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Pairs of Intertwined Integer Sequences

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Abstract

In previous work we computed the number $C_n(q)$ of ideals of codimension n of the algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ of two-variable Laurent polynomials over a finite field $\mathbb{F}_q$: it turned out that $C_n(q)$ is a palindromic polynomial of degree $2n$ in q, divisible by $(q-1)^2$. The quotient $P_n(q) = C_n(q)/(q-1)^2$ is a palindromic polynomial of degree $2n-2$. For each $n \geq 1$ let $\overline{P}_n(X) \in \mathbb{Z}[X]$ be the degree $n-1$ polynomial such that $\overline{P}_n(q+q^{-1}) = P_n(q)/q^{n-1}$. In this article we show that for every integer N the integer value $\overline{P}_n(N)$ is close to the value at N of the degree $n-1$ polynomial $F_{n-1}(X) = 1 + \sum_{k=1}^{n-1} \overline{T}_k(X)$, which is a sum of monic versions $\overline{T}_k(X)$ of Chebyshev polynomials of the first kind. We give a precise formula for $\overline{P}_n(X)$ as a linear combination of the $F_k(X)$, each appearance of the latter being indexed by an odd divisor of n. As a consequence, $\overline{P}_n(X) = F_{n-1}(X)$ if and only if n is a power of 2. We exhibit similar formulas for $C_n(q)$.

1 Introduction

In this article, for each integer N we produce a pair $(\overline{P}_n(N), (F_{n-1}(N))_{n\geq 1}$ of integer sequences. Both sequences, though of differing origin, consist of amazingly close elements. Some of them do even coincide: we have $\overline{P}_n(N) = F_{n-1}(N)$ whenever n is a power of 2. The reader is invited to examine Table 1 which exhibits such pairs for $N = 3, 4, 5$. In this table coinciding values have been set in boldface and values differing by 1 in italics.

n	$\overline{P}_n(3)$	$F_{n-1}(3)$	$\overline{P}_n(4)$	$F_{n-1}(4)$	$\overline{P}_n(5)$	$F_{n-1}(5)$
1	**1**	**1**	**1**	**1**	**1**	**1**
2	**4**	**4**	**5**	**5**	**6**	**6**
3	*10*	*11*	*18*	*19*	*28*	*29*
4	**29**	**29**	**71**	**71**	**139**	**139**
5	72	76	260	265	660	666
6	*200*	*199*	*990*	*989*	*3192*	*3191*
7	510	521	3672	3691	15260	15289
8	**1364**	**1364**	**13775**	**13775**	**73254**	**73254**
9	3546	3571	51343	51409	350848	350981
10	*9348*	*9349*	*191860*	*191861*	*1681650*	*1681651*
11	24400	24476	715770	716035	8056608	8057274
12	64090	64079	2672298	2672279	38604748	38604719
13	167562	167761	9972092	9973081	184963130	184966321
14	439200	439204	37220040	37220045	886226880	886226886
15	1149360	1149851	138903480	138907099	4246152960	4246168109
16	**3010349**	**3010349**	**518408351**	**518408351**	**20344613659**	**20344613659**

Table 1: Values of $\overline{P}_n(N)$ and $F_{n-1}(N)$ for $N = 3, 4, 5$.

Let us now describe the components of such pairs. Starting with $\overline{P}_n(N)$, let $\mathbb{F}_q$ be a finite field of cardinality q and $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ be the algebra of two-variable Laurent polynomials with coefficients in $\mathbb{F}_q$. For every integer $n \geq 1$, let $C_n(q)$ denote the number of ideals I of codimension n of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$, i.e., such that the quotient $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]/I$ is a $\mathbb{F}_q$-vector space of dimension n. (For algebraic geometers, $C_n(q)$ counts the $\mathbb{F}_q$-points of the Hilbert scheme of n points on the two-dimensional torus.) The authors [9] showed that $C_n(q)$ is a monic polynomial of degree $2n$ with integer coefficients in the variable q; moreover, $C_n(q)$ is palindromic and divisible by $(q-1)^2$. They [10, Th. 1.1] gave explicit formulas for the

coefficients of $C_n(q)$; also see § 5.2 below.

The polynomial $C_n(q)$ being divisible by $(q-1)^2$, we may consider the degree $2n-2$ polynomial $P_n(q) = C_n(q)/(q-1)^2$: it is monic, palindromic and has integer coefficients. One of the most striking features of $P_n(q)$ is that its coefficients are all *nonnegative*; actually, each coefficient of $P_n(q)$ can be expressed as the number of divisors of n in a certain real interval; see [10, Thm. 1.3] and § 4.1 below.

The authors [10, Thm. 1.6] also computed the values taken by $P_n(q)$ when q is a complex root of unity of order $1, 2, 3, 4$ or 6, equivalently when $q+q^{-1}$ takes one of the respective values $2, -2, -1, 0, 1$, thus recovering well-known arithmetical functions. For instance, $P_n(1) = \sigma(n)$ is equal to the sum of divisors of n and $P_n(-1) = r(n)/4$, where $r(n)$ is the number of representations of n as a sum of squares of two integers.

More generally, for every integer $n \geq 1$, the Laurent polynomial $P_n(q)/q^{n-1}$ takes integer values whenever $q + q^{-1} = N$ is an integer. Computing such values is the main objective of this paper. For simplicity we replace the Laurent polynomial $P_n(q)/q^{n-1}$ in the variable q by the degree $n-1$ polynomial $\overline{P}_n(X)$ in the variable X, defined by

$$\overline{P}_n(q+q^{-1}) = P_n(q)/q^{n-1}. \tag{1}$$

(This is possible since $P_n(q)$ is palindromic of degree $2n-2$.) We have thus reduced our problem to the computation of the values of $\overline{P}_n(X)$ at integers N.

Searching the *On-Line Encyclopedia of Integer Sequences* (OEIS) [12] for the sequence

$$(\overline{P}_1(5), \overline{P}_2(5), \overline{P}_3(5), \overline{P}_4(5), \overline{P}_5(5), \overline{P}_6(5)) = (1, 6, 28, 139, 660, 3192)$$

yielded, as of January 7 2025, what the OEIS called an "approximate match", namely the sequence of integers consisting of the first few coefficients of the power series expansion of the rational function

$$\frac{1+t}{1-5t+t^2}\,.$$

Replacing 5 by 3 and by 4 in this rational function has produced similar approximate matches for $\overline{P}_n(3)$ and $\overline{P}_n(4)$ respectively. In view of this, it was natural for us to replace $3, 4, 5$ by an indeterminate X as follows.

The second integer sequence mentioned above consists of integer values at N of the polynomials $F_k(X)$ defined by the following equality of formal power series:

$$\sum_{k\geq 0} F_k(X)\, t^k = \frac{1+t}{1-Xt+t^2}\,. \tag{2}$$

It turns out that each polynomial $F_k(X)$ is of degree $k-1$, has integer coefficients, and we have

$$F_k(X) = 1 + \sum_{m=1}^{k} \overline{T}_m(X), \tag{3}$$

which is the sum of the monic versions $\overline{T}_m(X)$ of the Chebyshev polynomials of the first kind defined in § 3.1 below.

With this notation we establish that each $\overline{P}_n(X)$ can be approximated by $F_{n-1}(X)$ in the sense that the difference between these two monic polynomials, both of degree $n-1$, is a polynomial of much lower degree (see Theorem 12 for a precise statement).

More interestingly, $\overline{P}_n(X)$ can be expressed as the following linear combination of $F_k(X)$, whose terms are indexed by the odd divisors d of n:

$$\overline{P}_n(X) = \sum_{\substack{d|n,\, d \text{ odd} \\ r_n(d)\geq 0}} F_{r_n(d)}(X) - \sum_{\substack{d|n,\, d \text{ odd} \\ r_n(d)<0}} F_{-r_n(d)-1}(X), \tag{4}$$

where $r_n(d) = n/d - (d+1)/2$; see Theorem 16. For $d = 1$ we have $r_n(1) = n-1$, which implies the presence of $F_{n-1}(X)$ in (4) for all $n \geq 1$.

Since the powers of 2 are the only integers with a unique odd divisor, namely $d = 1$, we have $\overline{P}_n(X) = F_{n-1}(X)$ if and only if n is a power of 2. This explains the coincidences observed in Table 1.

We deduce Formula (4) from the following new expression for $C_n(q)$:

$$C_n(q) = q^n \sum_{d|n,\, d \text{ odd}} \left(q^{\frac{n}{d} - \frac{d-1}{2}} + q^{-\frac{n}{d} + \frac{d-1}{2}} - q^{\frac{n}{d} - \frac{d+1}{2}} - q^{-\frac{n}{d} + \frac{d+1}{2}} \right). \tag{5}$$

See Theorem 23.

Formula (5) allows us to express the local zeta function and the Hasse–Weil zeta function of the Hilbert scheme of n points on the two-dimensional torus as products whose factors are indexed by the odd divisors of n (see § 5.3).

Let us detail the content of each section. In Section 2, for each $n \geq 1$ we define the degree $n-1$ polynomial $\overline{P}_n(X)$ by (1). As mentioned above, the integers of interest to us are the values $\overline{P}_n(N)$ at arbitrary integers N. Building on previous work [10], we show that the function $n \mapsto |\overline{P}_n(N)|$ is multiplicative if $N = -2, -1, 0$ or 2; for all remaining integer values of N it is not multiplicative.

Section 3 is devoted to the monic versions $\overline{T}_k(X)$ of the Chebyshev polynomials of the first kind and to the polynomials $F_k(X)$ defined above.

In Section 4 we state the results on $\overline{P}_n(X)$ mentioned above. Theorem 12 and Theorem 16 with Formula (4) are the main results. As special cases of (4) we obtain formulas for $P_n(X)$ in terms of the polynomials $F_k(X)$ for various families of integers n.

Section 5 is devoted to the polynomials $C_n(q)$: we prove (5) and state the formulas for the zeta functions mentioned above. In Section 6 we use the results of Section 5 to prove Theorem 16.

2 The polynomials $\overline{P}_n(X)$

2.1 Definition and basic properties

For every positive integer n we define the degree $n-1$ polynomial $\overline{P}_n(X)$ by

$$\overline{P}_n(q+q^{-1}) = P_n(q)/q^{n-1}, \tag{6}$$

where $P_n(q)$ was introduced in Section 1. This makes sense since $P_n(q)$ is a palindromic polynomial of degree $2n-2$. As $P_n(q)$ has integer coefficients, so does $\overline{P}_n(X)$.

As a consequence of [9, Cor. 1.4], we have the equality of formal power series

$$1 + (q+q^{-1}-2)\sum_{n\geq 1} \frac{P_n(q)}{q^{n-1}}\, t^n = \prod_{i\geq 1} \frac{(1-t^i)^2}{1-(q+q^{-1})t^i + t^{2i}}\,.$$

This translates into the following equality for the polynomials $\overline{P}_n(X)$:

$$1 + (X-2)\sum_{n\geq 1} \overline{P}_n(X)\, t^n = \prod_{i\geq 1} \frac{(1-t^i)^2}{1-Xt^i + t^{2i}}\,. \tag{7}$$

The authors [10, Thm. 1.3] established that each polynomial $P_n(q)$ is of the form

$$P_n(q) = a_{n,0}\, q^{n-1} + \sum_{i=1}^{n-1} a_{n,i}\, \left(q^{n+i-1} + q^{n-i-1}\right), \tag{8}$$

where the coefficients $a_{n,i}$ are all *nonnegative integers* (to be expressed explicitly in the proof of Lemma 14). It follows that for each positive integer n,

$$\overline{P}_n(X) = a_{n,0} + \sum_{i=1}^{n-1} a_{n,i}\, \overline{T}_i(X), \tag{9}$$

where $\overline{T}_i(X)$ is the monic Chebyshev polynomial of degree i defined by (10) below. Since the coefficients $a_{n,i}$ are nonnegative integers, each $\overline{P}_n(X)$ is a *sum* of the polynomials $\overline{T}_k(X)$.

A list of the polynomials $\overline{P}_n(X)$ for $1 \leq n \leq 12$ is given in Table 2. It is based on the corresponding list of $P_n(q)$ in Table 2 of either [9] or [10].

n	$\overline{P}_n(X)$
1	1
2	$X+1$
3	X^2+X-2
4	X^3+X^2-2X-1
5	$X^4+X^3-3X^2-3X$
6	$X^5+X^4-4X^3-3X^2+3X+2$
7	$X^6+X^5-5X^4-4X^3+5X^2+2X$
8	$X^7+X^6-6X^5-5X^4+10X^3+6X^2-4X-1$
9	$X^8+X^7-7X^6-6X^5+15X^4+9X^3-11X^2-X+3$
10	$X^9+X^8-8X^7-7X^6+21X^5+15X^4-20X^3-10X^2+5X$
11	$X^{10}+X^9-9X^8-8X^7+28X^6+21X^5$ $-36X^4-21X^3+18X^2+7X-2$
12	$X^{11}+X^{10}-10X^9-9X^8+36X^7+28X^6$ $-56X^5-35X^4+35X^3+16X^2-5X-2$

Table 2: The polynomials $\overline{P}_n(X)$.

The following result explains why all entries in Table 1 are positive.

Proposition 1. *Let N be an integer ≥ 2. Then $\overline{P}_n(N) > 0$ for all $n \geq 1$.*

Proof. This is a consequence of Proposition 6 below and of the fact mentioned above that each $\overline{P}_n(X)$ is a sum of the $\overline{T}_k(X)$. □

2.2 Multiplicative values of $|P_n(q)|$ at roots of unity

The content of this subsection is an addendum to [10, Th. 1.6]. It concerns the cases when $q+q^{-1}=N$ is an integer such that $-2 \leq N \leq 2$, equivalently when q is a root of unity of order $d = 1, 2, 3, 4$ or 6.

Proposition 2. *Let q be a complex root of unity of order $d = 1, 2, 3$ or 4. Then the sequence of nonnegative integers*

$$|P_n(q)| = \left|P_n(q)/q^{n-1}\right| = \left|\overline{P}_n(q+q^{-1})\right|$$

is multiplicative as a function of n.

Recall that a function $f(n)$ defined on positive integers is *multiplicative* if $f(mn) = f(m)f(n)$ whenever m and n are coprime.

Proof. When $q = 1$, by [9, Cor. 1.2] we have $\overline{P}_n(2) = P_n(1) = \sigma(n)$, the sum of divisors of n, which is well-known to be multiplicative. See Apostol [2, Sect. 2.13] and the OEIS [12] at A000203.

For $q = -1$ we have $\overline{P}_n(-2) = P_n(-1) = r(n)/4$, where $r(n)$ is the number of representations of n as a sum of squares of two integers. By the OEIS [12, A002654] the sequence $r(n)/4$ is multiplicative. Then so is $P_n(-1)$.

For $q = j$, a root of unity of order 3, we have $\overline{P}_n(-1) = P_n(j)/j^{n-1} = \lambda(n)$, where $\lambda(n)$ is the sequence A113063 in the OEIS [12]. Fine [6, Sect. 32, p. 79] states that $\lambda(n)$ is multiplicative. See also Caballero [5].

Let i be a square root of -1. It follows again from [10, Th. 1.6] that we have

$$|\overline{P}_n(0)| = |P_n(i)/i^{n-1}| = r'(n)/2,$$

where $r'(n)$ is the number of representations of n as a sum of a square and twice another square. The sequence of integers $r'(n)/2$ listed as A002325 in the OEIS [12] is known to be multiplicative. □

The case when $q = \omega$ is a root of unity of order 6 is slightly more involved.

Proposition 3. *Let ω be a root of unity of order 6. Then for all pairs (m, n) of coprime positive integers we have*

$$|P_m(\omega)||P_n(\omega)| = |P_{mn}(\omega)| \text{ when } (m,n) \equiv (0,1), (1,1) \text{ or } (1,2) \pmod 3,$$

$$|P_m(\omega)||P_n(\omega)| = 2\,|P_{mn}(\omega)| \quad \text{when } (m,n) \equiv (0,2) \pmod 3,$$

$$|P_m(\omega)||P_n(\omega)| = 4\,|P_{mn}(\omega)| \quad \text{when } (m,n) \equiv (2,2) \pmod 3.$$

Proof. This is a consequence of the multiplicativity of the sequence $r(n)/4$ observed above and of the following reformulation of a result due to the authors [8, Th. 1.1] and [10, Th. 1.6 (d)]:

$$|P_n(\omega)| = \begin{cases} r(n), & \text{if } n \equiv 0 \pmod 3; \\ r(n)/4, & \text{if } n \equiv 1 \pmod 3; \\ r(n)/2, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

□

We have the following consequence of Propositions 2 and 3.

Corollary 4. *Given a pair (m, n) of coprime positive integers, the polynomial $\overline{P}_{mn}(X)^2 - \overline{P}_m(X)^2\overline{P}_n(X)^2$ is divisible by $X(X+1)(X^2-4)$, and moreover by $X-1$ if $(m,n) \equiv (0,1), (1,1)$ or $(1,2) \pmod 3$.*

We may wonder whether there are other integers N such that $n \mapsto |\overline{P}_n(N)|$ is a multiplicative function. The answer is no.

Proposition 5. *The function $n \mapsto |\overline{P}_n(N)|$ is multiplicative if and only if $N = -2, -1, 0$ or 2.*

Proof. The "if direction" is a reformulation of Proposition 2. Conversely, if $n \mapsto |\overline{P}_n(N)|$ is multiplicative, then we have $|\overline{P}_6(N)| = |\overline{P}_2(N)||\overline{P}_3(N)|$ as a special case. Now the polynomial $\overline{P}_6(X)^2 - \overline{P}_2(X)^2\,\overline{P}_3(X)^2$ can be factored as

$$\overline{P}_6(X)^2 - \overline{P}_2(X)^2\,\overline{P}_3(X)^2 = X(X-1)^3(X+1)^3(X-2)(X+2)^2.$$

This implies that multiplicativity holds only for integers N such that $|N| \leq 2$. We rule out the case $N = 1$ in view of Proposition 3. □

3 The polynomials $F_n(X)$

Let us define the monic Chebyshev polynomials and the polynomials $F_n(X)$ mentioned in the introduction.

3.1 Monic Chebyshev polynomials

Given an integer $k \geq 0$, the *standard Chebyshev polynomial of the first kind* $T_k(X)$ is the degree k polynomial defined by

$$\cos(k\theta) = T_k(\cos(\theta)),$$

equivalently by

$$\frac{q^k + q^{-k}}{2} = T_k\left(\frac{q + q^{-1}}{2}\right).$$

Rivlin [13] gave a useful overview of the important properties of Chebyshev polynomials.

We are interested in the following variant of $T_k(X)$: let $\overline{T}_k(X)$ be the degree k polynomial defined by

$$q^k + q^{-k} = \overline{T}_k(q + q^{-1}). \tag{10}$$

Both $T_k(X)$ and $\overline{T}_k(X)$ have integer coefficients. They are clearly related by

$$\overline{T}_k(X) = 2T_k(X/2). \tag{11}$$

The polynomials $\overline{T}_k(X)$ are monic for all $k \geq 1$. That is why we call them *monic Chebyshev polynomials*. Abramowitz et al. [1, p. 778, Eq. 22.5.11] denote the polynomials $\overline{T}_k(X)$ by C_k; Horadam [7] calls them *Vieta–Lucas polynomials* and denotes them by $v_n(x)$.

Table 3 lists the polynomials $\overline{T}_k(X)$ for $1 \leq k \leq 12$.

k	$\overline{T}_k(X)$
0	2
1	X
2	X^2-2
3	X^3-3X
4	X^4-4X^2+2
5	X^5-5X^3+5X
6	$X^6-6X^4+9X^2-2$
7	$X^7-7X^5+14X^3-7X$
8	$X^8-8X^6+20X^4-16X^2+2$
9	$X^9-9X^7+27X^5-30X^3+9X$
10	$X^{10}-10X^8+35X^6-50X^4+25X^2-2$
11	$X^{11}-11X^9+44X^7-77X^5+55X^3-11X$
12	$X^{12}-12X^{10}+54X^8-112X^6+105X^4-36X^2+2$

Table 3: The monic Chebyshev polynomials $\overline{T}_k(X)$.

Let us state a few properties of the polynomials $\overline{T}_k(X)$ which can be easily derived from well-known properties of $T_k(X)$.

First, the polynomials $\overline{T}_k(X)$ can be defined inductively by $\overline{T}_0(X)=2$, $\overline{T}_1(X)=X$, and for $k\geq 1$ by

$$\overline{T}_{k+1}(X)=X\overline{T}_k(X)-\overline{T}_{k-1}(X). \tag{12}$$

The previous linear recurrence relation can be written in matrix form as

$$\begin{pmatrix}\overline{T}_{k+1}(X)\\ \overline{T}_k(X)\end{pmatrix}=M\begin{pmatrix}\overline{T}_k(X)\\ \overline{T}_{k-1}(X)\end{pmatrix},\ \text{ where } M=\begin{pmatrix}X & -1\\ 1 & 0\end{pmatrix}. \tag{13}$$

Next, the generating function of the polynomials $\overline{T}_k(X)$ is the power series expansion of the following rational function:

$$\sum_{k\geq 0}\overline{T}_k(X)\,t^k=\frac{2-Xt}{1-Xt+t^2}. \tag{14}$$

We have the following expression of $\overline{T}_n(X)$ in terms of powers of X:

$$\overline{T}_n(X)=\sum_{m=0}^{[n/2]}(-1)^m\left(\binom{n-m}{m}-\binom{n-m-1}{m-1}\right)X^{n-2m}.$$

We also have a positivity result for the polynomials $\overline{T}_k(X)$.

Proposition 6. *For each $k \geq 0$, we have $\overline{T}_k(x) > 0$ if x is a real number ≥ 2.*

Proof. On one hand, as is well known, the zeros of the standard Chebyshev polynomials $T_k(X)$ all lie in the open interval $(-1, 1)$. Hence, in view of (11) the zeros of $\overline{T}_k(X)$ lie in $(-2, 2)$. On the other, since $\overline{T}_k(X)$ is monic, $\lim_{x\to+\infty} \overline{T}_k(x) = +\infty$. Combining these two facts implies the desired positivity. □

We close our summary on $\overline{T}_k(X)$ with the following result, which may be of independent interest.

Proposition 7. *For all $k \geq 0$ we have $\overline{T}_k(X) = \mathrm{tr}(M^k)$, where M is the 2×2-matrix defined in (13).*

Proof. First, observe that $\mathrm{tr}(M) = X = \overline{T}_1(X)$ and $\mathrm{tr}(M^0) = \mathrm{tr}(\mathrm{Id}) = 2 = \overline{T}_0(X)$, where Id is the identity matrix. In order to complete the proof of the desired equality, it suffices to show that the sequence $(\mathrm{tr}(M^k))_{k\geq 0}$ satisfies the linear recurrence relation (12). By the Cayley–Hamilton theorem applied to the matrix M, we have

$$M^2 = \mathrm{tr}(M)M - \det(M)\mathrm{Id} = XM - \mathrm{Id}.$$

Now multiply both sides by M^{k-1} and take traces. □

3.2 Definition and basic properties of $F_n(X)$

We define the sequence of polynomials $(F_n(X))_{n\geq 0}$ inductively by $F_0(X) = 1$ and by

$$F_n(X) = F_{n-1}(X) + \overline{T}_n(X) \tag{15}$$

for $n \geq 1$. In other words,

$$F_n(X) = 1 + \sum_{k=1}^{n} \overline{T}_k(X). \tag{16}$$

It is clear that each $F_n(X)$ has integer coefficients and is monic of degree n. We give a list of the polynomials $F_n(X)$ for $n \leq 11$ in Table 4.

Our first result is the following.

Proposition 8. *The generating function of the polynomials $F_n(X)$ is the power series expansion of the following rational function:*

$$\sum_{n\geq 0} F_n(X)\, t^n = \frac{1+t}{1-Xt+t^2}\,.$$

Proof. Using (14), we obtain

$$\begin{aligned} 1 + \sum_{k\geq 1} \overline{T}_k(X)\, t^k &= \frac{2-Xt}{1-Xt+t^2} + 1 - \overline{T}_0(X) \\ &= \frac{2-Xt}{1-Xt+t^2} - 1 = \frac{1-t^2}{1-Xt+t^2}\,. \end{aligned}$$

n	$F_n(X)$
0	1
1	$X+1$
2	X^2+X-1
3	X^3+X^2-2X-1
4	$X^4+X^3-3X^2-2X+1$
5	$X^5+X^4-4X^3-3X^2+3X+1$
6	$X^6+X^5-5X^4-4X^3+6X^2+3X-1$
7	$X^7+X^6-6X^5-5X^4+10X^3+6X^2-4X-1$
8	$X^8+X^7-7X^6-6X^5+15X^4+10X^3-10X^2-4X+1$
9	$X^9+X^8-8X^7-7X^6+21X^5+15X^4$ $-20X^3-10X^2+5X+1$
10	$X^{10}+X^9-9X^8-8X^7+28X^6+21X^5$ $-35X^4-20X^3+15X^2+5X-1$
11	$X^{11}+X^{10}-10X^9-9X^8+36X^7+28X^6$ $-56X^5-35X^4+35X^3+15X^2-6X-1$

Table 4: The polynomials $F_n(X)$.

Therefore,

$$\begin{aligned}\sum_{n\geq 0} F_n(X)\,t^n &= \sum_{n\geq 0}\left(1+\sum_{m=1}^{n}\overline{T}_m(X)\right)t^n\\ &= \left(\sum_{\ell\geq 0} t^\ell\right)\left(1+\sum_{k\geq 1}\overline{T}_k(X)\,t^k\right)\\ &= \frac{1}{1-t}\cdot\frac{1-t^2}{1-Xt+t^2}=\frac{1+t}{1-Xt+t^2}\,.\end{aligned}$$

□

Setting $X=0$ in the formula of Proposition 8 yields the *constant term* of $F_n(X)$: for all $m\geq 0$ we have

$$F_{2m}(0)=F_{2m+1}(0)=(-1)^m.$$

For the remaining coefficients of $F_n(X)$ we have the following expression.

Proposition 9. *For $n \geq 1$ we have*

$$F_n(X) = \sum_{0 \leq m \leq n/2} (-1)^m \binom{n-m}{m} X^{n-2m} + \sum_{0 \leq m \leq (n-1)/2} (-1)^m \binom{n-m-1}{m} X^{n-2m-1}.$$

Proof. Expanding the following fraction, we obtain

$$\begin{aligned} \frac{1}{1 - Xt + t^2} &= 1 + \sum_{r \geq 1} t^r (X - t)^r \\ &= 1 + \sum_{r \geq 1} \sum_{s=0}^{r} (-1)^s \binom{r}{s} X^{r-s} t^{r+s} \\ &= 1 + \sum_{n \geq 1} \sum_{0 \leq m \leq n/2} (-1)^m \binom{n-m}{m} X^{n-2m} t^n. \end{aligned}$$

In view of Proposition 8, multiplying by $1 + t$ yields the desired formula for $F_n(X)$. □

Corollary 10. *For $n \geq 5$ we have*

$$\begin{aligned} F_n(X) &= X^n + X^{n-1} - (n-1)X^{n-2} - (n-2)X^{n-3} \\ &\quad + \frac{(n-2)(n-3)}{2} X^{n-4} + \frac{(n-3)(n-4)}{2} X^{n-5} \\ &\quad + \text{monomials of degree} \leq n - 6. \end{aligned}$$

In Section 6 we need the following linear recurrence relation satisfied by the polynomials $F_k(X)$.

Proposition 11. *For $k \geq 1$ we have $F_{k+1}(X) = XF_k(X) - F_{k-1}(X)$.*

Proof. The rational function expressing the generating function in Proposition 8 has the same denominator as the rational function in (14). Therefore the polynomials $F_k(X)$ satisfy the same linear recurrence relation as the polynomials $\overline{T}_k(X)$ in (12); see, e.g., Berstel et al. [3, Chap. 6, Prop. 1.2 and Cor. 1.3]. □

4 Results for the polynomials $\overline{P}_n(X)$

Now we establish a connection between the polynomials $\overline{P}_n(X)$ and the polynomials $F_k(X)$. We start with an approximation result.

4.1 Approximating $\overline{P}_n(X)$ by $F_{n-1}(X)$

The aim of this subsection is the following result.

Theorem 12. *For all $n \geq 2$, the difference $\overline{P}_n(X) - F_{n-1}(X)$ is a polynomial whose degree is smaller than $n/2 - 1$.*

Recall that $\overline{P}_n(X)$ and $F_{n-1}(X)$ are both of degree $n-1$. The theorem states that they coincide in every degree $\geq n/2 - 1$.

The bound on degrees is sharp as can be seen from Table 5: indeed, $\overline{P}_9(X) - F_8(X) = -F_3(X) + F_1(X)$ is of degree $3 < 9/2 - 1$; similarly, $\overline{P}_{11}(X) - F_{10}(X) = -F_4(X)$ is of degree $4 < 11/2 - 1$, and $\overline{P}_{15}(X) - F_{14}(X)$ is of degree $6 < 15/2 - 1$.

Together with Corollary 10 we derive the following.

Corollary 13. *For all $n \geq 10$ we have*

$$\begin{aligned}\overline{P}_n(X) &= X^{n-1} + X^{n-2} - (n-2)X^{n-3} - (n-3)X^{n-4} \\ &\quad + \frac{(n-3)(n-4)}{2}X^{n-5} + \frac{(n-4)(n-5)}{2}X^{n-6} \\ &\quad + \text{monomials of degree} \leq n-7.\end{aligned}$$

Recall the (nonnegative) coefficients $a_{n,i}$ of $\overline{P}_n(X)$ appearing in (8) and (9). For the proof of Theorem 12 we need the following result.

Lemma 14. *We have $a_{n,i} = 1$ for all i such that $n/2 - 1 \leq i \leq n-1$.*

Proof. Given $0 \leq i \leq n-1$, the authors [10, Thm. 1.3] showed that the coefficient $a_{n,i}$ is equal to the number of divisors of n belonging to the half-open interval

$$I_{n,i} = \left(\frac{i + \sqrt{2n + i^2}}{2}, i + \sqrt{2n + i^2} \right].$$

It is enough to show that n is the only divisor of n belonging to $I_{n,i}$. Since all other divisor of n are $\leq n/2$, it suffices to check that $n \leq i + \sqrt{2n + i^2}$ and $n/2 \leq (i + \sqrt{2n + i^2})/2$; the latter inequality is equivalent to the former. They are both equivalent to $n(n-2i) = (n-i)^2 - i^2 \leq 2n$, which simplifies into $n - 2i \leq 2$ hence to $i \geq (n-2)/2 = n/2 - 1$. □

Proof of Theorem 12. By (16) and (9) we have

$$\begin{aligned}\overline{P}_n(X) - F_{n-1}(X) &= (a_{n,0} - 1) + \sum_{1 \leq i \leq n-1} (a_{n,i} - 1)\,\overline{T}_i(X) \\ &= (a_{n,0} - 1) + \sum_{1 \leq i < n/2-1} (a_{n,i} - 1)\,\overline{T}_i(X).\end{aligned}$$

The last equality is a consequence of Lemma 14. Since $\overline{T}_i(X)$ is of degree i, we obtain the desired result. □

Remark 15. The sequence of integers $\overline{P}_n(3)$ is Sequence A329156 in the OEIS [12]. The sequence $(F_{n-1}(3))_{n\geq 1}$ is A002878: we have $F_{n-1}(3) = L(2n+1)$, where $L(n)$ is the Lucas sequence defined by $L(n) = L(n-1) + L(n-2)$, $L(0) = 2$, and $L(1) = 1$.

The sequence $(\overline{P}_n(4))_{n\geq 1}$ is now referenced as A386706 in the OEIS [12]. According to Kimberling's remark in the OEIS [12, A001834], the sequence $(F_{n-1}(4))_{n\geq 1}$ consist of the numerators p_n of the even-rank convergents p_n/q_n to $\sqrt{3}$; these are the convergents that are smaller than $\sqrt{3}$.

The sequence $(\overline{P}_n(5))_{n\geq 1}$ is now A387017 and the sequence $(F_{n-1}(5))_{n\geq 1}$ is A030221 in the OEIS [12]. The latter consists of the values of Chebyshev even-indexed U-polynomials at $\sqrt{7}/2$.

4.2 Odd divisors

Now we give a formula for $\overline{P}_n(X)$ as a linear combination of the polynomials $F_k(X)$, each appearance of $F_k(X)$ being indexed by an odd divisor of n.

Recall that if we express n (≥ 1) as the product $n = 2^a p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $p_1, p_2, \ldots, p_r$ are distinct odd primes and $a, a_1, a_2, \ldots, a_r$ are nonnegative integers, then the number of *odd divisors* of n (including 1) is equal to

$$(a_1+1)(a_2+1)\cdots(a_r+1) \geq 1.$$

Observe that $(a_1+1)(a_2+1)\cdots(a_r+1) = 1$ if and only if n is a power of 2; otherwise, the number of odd divisors of n is at least 2.

For $n \geq 1$ and every odd divisor d of n, define $r_n(d)$ to be the integer

$$r_n(d) = \frac{n}{d} - \frac{d+1}{2}. \tag{17}$$

When d is the trivial divisor 1, we have $r_n(1) = n-1$.

Let us state our main result for $\overline{P}_n(X)$.

Theorem 16. *For all $n \geq 1$,*

$$\overline{P}_n(X) = \sum_{\substack{d|n,\, d \text{ odd} \\ r_n(d)\geq 0}} F_{r_n(d)}(X) - \sum_{\substack{d|n,\, d \text{ odd} \\ r_n(d)<0}} F_{-r_n(d)-1}(X).$$

The number of terms $F_r(X)$ in the previous formula is clearly equal to the number of odd divisors of n. The proof of Theorem 16 will be given in Section 6.

In Table 5 each polynomial $\overline{P}_n(X)$ $(1 \leq n \leq 16)$ is expressed as a sum of the polynomials $\overline{T}_k(X)$ as well as a linear combination of the polynomials $F_k(X)$. In order to save space we left out the indeterminate X from the table.

n	As sums of $\overline{T}_k$	In terms of F_k
1	1	F_0
2	$1+\overline{T}_1$	F_1
3	$\overline{T}_1+\overline{T}_2$	F_2-F_0
4	$1+\overline{T}_1+\overline{T}_2+\overline{T}_3$	F_3
5	$\overline{T}_2+\overline{T}_3+\overline{T}_4$	F_4-F_1
6	$\overline{T}_0+\overline{T}_1+\overline{T}_2+\overline{T}_3+\overline{T}_4+\overline{T}_5$	F_5+F_0
7	$\overline{T}_3+\overline{T}_4+\overline{T}_5+\overline{T}_6$	F_6-F_2
8	$1+\overline{T}_1+\overline{T}_2+\overline{T}_3+\overline{T}_4+\overline{T}_5+\overline{T}_6+\overline{T}_7$	F_7
9	$1+\overline{T}_1+\overline{T}_4+\overline{T}_5+\overline{T}_6+\overline{T}_7+\overline{T}_8$	$F_8-F_3+F_1$
10	$\overline{T}_1+\overline{T}_2+\overline{T}_3+\overline{T}_4+\overline{T}_5+\overline{T}_6+\overline{T}_7+\overline{T}_8+\overline{T}_9$	F_9-F_0
11	$\overline{T}_5+\overline{T}_6+\overline{T}_7+\overline{T}_8+\overline{T}_9+\overline{T}_{10}$	$F_{10}-F_4$
12	$\overline{T}_0+2\overline{T}_1+2\overline{T}_2+\overline{T}_3+\overline{T}_4+\overline{T}_5$ $+\overline{T}_6+\overline{T}_7+\overline{T}_8+\overline{T}_9+\overline{T}_{10}+\overline{T}_{11}$	$F_{11}+F_2$
13	$\overline{T}_6+\overline{T}_7+\overline{T}_8+\overline{T}_9+\overline{T}_{10}+\overline{T}_{11}+\overline{T}_{12}$	$F_{12}-F_5$
14	$\overline{T}_2+\overline{T}_3+\overline{T}_4+\overline{T}_5+\overline{T}_6+\overline{T}_7$ $+\overline{T}_8+\overline{T}_9+\overline{T}_{10}+\overline{T}_{11}+\overline{T}_{12}+\overline{T}_{13}$	$F_{13}-F_1$
15	$\overline{T}_0+\overline{T}_1+\overline{T}_2+\overline{T}_3+\overline{T}_7+\overline{T}_8$ $+\overline{T}_9+\overline{T}_{10}+\overline{T}_{11}+\overline{T}_{12}+\overline{T}_{13}+\overline{T}_{14}$	$F_{14}(X)-F_6(X)$ $+F_3(X)+F_0(X)$
16	$1+\overline{T}_1+\overline{T}_2+\overline{T}_3+\overline{T}_4+\overline{T}_5+\overline{T}_6+\overline{T}_7$ $+\overline{T}_8+\overline{T}_9+\overline{T}_{10}+\overline{T}_{11}+\overline{T}_{12}+\overline{T}_{13}+\overline{T}_{14}+\overline{T}_{15}$	F_{15}

Table 5: Expressing $\overline{P}_n(X)$ in terms of $\overline{T}_k(X)$ and of $F_k(X)$.

Since $d=1$ is an odd divisor for every $n\geq 1$ and $r_n(1)=n-1\geq 0$, the polynomial $F_{n-1}(X)$ always appears, with a positive sign, in the formula of Theorem 16.

We remarked above that only powers of 2 have a unique odd divisor, namely $d=1$. This implies the following consequence of Theorem 16, which explains the coincidences $\overline{P}_n(N)=F_{n-1}(N)$ (set in boldface) observed in Table 1 for $N=3$, 4, and 5.

Corollary 17. *We have $\overline{P}_n(X)=F_{n-1}(X)$ if and only if n is a power of 2.*

As special cases of Theorem 16, we obtain the following formulas for $\overline{P}_n(X)$.

4.2.1 Two odd divisors

If $n = p$ is an odd prime, then it has two odd divisors, namely 1 and p, so that the formula for $\overline{P}_p(X)$ has two summands. We have

$$\overline{P}_p(X) = F_{p-1}(X) - F_{(p-3)/2}(X) = \sum_{m=(p-1)/2}^{p-1} \overline{T}_m(X).$$

Now consider the case $n = 2^a p$, where $a \geq 1$ and p is an odd prime. Such an integer n also has exactly two odd divisors.

If $r_n(p) = 2^a - (p+1)/2 \geq 0$, which is equivalent to $p \leq 2^{a+1} - 1$, then

$$\overline{P}_{2^a p}(X) = F_{2^a p - 1}(X) + F_{2^a - (p+1)/2}(X).$$

For instance, $\overline{P}_6(X) = F_5(X) + F_0(X)$ and $\overline{P}_{12}(X) = F_{11}(X) + F_2(X)$

If $r_n(p) = 2^a - (p+1)/2 < 0$, which is equivalent to $p > 2^{a+1} - 1$, then

$$\overline{P}_{2^a p}(X) = F_{2^a p - 1}(X) - F_{(p-1)/2 - 2^a}(X).$$

As special cases, $\overline{P}_{10}(X) = F_9(X) - F_0(X)$ and $\overline{P}_{14}(X) = F_{13}(X) - F_1(X)$.

4.2.2 Three odd divisors

If $n = p^2$ for some odd prime p, then

$$\overline{P}_{p^2}(X) = F_{p^2-1}(X) - F_{(p^2-3)/2}(X) + F_{(p-1)/2}(X).$$

For instance, $\overline{P}_9(X) = F_8(X) - F_3(X) + F_1(X)$.

4.2.3 Four odd divisors

If $n = p^3$ for some odd prime p, then n has four odd divisors. We have

$$\overline{P}_{p^3}(X) = F_{p^3-1}(X) - F_{(p^3-3)/2}(X) + F_{(2p^2-p-1)/2}(X) - F_{(p^2-2p-1)/2}(X).$$

For $p = 3$, we obtain $\overline{P}_{27}(X) = F_{26}(X) - F_{12}(X) + F_7(X) - F_1(X)$.

Similarly, consider an integer of the form $n = 3p$, where p is an odd prime ≥ 7. Then n has four odd divisors, namely $1, 3, p, 3p$, and

$$\overline{P}_{3p}(X) = F_{3p-1}(X) - F_{3(p-1)/2} + F_{p-2}(X) - F_{(p-7)/2}\,.$$

For $p = 5$ and $n = 15$, we have $\overline{P}_{15}(X) = F_{14}(X) - F_6(X) + F_3(X) + F_0(X)$.

Remark 18. In Table 1 we set the values $\overline{P}_n(N)$ and $F_{n-1}(N)$ differing by 1 in italics, which occurs for $n = 3$, 6, and 10. These quasi-coincidences between values can be explained as follows.

We have $\overline{P}_n(X) = F_{n-1}(X) + 1 = F_{n-1}(X) + F_0(X)$ if and only if the integer n is an even *perfect* number; equivalently, $n = 2^a p$, where $p = 2^{a+1} - 1$ is a *Mersenne prime* such as $3, 7, 31, 127$ (see the OEIS [12, A000668]). The corresponding n are $n = 6, 28, 496, 8128$.

Similarly, $\overline{P}_n(X) = F_{n-1}(X) - 1 = F_{n-1}(X) - F_0(X)$ if and only if $n = 2^a p$, where $p = 2^{a+1} + 1$ is a *Fermat prime* such as 3, 5, 17, 257 (see the OEIS [12, A000215]). The corresponding n are 3, 10, 136, 32896.

In general, the formula for $\overline{P}_n(X)$ in Theorem 16 contains $\pm F_0(X)$ as a summand if and only if n is *triangular*, i.e., $n = r(r+1)/2$ for some $r \geq 1$. The smallest triangular integers are 1, 3, 6, 10, 15, 21, 28, 36.

Remark 19. For which natural numbers n do we have $\overline{P}_n(X) = F_{n-1}(X) \pm F_1(X)$?

We have $\overline{P}_n(X) = F_{n-1}(X) + F_1(X)$ if and only if $n = 2^a p$, where p is a prime of the form $p = 2^{a+1} - 3$. Primes $p = 5$, 13, 29, 61 are of this form; the corresponding n are $n = 20$, 104, 464, 1952 (see A050415 and A181703 in the OEIS [12]).

Similarly, $\overline{P}_n(X) = F_{n-1}(X) - F_1(X)$ if and only if $n = 2^a p$, where p is a prime of the form $p = 2^{a+1} + 3$. Primes $p = 5$, 7, 11, 19, 67 are of this form (see the OEIS [12, A057733]); they correspond to $n = 5$, 14, 44, 152, 2144.

More generally, $F_1(X)$ appears in the formula for $\overline{P}_n(X)$ in Theorem 16 if and only if $n = r(r+3)/2$ for some $r \geq 1$; see the OEIS [12, A000096] for a list of such n.

5 Formulas for $C_n(q)$

In order to prove Theorem 16 we need new expressions for the polynomial $C_n(q)$. Recall that $C_n(q)$ and $\overline{P}_n(X)$ are related by the equalities

$$\frac{C_n(q)}{q^n} = \frac{(q-1)^2}{q} \frac{P_n(q)}{q^{n-1}} = \frac{(q-1)^2}{q} \overline{P}_n(q + q^{-1}).$$

5.1 Increasing sequences

We need the following material borrowed from Mason [11].

We call *increasing sequence* every finite nonempty ordered set of consecutive integers $\{a+1, \ldots, a+h\}$ with smallest element $a+1$ and largest element $a+h$. Here a is an integer (positive, negative or zero) and h is a positive integer. We denote this set by $\mathrm{IS}(a, h)$; it has h elements. An increasing sequence $\mathrm{IS}(a, h)$ is called *positive* if $a \geq 0$ and *negative* if $a < 0$. It is called *odd* if h is odd and *even* if h is even.

We say that an integer $n \geq 1$ is *represented* by $\mathrm{IS}(a, h)$, and we write

$$\mathrm{IS}(a, h) \models n,$$

if n is equal to the sum of all elements of $\mathrm{IS}(a,h)$.

There is an involution $\mathrm{IS}(a,h) \mapsto \mathrm{IS}(\breve{a},\breve{h})$ on increasing sequences determined by

$$a + \breve{a} + 1 = 0 \quad \text{and} \quad \breve{a} + \breve{h} = a + h. \tag{18}$$

If $\mathrm{IS}(a,h)$ represents n, so does $\mathrm{IS}(\breve{a},\breve{h})$. Since $\breve{h} - h = a - \breve{a} = 2a + 1$, we conclude that h and $\breve{h}$ are of opposite parity. In other words, $\mathrm{IS}(a,h)$ is even (resp., odd) if and only if $\mathrm{IS}(\breve{a},\breve{h})$ is odd (resp., even). Moreover, if $\mathrm{IS}(a,h)$ is positive (resp., negative), then $\mathrm{IS}(\breve{a},\breve{h})$ is negative (resp., positive).

Let $n \geq 1$ be a positive integer. With every *odd* divisor d of n we associate the odd increasing sequence $\mathrm{IS}(a,h)$, where

$$a = \frac{n}{d} - \frac{d+1}{2} \quad \text{and} \quad h = d. \tag{19}$$

It is centered around n/d, namely we have

$$\mathrm{IS}(a,h) = \left\{ \frac{n}{d} - \frac{d-1}{2}, \ldots, \frac{n}{d} - 1, \frac{n}{d}, \frac{n}{d} + 1, \ldots, \frac{n}{d} + \frac{d-1}{2} \right\}. \tag{20}$$

This odd increasing sequence may be positive or negative, depending on d. We have $\mathrm{IS}(a,h) \models n$.

Starting from $a = n/d - (d+1)/2$ and $h = d$, we have

$$\breve{a} = -\frac{n}{d} + \frac{d-1}{2} \quad \text{and} \quad \breve{h} = \frac{2n}{d}. \tag{21}$$

We thus obtain the even increasing sequence

$$\mathrm{IS}(\breve{a},\breve{h}) = \left\{ -\frac{n}{d} + \frac{d+1}{2}, \ldots, \frac{n}{d} + \frac{d-1}{2} \right\} \models n. \tag{22}$$

Clearly, if $\mathrm{IS}(a,h)$ is positive, then $\mathrm{IS}(a,h) \subset \mathrm{IS}(\breve{a},\breve{h})$ and

$$\mathrm{IS}(\breve{a},\breve{h}) \setminus \mathrm{IS}(a,h) = \left\{ -\frac{n}{d} + \frac{d+1}{2}, \ldots, \frac{n}{d} - \frac{d+1}{2} \right\}. \tag{23}$$

If $\mathrm{IS}(a,h)$ is negative, then $\mathrm{IS}(\breve{a},\breve{h}) \subset \mathrm{IS}(a,h)$ and

$$\mathrm{IS}(a,h) \setminus \mathrm{IS}(\breve{a},\breve{h}) = \left\{ \frac{n}{d} - \frac{d-1}{2}, \ldots, -\frac{n}{d} + \frac{d-1}{2} \right\}. \tag{24}$$

For both pairs (a,h) and $(\breve{a},\breve{h})$ we have the same relation, namely

$$\frac{h(h+2a+1)}{2} = n = \frac{\breve{h}(\breve{h}+2\breve{a}+1)}{2}. \tag{25}$$

Conversely, starting from an increasing sequence $\mathrm{IS}(a,h)$ we obtain an odd divisor d of n as follows: d is equal to the only *odd* element of $\{h, \check{h}\}$. In this way we have a bijection between the set of odd divisors of n and the set of positive increasing sequences representing n, which is in bijection with the set of negative increasing sequences representing n.

Suppose that given $n \geq 1$ we have a pair (a,h) of integers with $h \geq 1$ verifying Equation (25), which we rewrite as $h(h+2a+1) = 2n$. If h is odd, hence coprime to 2, then $d = h$ is an odd divisor of n and $a = n/d - (d+1)/2$, as in (19). If h is even, then $d = h + 2a + 1$ is odd; being a divisor of $2n$, it is an odd divisor of n; in this case $h = 2n/d$ and $a = -n/d + (d-1)/2$, as in (21).

Example 20. If n is a power of 2, then it has only one odd divisor, namely $d = 1$; the corresponding positive increasing sequence is

$$\mathrm{IS}(n-1, 1) = \{n\}$$

and its negative counterpart is

$$\mathrm{IS}(-n, 2n) = \{-(n-1), \ldots, -1, 0, 1, \ldots, n-1, n\}.$$

Example 21. Every other integer n has odd divisors $d \neq 1$, hence more than one positive (or negative) increasing sequences. For instance, consider $n = 18$ with odd divisors $d = 3$ and $d = 9$.

If $d = 3$, then $a = 4$, $h = 3$, $\check{a} = -5$, $\check{h} = 12$, and

$$\mathrm{IS}(4, 3) = \{5, 6, 7\} \quad \text{and} \quad \mathrm{IS}(-5, 12) = \{-4, -3, \ldots, 3, 4, 5, 6, 7\}.$$

If $d = 9$, then $a = -3$, $h = 9$, $\check{a} = 2$, $\check{h} = 4$, and

$$\mathrm{IS}(-3, 9) = \{-2, -1, 0, 1, 2, 3, 4, 5, 6\} \quad \text{and} \quad \mathrm{IS}(2, 4) = \{3, 4, 5, 6\}.$$

5.2 New formulas for $C_n(q)$

The authors [10, Th. 1.1] gave the following explicit expression for $C_n(q)$:

$$\frac{C_n(q)}{q^n} = c_{n,0} + \sum_{i=1}^{n} c_{n,i} \left(q^i + q^{-i}\right),$$

where

$$c_{n,0} = 2(-1)^r \tag{26}$$

if $n = r(r+1)/2$ for some positive integer r and $c_{n,0} = 0$ otherwise. For the remaining coefficients $c_{n,i}$ $(1 \leq i \leq n)$ we have

$$c_{n,i} = \begin{cases} (-1)^k, & \text{if } n = k(k+2i+1)/2 \text{ for some integer } k \geq 1; \\ (-1)^{k-1}, & \text{if } n = k(k+2i-1)/2 \text{ for some integer } k \geq 1; \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

Observe that in (27) the first two conditions are mutually exclusive, as shown by the authors [10, Lemma 3.5 (ii)]. This may also be verified as follows: suppose

$$k(k+2i+1) = h(h+2i-1)$$

with $i, h, k \geq 1$; then $k^2 + 2ik + k = h^2 + 2ih - h$, which can be rewritten as

$$(k-h+1)(k+h) + 2i(k-h) = 0.$$

It is readily seen that the left-hand side cannot vanish, neither in the case $k \geq h$, nor in the case $k < h$.

Lemma 22. *We have*

$$\frac{C_n(q)}{q^n} = \sum (-1)^h \left(q^a + q^{-a}\right),$$

where the summation is over all $h \in \mathbb{Z}_{\geq 1}$ and $a \in \mathbb{Z}$ such that $n = h(h+1+2a)/2$. Moreover, if n has such a representation and $|a|$ is given, then the pair (a, h) is unique.

Proof. It follows from (26) that both sides have the same constant term.

Henceforth we concentrate on the case when a is a nonzero integer. Let $i \geq 1$. Suppose that $n = k(k+2i+1)/2$ with $k \geq 1$ as in the first row of (27); then $n = h(h+2a+1)/2$ with $h = k \geq 1$, $a = i \in \mathbb{Z} \setminus \{0\}$, and $(-1)^k = (-1)^h$.

Now suppose that $n = k(k+2i-1)/2$ with $k \geq 1$ as in the second row of (27); then $n = h(h+2a+1)/2$ with $h = k+2i-1 \geq 1$, $a = -i \in \mathbb{Z} \setminus \{0\}$, and $(-1)^{k-1} = (-1)^h$.

Conversely, let $n = h(h+2a+1)/2$ with $h \geq 1$, $i \in \mathbb{Z}$, and $a \neq 0$. Then either $a \geq 1$, and then $n = k(k+2i+1)/2$ with $k = h$, $i = a \geq 1$, and $(-1)^k = (-1)^h$; or $a < 0$, and then $n = k(k+2i-1)/2$, $k = h+2a+1 \geq 1$, $i = -a > 0$, and $(-1)^{k-1} = (-1)^h$.

This proves the lemma in view of the observation made just before its statement. □

We have the following new formulas for $C_n(q)$.

Theorem 23. *For each integer $n \geq 1$ we have*

$$\frac{C_n(q)}{q^n} = \sum (-1)^h \left(q^a + q^{-a}\right),$$

where the summation is taken over all increasing sequences $\mathrm{IS}(a, h)$ such that $\mathrm{IS}(a, h) \models n$. Moreover, for given $|a|$, there is a unique increasing sequence $\mathrm{IS}(a, h)$ with $\mathrm{IS}(a, h) \models n$.

We also have

$$\frac{C_n(q)}{q^n} = \sum_{d|n,\, d \,\mathrm{odd}} \left(q^{\frac{n}{d} - \frac{d-1}{2}} + q^{-\frac{n}{d} + \frac{d-1}{2}} - q^{\frac{n}{d} - \frac{d+1}{2}} - q^{-\frac{n}{d} + \frac{d+1}{2}}\right).$$

In the previous sum, all monomials $\neq q^0$ are distinct.

Proof. The first formula follows from Lemma 22 since the sum of the elements in $\mathrm{IS}(a,h)$ is equal to

$$a+1+a+2+\cdots+a+h = ha+h(h+1)/2 = h(h+2a+1)/2.$$

As in § 5.1, with each odd divisor d of n we associate the two increasing sequences $\mathrm{IS}(a,h)$ and $\mathrm{IS}(\check{a},\check{h})$, respectively odd and even, defined by (20) and (22). All increasing sequences representing n are obtained in this way. The second formula then follows from the first one: the first two terms correspond to $\mathrm{IS}(\check{a},\check{h})$ and the last two to $\mathrm{IS}(a,h)$. □

The previous result immediately implies the following.

Corollary 24. *The sum of the absolute values of the coefficients of $C_n(q)$ is equal to four times the number of odd divisors of n.*

5.3 Zeta functions

The following consequences of Theorem 23 may be of interest to algebraic geometers.

Recall that the *local zeta function* $Z_{\mathcal{H}^n/\mathbb{F}_q}(t)$ of the Hilbert scheme $\mathcal{H}^n$ of n points on the two-dimensional torus over the finite field $\mathbb{F}_q$ is the formal power series

$$Z_{X/\mathbb{F}_q}(t) = \exp\left(\sum_{m\geq 1} C_n(q^m)\,\frac{t^m}{m}\right). \tag{28}$$

The authors [10, Th. 2.1] showed that $Z_{X/\mathbb{F}_q}(t)$ is equal to the formal power series expansion of the following rational function:

$$Z_{\mathcal{H}^n/\mathbb{F}_q}(t) = \frac{1}{(1-q^nt)^{c_{n,0}}}\prod_{i=1}^{n}\frac{1}{[(1-q^{n+i}t)(1-q^{n-i}t)]^{c_{n,i}}},$$

where $c_{n,i}$ are the coefficients of $C_n(q)$ defined in § 5.2. Rewriting this formula in view of Theorem 23 yields the following statement, where $r_n(d) = n/d-(d+1)/2$ is the integer introduced in (17).

Proposition 25. *For $n\geq 1$ we have*

$$Z_{\mathcal{H}^n/\mathbb{F}_q}(t) = \prod_{d|n,\,d\,\mathrm{odd}}\frac{(1-q^{n+r_n(d)}t)(1-q^{n-r_n(d)}t)}{(1-q^{n+r_n(d)+1}t)(1-q^{n-r_n(d)-1}t)}.$$

The right-hand side is a rational function whose numerator and denominator are factored into degree-one polynomials. Clearly, the number of such factors in the numerator (resp., in the denominator) is equal to twice the number of odd divisors of n.

Let us display the product formula for $Z_{\mathcal{H}^n/\mathbb{F}_q}(t)$ in two special cases. For $n=4$, which has only 1 as an odd divisor, we have $r_4(1)=3$. Hence,

$$Z_{\mathcal{H}^4/\mathbb{F}_q}(t) = \frac{(1-q^7t)(1-qt)}{(1-q^8t)(1-t)}.$$

For $n = 3$, which has two odd divisors, we have $r_3(1) = 2$ and $r_3(3) = -1$. Hence,

$$Z_{\mathcal{H}^3/\mathbb{F}_q}(t) = \frac{(1-q^5t)(1-qt)}{(1-q^6t)(1-t)} \cdot \frac{(1-q^2t)(1-q^4t)}{(1-q^3t)^2}.$$

Proposition 25 allows us to give a similar formula for the *Hasse–Weil zeta function* $\zeta_{\mathcal{H}^n}(s)$ of the above Hilbert scheme; this zeta function is defined over the complex numbers by

$$\zeta_{\mathcal{H}^n}(s) = \prod_{p \text{ prime}} Z_{\mathcal{H}^n/\mathbb{F}_p}(p^{-s}), \qquad (s \in \mathbb{C})$$

where the product is taken over all prime integers p. As an immediate consequence of the formula for the local zeta functions $Z_{\mathcal{H}^n/\mathbb{F}_q}(t)$ we have

$$\zeta_{\mathcal{H}^n}(s) = \prod_{d|n,\, d \text{ odd}} \frac{\zeta(s-n-r_n(d))\,\zeta(s-n+r_n(d))}{\zeta(s-n-r_n(d)-1)\,\zeta(s-n+r_n(d)+1)}\,, \tag{29}$$

where $\zeta(s)$ is Riemann's zeta function. Recall from [10, Sect. 2] that the Hasse–Weil zeta function satisfies the simple functional equation

$$\zeta_{\mathcal{H}^n}(s) = \zeta_{\mathcal{H}^n}(2n-s). \tag{30}$$

6 Proof of Theorem 16

The following lemma exhibits a crucial link between the polynomials $F_r(X)$ and the expression in parentheses in the second formula of Theorem 23.

Lemma 26. *For $r \geq 0$ we have*

$$q^{r+1} + q^{-r-1} - q^r - q^{-r} = q^{-1}(q-1)^2 F_r(q+q^{-1}).$$

Proof. The formula can be rephrased in terms of polynomials in the indeterminate X as $\overline{T}_{r+1}(X) - \overline{T}_r(X) = (X-2)F_r(X)$. By (15) the latter is equivalent to

$$\big(F_{r+1}(X) - F_r(X)\big) - \big(F_r(X) - F_{r-1}(X)\big) = XF_r(X) - 2F_r(X),$$

hence to the linear recurrence relation of Proposition 11. □

Proof of Theorem 16. It follows from the second formula of Theorem 23 that

$$q^{-1}(q-1)^2 \frac{P_n(q)}{q^{n-1}} = \frac{C_n(q)}{q^n} = \sum_{d|n,\, d \text{ odd}} \left(q^{\frac{n}{d} - \frac{d-1}{2}} + q^{-\frac{n}{d} + \frac{d-1}{2}} - q^{\frac{n}{d} - \frac{d+1}{2}} - q^{-\frac{n}{d} + \frac{d+1}{2}} \right).$$

We partition the previous sum into two sums depending on the sign of $r_n(d) = n/d-(d+1)/2$. We obtain

$$\begin{aligned} q^{-1}(q-1)^2\frac{P_n(q)}{q^{n-1}} &= \sum_{\substack{d|n,\, d\,\mathrm{odd}\\ r_n(d)\geq 0}} \left(q^{\frac{n}{d}-\frac{d-1}{2}} + q^{-\frac{n}{d}+\frac{d-1}{2}} - q^{\frac{n}{d}-\frac{d+1}{2}} - q^{-\frac{n}{d}+\frac{d+1}{2}}\right) \\ &\quad + \sum_{\substack{d|n,\, d\,\mathrm{odd}\\ r_n(d)< 0}} \left(q^{\frac{n}{d}-\frac{d-1}{2}} + q^{-\frac{n}{d}+\frac{d-1}{2}} - q^{\frac{n}{d}-\frac{d+1}{2}} - q^{-\frac{n}{d}+\frac{d+1}{2}}\right). \end{aligned}$$

Now replacing r by $r_n(d)$ in Lemma 26, we see that the sum of monomials corresponding to $r_n(d) \geq 0$ is equal to

$$q^{-1}(q-1)^2 \sum_{\substack{d|n,\, d\,\mathrm{odd}\\ r_n(d)\geq 0}} F_{r_n(d)}(q+q^{-1}).$$

Replacing r by $-r_n(d)-1$ in Lemma 26, we observe that the sum of monomials corresponding to $r_n(d) < 0$, equivalently to $-r_n(d) - 1 \geq 0$, is equal to

$$q^{-1}(q-1)^2 \sum_{\substack{d|n,\, d\,\mathrm{odd}\\ r_n(d)< 0}} F_{-r_n(d)-1}(q+q^{-1}).$$

To conclude, it suffices to divide by $q^{-1}(q-1)^2$ and to appeal to the equality $P_n(q)/q^{n-1} = \overline{P}_n(q+q^{-1})$. □

We close this section with another formula for $P_n(q)/q^{n-1}$.

Proposition 27. *We have*

$$\frac{P_n(q)}{q^{n-1}} = \sum_{\substack{d|n,\, d\,\mathrm{odd}\\ r_n(d)\geq 0}} \sum_{i\in \mathrm{IS}(\check{a},\check{h})\setminus \mathrm{IS}(a,h)} q^i - \sum_{\substack{d|n,\, d\,\mathrm{odd}\\ r_n(d)< 0}} \sum_{i\in \mathrm{IS}(a,h)\setminus \mathrm{IS}(\check{a},\check{h})} q^i,$$

where $a = r_n(d)$ *and* $\check{a} = -r_n(d) - 1$, *and* $\mathrm{IS}(a,h)$ *and* $\mathrm{IS}(\check{a},\check{h})$ *are defined respectively by* (20) *and* (22).

Proof. It follows from Theorem 16 and the explicit descriptions (23) and (24). □

Remark 28. Sums of powers of q indexed by odd divisors have also appeared in work of Caballero [4] (especially in Chapter II).

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