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Enumeration of Certain k-Block Partitions of Barry

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Abstract

The OEIS sequence A080416, proposed by Barry, is a Stirling-like number triangle defined by a paired decomposition of $C(n+3,3)$. The recurrence formula for this sequence can be derived from an article by Corcino et al. However, explicit formulas have not been discovered. In this paper, we derive two formulas for calculating the members of this sequence. One of them involves matrix multiplication, while the other one does not require it. We also generalize the approach for sequences defined by counting North-East lattice paths with weights.

1 Introduction

The sequence A080416 from the *On-Line Encyclopedia of Integer Sequences* (OEIS) was first proposed by Barry in 2003 [3]. As stated by Gonzalez, it "gives the number of ways to construct pairs of k-block partitions from 1 to n such that the sum of the minima of the i^{th} block of the first partition and the $(k-i+1)^{\text{th}}$ block of the second partition is $n+1$" [3].

In 2022, Lavrov [2, 3] noted that the results of a paper by Corcino et al. [1] imply the following recursive formula for this sequence:

$$s(\beta,n,k) = \begin{cases} 1, & \text{if } n = k = 0; \\ 0, & \text{if } n = 0 \text{ XOR } k = 0; \\ s(\beta+1,n-1,k-1) + \beta\cdot k\cdot s(\beta,n-1,k), & \text{otherwise.} \end{cases} \qquad (1)$$

The triangle given by $s(0,n,k)$ looks as follows [2]:

$$\begin{matrix} s(0,2,2) & & & & \\ s(0,3,2) & s(0,3,3) & & & \\ s(0,4,2) & s(0,4,3) & s(0,4,4) & & \\ s(0,5,2) & s(0,5,3) & s(0,5,4) & s(0,5,5) & \\ s(0,6,2) & s(0,6,3) & s(0,6,4) & s(0,6,5) & s(0,6,6) \end{matrix} = \begin{matrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 4 & 1 & & \\ 1 & 12 & 10 & 1 & \\ 1 & 32 & 67 & 20 & 1 \end{matrix}. \tag{2}$$

We will start from the recursive formula (1) and derive two non-recursive formulas. The first of them involves matrix multiplication, while the second one is matrix-free.

2 Deriving a matrix-based formula

Formula (1) represents recurrence relations in three dimensions. However, if we fix k, we notice that all the recurrence relations needed to calculate $s(0,n,k)$ lie in one plane. Thus, for $s(0,n,k)$ with fixed k, we can reconstruct the recurrence in two dimensions.

Let $j = n - k$, $a = k + \beta$, and $S_a(k,j) = s(a-k, j+k, k)$. The (k,j) coordinate system allows us to use North-East lattice paths with weights 1 (East) and $k(a-k)$ (North):

$$S_a(k,j) = \begin{cases} 1, & \text{if } j = k = 0; \\ 0, & \text{if } j < 0 \text{ or } k < 0; \\ S_a(k-1,j) + k \cdot (a-k) \cdot S_a(k,j-1), & \text{otherwise.} \end{cases}$$

Thus,

$$s(0,n,a) = S_a(a,n-a) = S_a(a-1,n-a) + 0 = S_a(a-1,n-a).$$

We can evaluate $S_a(a-1,n-a)$ by calculating the values of $S_a(k,j)$ at diagonals starting from the $(a-1)$-element diagonal $\{(1,0),(2,-1),\ldots,(a-1,-a+2)\}$ (we choose the first element of this diagonal to be at $(1,0)$ due to the fact that $S_a(1,0) = S_a(0,0)$, while $S_a(0,j) = 0$ for all $j \neq 0$). The corresponding values of $S_a(k,j)$ for this diagonal can be accumulated in the vector

$$c_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

The values of $S_a(k,j)$ for the higher diagonal $\{(1,1),(2,0),\ldots,(a-1,-a+3)\}$ can be calculated by matrix-vector multiplication:

$$c_1 = A \cdot c_0 = \begin{bmatrix} a-1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 1 & 2(a-2) & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 1 & 3(a-3) & \cdots & 0 & 0 & \cdots \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & k(a-k) & \cdots \\ \vdots & & & & & & \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{3}$$

In formula (3), k is the column number of the $(a-1) \times (a-1)$ matrix A.

The value of $S_a(a-1, n-a)$ is the last entry of the vector c_{n-2}:

$$c_{n-2} = A^{n-2} \cdot c_0.$$

Thus, for all $n \geq a \geq 2$:

$$s(0,n,a) = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a-1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 1 & 2(a-2) & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 1 & 3(a-3) & \cdots & 0 & 0 & \cdots \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & k(a-k) & \cdots \\ \vdots & & & & & & \end{bmatrix}^{n-2} \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{4}$$

Using formula (4), we evaluated the first entries of the sequence and listed them in Table 1.

n \ a	2	3	4	5	6	7	8	9	10	11
2	1									
3	1	1								
4	1	4	1							
5	1	12	10	1						
6	1	32	67	20	1					
7	1	80	376	252	35	1				
8	1	192	1909	2560	742	56	1			
9	1	448	9094	22928	12346	1848	84	1		
10	1	1024	41479	189120	177599	46912	4074	120	1	
11	1	2304	183412	1472704	2318149	1013936	149644	8184	165	1

Table 1: Triangular array of the numbers $s(0, n, a)$ up to $n = 11$.

In the OEIS [3], there is a list of values of the first 50 members of the sequence, and they are the same as given in Table 1.

3 Deriving a matrix-free formula

The value of $s(0, n, a)$ is given in the $(a-1, 1)$ entry (the bottom left corner) of the $(a-1) \times (a-1)$ matrix A^{n-2}.

Let C_w denote a matrix in which all entries located w positions to the left of the main diagonal are 1, while all other entries are 0. In this case,

$$A = D + C_1 = \mathrm{diag}(a-1, 2(a-2), \ldots, i(a-i), \ldots) + C_1,$$

where i is the row number.

The matrices C_w have the following property:

$$C_w = C_1^w. \tag{5}$$

When we expand $(D + C_1)^{n-2}$, we are interested only in those terms that change the $(a-1, 1)$ entry of the final matrix and thus only in those of them that have $a-2$ factors of C_1 (due to formula (5)). Thus, $s(0, n, a)$ is the $(a-1, 1)$ entry of the matrix that is the sum of all possible products of the following form:

$$\underbrace{D \cdots D}_{t_1} \cdot C_1 \underbrace{D \cdots D}_{t_2} \cdot C_1 \cdots C_1 \underbrace{D \cdots D}_{t_{a-1}} = D^{t_1} \cdot C_1 \cdot D^{t_2} \cdot C_1 \cdots C_1 \cdot D^{t_{a-1}}, \tag{6}$$

where $\sum t_m = n - a$.

This leads to investigating a product of matrices of the form $C_1 \cdot D^m$. For this purpose, let us call a matrix that was derived from a diagonal matrix by shifting all its entries by w positions downwards a *w-shifted diagonal matrix*, and let $\mathrm{sdiag}_w(d_i)$ denote it, where d_i is a formula for evaluating the element of the i^{th} column on the shifted diagonal.

With this notation,

$$\begin{aligned}
\mathrm{sdiag}_p(x_i) \cdot \mathrm{sdiag}_r(y_i) &= \mathrm{sdiag}_{p+r}(x_{i+r} y_i),\\
C_w &= \mathrm{sdiag}_w(1),\\
D &= \mathrm{sdiag}_0(i(a-i)),\\
C_1 \cdot D &= \mathrm{sdiag}_1(i(a-i)),\\
C_1 \cdot D^t &= \mathrm{sdiag}_1(i^t (a-i)^t).
\end{aligned}$$

Then (6) is equivalent to

$$D^{t_1} \prod_{m=2}^{a-1} \mathrm{sdiag}_1(i^{t_m} (a-i)^{t_m})$$

that can be simplified to

$$\mathrm{sdiag}_{a-2}\Big(\prod_{i=1}^{a-1} i^{t_{a-i}} (a-i)^{t_{a-i}}\Big). \tag{7}$$

Let $p_i = t_{a-i}$. Then the $(a-1,1)$ entry of matrix (7) is

$$\prod_{i=1}^{a-1} i^{p_i}(a-i)^{p_i}.$$

The value of $s(0,n,a)$ is a sum of such products for all possible p_i ($p_i \geq 0$, while $\sum_{i=1}^{a-1} p_i = n-a$). Thus,

$$s(0,n,a) = \sum_{\sum p_i = n-a} \Big(\prod_{i=1}^{a-1} i^{p_i}(a-i)^{p_i}\Big). \tag{8}$$

Using formula (8), we get for $a = 3$ and 4:

$$\begin{aligned} s(0,n,3) &= (n-2)\cdot 2^{n-3}, \\ s(0,n,4) &= 3^{n-4}\sum_{p_2=0}^{n-4}\Big((\frac{4}{3})^{p_2}\cdot(n-3-p_2)\Big). \end{aligned}$$

4 Generalization of the approach

The derivation of formulas (4) and (8) can be viewed as a special case of investigating the sequence $S_a(a,j)$ with corresponding North-East lattice paths in the (x,y) coordinate system with weights $N(x)$ and $E(x)$ (that may include a and j in their formulas) for North and East directions:

$$S_a(x,y) = N(x)\cdot S_a(x,y-1) + E(x)\cdot S_a(x-1,y).$$

If we assume the offset $S_a(0,0) = b$, then a matrix-based formula (using $(a+1)\times(a+1)$ matrices, because for the general case, our first diagonal starts at $(0,0)$ and ends at $(a,-a)$) is

$$S_a(a,j) = b\begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix}(\mathrm{sdiag}_0(N(i-1)) + \mathrm{sdiag}_1(E(i)))^{a+j}\begin{bmatrix}1\\0\\\vdots\\0\\0\end{bmatrix}, \tag{9}$$

while a matrix-free formula is

$$S_a(a,j) = b\Big(\prod_{i=1}^{a} E(i)\Big)\sum_{\sum_{i=1}^{a+1} p_i = j}\Big(\prod_{i=1}^{a+1} N^{p_i}(i-1)\Big). \tag{10}$$

In our case (A080416), $j = n-a$, $E(i) = b = 1$, $N(i-1) = (i-1)(a+1-i)$.

Formulas (9) and (10) can be applied to sequences defined by counting North-East lattice paths with weights. Some examples of such sequences are given in Table 2.

b	$N(i)$	$E(i)$	Description	Sequence
1	const	const	triangles of coefficients in expansion of $(E + N \cdot x)^n$	A013609–A013628, A027465–A027467, A038207, A038208, A038210, A038212, A038214–A038218, A038220–A038222, A038224, A038226–A038234, A038236, A038238–A038248, A038250–A038260, A038262–A038338, A123187, A147716, A303901, A317498
1	i	1	Stirling numbers of the second kind	A008277
1	$i+2$	1	a matrix product of the Stirling numbers of the second kind with the binomial coefficients	A126351
1	$i+1$	1	the reflected triangle of Stirling numbers of the second kind	A008278
1	$i+1$	$i+1$	$\left\{ {n \atop n-k+1} \right\} (n-k+1)!$	A090582
1	$i+1$	i	$\left\{ {n \atop k} \right\} (k-1)!$	A028246, A130850
1	2^i	1	the triangle of Gaussian binomial coefficients $[n, k]$ for $q = 2$	A022166
1	2^i	2^i		A108084
1	1	$j+i$	$\frac{n!}{(n-k)!} \binom{n}{k}$	A144084
1	1	$a+j+i$	$\frac{(n+k)!}{(n-k)!\, k!}$	A113025
1	$a+i$	1		A143395

Table 2: Sequences defined by counting North-East lattice paths with weights.

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