ and the Fibonacci sequence $(F_n)_{n\geq 0}$, we showed that $(C_n F_{n+1})_{n\geq 2}$ is log-balanced. For the sequences $(C_n)_{n\geq 0}$ and $(F_n)_{n\geq 0}$, set

$$a_n = \sum_{k=0}^{n} \binom{n}{k} C_{n-k} F_k.$$

The sequence $(a_n)_{n\geq 0}$ is sequence A277220 in the OEIS [13]. The authors plan to study the log-behavior of the sequence $(a_n)_{n\geq 0}$ and propose the following hypothesis.

Conjecture 26. For the sequence $(a_n)_{n\geq 0}$, $(a_n)_{n\geq 1}$ is log-balanced.

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