Journal of Integer Sequences, Vol. 29 (2026),
Article 26.5.3

# An Explicit Formula and a Recurrence for OEIS A265233

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**Abstract**

We give an explicit formula for OEIS sequence A265233 as a binomial sum involving the Franel numbers. The sequence counts three-row arrays that contain equally many copies of each of three symbols and have no equal vertically adjacent entries. Reading the array row by row, we label the symbols in order of first occurrence. A decomposition of the column-counting polynomial yields the explicit formula and a proof of Mathar's conjectured recurrence. We then derive the hypergeometric generating function recorded by van Hoeij and the asymptotic formula conjectured by Kotesovec. The latter follows from a logarithmic singular expansion with an explicit error estimate.

## 1 Introduction

Let $a_n$ count the $3 \times n$ arrays over $\{0, 1, 2\}$ with the following properties:

(a) Each symbol occurs exactly $n$ times in the array.

(b) Vertically adjacent entries are distinct. Thus a column $(c_1, c_2, c_3)^{\mathsf{T}}$ satisfies $c_1 \neq c_2$ and $c_2 \neq c_3$, while equality between $c_1$ and $c_3$ is allowed.

(c) When we read the entries from left to right across the first row, then the second, and then the third, the first occurrences of the symbols appear in the order $0, 1, 2$.

We include the empty array by setting $a_0 = 1$. These numbers form sequence A265233 in the On-Line Encyclopedia of Integer Sequences (OEIS) [4]. For example, the seven arrays counted by $a_2$ are

$$\begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 2 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{pmatrix}.$$

We give an explicit binomial-sum formula for $a_n$ and use the formula to prove the recurrence conjectured by Mathar in 2015. The formula expresses the array count in terms of the Franel numbers, which are sums of cubes of binomial coefficients. Their classical recurrence then gives the recurrence for $a_n$.

**Theorem 1.** *The sequence $(a_n)_{n\geq 0}$ has the following properties.*

*(a) For every integer $n \geq 1$, we have*

$$a_n = \frac{1}{6} \sum_{m=0}^{n} \binom{n}{m} 4^{n-m} \sum_{k=0}^{m} \binom{m}{k}^3. \tag{1}$$

*(b) For every integer $n \geq 4$, the sequence satisfies the recurrence*

$$n^2 a_n - (19n^2 - 19n + 6)a_{n-1} + 96(n-1)^2 a_{n-2} - 144(n-1)(n-2)a_{n-3} = 0, \tag{2}$$

*with initial values $a_0 = 1$, $a_1 = 1$, $a_2 = 7$, and $a_3 = 56$.*

The OEIS entry also contains an asymptotic formula conjectured by Kotesovec in 2023 and a hypergeometric generating function recorded by van Hoeij in 2024, conditional on Mathar's recurrence. We derive these as consequences of Theorem 1. For $|z| < 1$ and $c \notin \{0, -1, -2, \ldots\}$, we define the *Gauss hypergeometric function* by

$${}_2F_1(a, b; c; z) = \sum_{k\geq 0} \frac{(a)_k (b)_k}{(c)_k \, k!} z^k,$$

where we define the *Pochhammer symbol* by $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geq 1$.

**Corollary 2.** *The sequence $(a_n)_{n\geq 0}$ satisfies the following formulas.*

*(a) The ordinary generating function for $(a_n)_{n\geq 0}$ is*

$$A(x) = \sum_{n\geq 0} a_n x^n = \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 27x(1-4x)^2\right) + 5}{6}. \tag{3}$$

*(b) As $n \to \infty$, we have*

$$a_n \sim \frac{\sqrt{3}}{6\pi} \frac{12^n}{n} = \frac{2^{2n-1} 3^{n-1/2}}{\pi n}. \tag{4}$$

## 2 The column polynomial and the explicit formula

Let $b_n$ count the arrays satisfying conditions (a) and (b), with no restriction on the order of first occurrences. For $n \geq 1$, every array uses all three symbols, so its six relabelings are distinct. Exactly one of them satisfies condition (c): relabel the first symbol encountered as 0, the next new symbol as 1, and the remaining symbol as 2. Hence

$$b_n = 6a_n \qquad (n \geq 1). \tag{5}$$

The empty array gives $a_0 = b_0 = 1$.

Encode a column by the monomial $x^{e_0}y^{e_1}z^{e_2}$, where the exponent $e_s$ counts the entries equal to $s$. For example, the column $(0, 1, 0)^{\mathsf{T}}$ contributes $x^2y$. There are twelve allowed columns: six use all three symbols, and six have equal top and bottom entries. Their counting polynomial is

$$\begin{aligned} P(x, y, z) &= x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2 + 6xyz \\ &= (x + y)(y + z)(z + x) + 4xyz. \end{aligned} \tag{6}$$

Write $[x^iy^jz^k]Q$ for the coefficient of $x^iy^jz^k$ in a polynomial or formal power series $Q$. Since there are no horizontal adjacency restrictions, we multiply the column weights and impose condition (a) to obtain

$$b_n = [x^ny^nz^n]P(x, y, z)^n. \tag{7}$$

The *Franel numbers*, sequence A000172, are

$$f_m = \sum_{k=0}^{m} \binom{m}{k}^3 \qquad (m \geq 0).$$

The following coefficient identity connects them to the array model.

**Lemma 3.** *For every integer $m \geq 0$, we have*

$$[x^my^mz^m]\big((x + y)(y + z)(z + x)\big)^m = f_m. \tag{8}$$

*Proof.* Expand the three factors as

$$\begin{aligned} (x + y)^m &= \sum_{i=0}^{m} \binom{m}{i} x^iy^{m-i}, \\ (y + z)^m &= \sum_{j=0}^{m} \binom{m}{j} y^jz^{m-j}, \\ (z + x)^m &= \sum_{k=0}^{m} \binom{m}{k} z^kx^{m-k}. \end{aligned}$$

To obtain $x^my^mz^m$, we require $i = k$, $j = i$, and $k = j$. The coefficient is therefore $\sum_{k=0}^{m} \binom{m}{k}^3$. □

*Proof of the explicit formula in Theorem 1.* Apply the binomial theorem to the decomposition in (6). Equations (7) and (8) give

$$\begin{aligned} b_n &= [x^n y^n z^n] \sum_{m=0}^{n} \binom{n}{m} (4xyz)^{n-m} \big((x+y)(y+z)(z+x)\big)^m \\ &= \sum_{m=0}^{n} \binom{n}{m} 4^{n-m} f_m. \end{aligned} \tag{9}$$

For $n \geq 1$, divide by 6 and use the definition of $f_m$ to obtain (1). □

The formula, together with $a_0 = 1$, gives the first ten terms

$$(a_n)_{n=0}^{9} = (1, 1, 7, 56, 495, 4686, 46456, 475392, 4976271, 52977890),$$

in agreement with A265233.

# 3 The recurrence

Let $\mathcal{F}(t) = \sum_{m\geq 0} f_m t^m$ and $B(x) = \sum_{n\geq 0} b_n x^n$ denote the ordinary generating functions for $(f_m)_{m\geq 0}$ and $(b_n)_{n\geq 0}$, respectively. For every integer $m \geq 0$, the binomial identity

$$\sum_{n\geq m} \binom{n}{m} 4^{n-m} x^n = \frac{x^m}{(1-4x)^{m+1}}$$

and (9) give the binomial-transform identity

$$B(x) = \frac{1}{1-4x} \mathcal{F}\left(\frac{x}{1-4x}\right). \tag{10}$$

The Franel numbers satisfy

$$m^2 f_m - (7m^2 - 7m + 2) f_{m-1} - 8(m-1)^2 f_{m-2} = 0 \qquad (m \geq 2), \tag{11}$$

with $f_0 = 1$ and $f_1 = 2$. Strehl [5, p. 4, Eq. (2)] discusses this classical recurrence and its relation to transforms of sequences. Cusick [1] gives a general method for deriving recurrences for sums of powers of binomial coefficients. The recurrence (11) and the initial values are equivalent to

$$\begin{aligned} t(1+t)(1-8t)\mathcal{F}''(t) + (1-14t-24t^2)\mathcal{F}'(t) \\ -2(1+4t)\mathcal{F}(t) = 0, \end{aligned} \tag{12}$$

together with $\mathcal{F}(0) = 1$.

*Proof of the recurrence in Theorem 1.* Set $t = x/(1-4x)$. By (10), we have $\mathcal{F}(t) = (1-4x)B(x)$. The chain rule gives

$$\begin{aligned}\mathcal{F}'(t) &= (1-4x)^3 B'(x) - 4(1-4x)^2 B(x),\\ \mathcal{F}''(t) &= (1-4x)^5 B''(x) - 16(1-4x)^4 B'(x) + 32(1-4x)^3 B(x).\end{aligned}$$

Substitution into (12) and division by $1-4x$ yield

$$\begin{aligned}(x - 19x^2 + 96x^3 - 144x^4)B''(x) + (1 - 38x + 288x^2 - 576x^3)B'(x)\\ +(-6 + 96x - 288x^2)B(x) = 0.\end{aligned} \tag{13}$$

For $n \geq 3$, extracting the coefficient of $x^{n-1}$ gives

$$\begin{aligned}n^2 b_n - (19n^2 - 19n + 6)b_{n-1} + 96(n-1)^2 b_{n-2}\\ -144(n-1)(n-2)b_{n-3} = 0.\end{aligned} \tag{14}$$

For $n \geq 4$, all four subscripts are positive, so equation (5) converts (14) into (2). The initial values follow from (1) and $a_0 = 1$. □

# 4 The hypergeometric generating function

Write $\Phi(z) = {}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; z)$. In the OEIS entry A000172 [4], Somos recorded the generating function for the Franel numbers on December 17, 2010:

$$\mathcal{F}(t) = \frac{1}{1-2t}\Phi\left(\frac{27t^2}{(1-2t)^3}\right). \tag{15}$$

The right-hand side also satisfies (12) and has constant term 1, which provides a direct verification of the identity. Combining (15) with (10) gives

$$B(x) = \frac{1}{1-6x}\Phi\left(\frac{27x^2(1-4x)}{(1-6x)^3}\right).$$

We obtain the form in (3) directly from the differential equation for $B$.

*Proof of* (3). The defining series for $\Phi$ satisfies the Gauss differential equation

$$z(1-z)\Phi''(z) + (1-2z)\Phi'(z) - \frac{2}{9}\Phi(z) = 0. \tag{16}$$

Set

$$w(x) = 27x(1-4x)^2, \qquad p(x) = x(1-3x)(1-4x)(1-12x),$$

and write

$$\mathcal{L}(y) = p(x)y'' + p'(x)y' - 6(1-4x)(1-12x)y.$$

Thus equation (13) is $\mathcal{L}(B) = 0$. Direct differentiation and (16) give

$$\mathcal{L}(\Phi \circ w) = 27(1-4x)(1-12x)\big(w(x)(1-w(x))\Phi''(w(x)) + (1-2w(x))\Phi'(w(x)) - \tfrac{2}{9}\Phi(w(x))\big) = 0. \tag{17}$$

By (17), both $B(x)$ and $\Phi(w(x))$ solve the same differential equation, and both have constant term 1. In (13), equating coefficients of $x^{n-1}$ determines the coefficient of $x^n$ in a solution from its preceding coefficients, since the multiplier of that coefficient is $n^2 \neq 0$ for every $n \geq 1$. Consequently,

$$B(x) = \Phi\big(27x(1-4x)^2\big) \tag{18}$$

as power series at $x = 0$. Finally, the relations $a_0 = b_0 = 1$ and (5) give $A(x) = (B(x)+5)/6$, proving (3). □

# 5 The asymptotic formula

*Proof of* (4). Put $\rho = 1/12$. We first justify the analytic continuation needed for coefficient extraction. Since there are twelve allowed columns, we have $0 \leq b_n \leq 12^n$, so the series $B$ is analytic in $|x| < \rho$. The leading coefficient in (13) factors as

$$p(x) = x(1-3x)(1-4x)(1-12x).$$

Fix $R$ with $\rho < R < 1/4$. Apart from 0 and $\rho$, every point of $|x| < R$ is an ordinary point of the differential equation. The power series solution is analytic at 0; continuation of solutions of the linear differential equation therefore extends $B$ to the slit disk

$$D_R = \{x \in \mathbb{C} : |x| < R\} \setminus [\rho, R).$$

The extension is single-valued because each loop in $D_R \setminus \{0\}$ is homotopic to an integer number of turns around 0, where the original solution is analytic. In particular, the function $B$ is analytic in the $\Delta$-*domain*

$$\Delta = \{x : |x| < R,\ x \neq \rho,\ |\arg(x-\rho)| > \theta\}, \qquad 0 < \theta < \pi/2.$$

The sum of the numerator parameters of $\Phi$ equals its denominator parameter, so the function is *zero-balanced*. The corresponding logarithmic expansion in the NIST Digital Library of Mathematical Functions [3, Eq. (15.8.10), with $m = 0$] gives

$$\Phi(z) = -c\log(1-z) + C_0 + O\big(|1-z|\big(1+|\log(1-z)|\big)\big), \tag{19}$$

where

$$c = \frac{1}{\Gamma(1/3)\Gamma(2/3)} = \frac{\sqrt{3}}{2\pi}$$

and the quantity $C_0$ is constant. Here the symbol $\Gamma$ denotes the gamma function, and Euler's reflection formula [3, Eq. (5.5.3)] gives the displayed value of $c$. The regularized hypergeometric function used in the zero-balanced expansion equals $\Phi$, since its normalizing factor is $\Gamma(1) = 1$. The full zero-balanced expansion converges and has analytic coefficient functions. Consequently, the expansion continues with the chosen logarithm from a slit neighborhood of $z = 1$.

The exact identity

$$1 - w(x) = (1 - 3x)(1 - 12x)^2 \tag{20}$$

shows that, with the logarithms real for $0 < x < \rho$ and continued into $D_R$, we have

$$\log(1 - w(x)) = \log(1 - 3x) + 2\log(1 - 12x).$$

Substituting (20) into (19) and using (18) yield

$$B(x) = -\frac{\sqrt{3}}{\pi}\log(1 - 12x) + H(x) + E(x), \tag{21}$$

where the function $H(x) = C_0 - c\log(1 - 3x)$ is analytic for $|x| < 1/3$. The remainder $E$ is analytic in $\Delta$ and satisfies

$$E(x) = O\big(|1 - 12x|^2\big(1 + |\log(1 - 12x)|\big)\big)$$

as $x \to \rho$ within $\Delta$. The logarithmic term in (21) also shows that $\rho$ is a singularity. All other points on $|x| = \rho$ are ordinary points of (13). Hence the point $\rho$ is the unique singularity of minimum modulus.

For $n \geq 1$, the coefficients of $-\log(1-12x)$ and $H(x)$ are $12^n/n$ and $c\,3^n/n$, respectively. Apply the error-transfer theorem of Flajolet and Sedgewick [2, Theorem VI.3(i)] to $E$ after the change of variable $u = 12x$, with the parameters $\alpha = -2$ and $\beta = 1$. We obtain

$$[x^n]E(x) = O\left(\frac{12^n \log n}{n^3}\right).$$

The contribution from $H$ is exponentially smaller, so we obtain

$$b_n = \frac{\sqrt{3}}{\pi}\frac{12^n}{n} + O\left(\frac{12^n \log n}{n^3}\right).$$

Division by 6 proves (4). □

# 6 Acknowledgments

The author thanks Mathar, Kotesovec, and van Hoeij for the formulas recorded in the OEIS, and the two referees for their comments on the manuscript.

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2020 *Mathematics Subject Classification*: Primary 05A15; Secondary 05A19, 11B65, 33C20.
*Keywords:* Franel number, binomial sum, recurrence, generating function, asymptotic formula.


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(Concerned with sequences A000172 and A265233.)

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Received March 23 2026; revised versions received March 26 2026; September 18 2026. Published in *Journal of Integer Sequences*, September 22 2026.


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