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Some Generalizations of the Bridge and Torch Problem

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Abstract

For the classic bridge and torch problem with crossing times $\{1, \ldots, n\}$, the optimal crossing time $T(n)$ is already known: it is recorded in A078476, and it follows from the optimal recursive procedure of Rote. We give a short derivation of the closed form

$$T(n) = \frac{n^2}{4} + 3n - 5 + \frac{(-1)^n - 1}{8},$$

where $n \geq 2$. We also determine a secondary quantity exactly: the least number of trips made by the fastest person in a schedule of minimum total time is $2\lceil \frac{n}{2} \rceil - 1$. We then generalize the problem to a bridge of capacity 3, and prove that

$$T_3(n) = \frac{n^2}{6} + 2n - \frac{181}{36} + \frac{(-1)^n}{4} - \frac{2}{9}\cos\left(\frac{2n\pi}{3}\right)$$

for all $n \geq 7$. The resulting values form A392834. Lastly, we explore the problem on star graphs, where we determine the optimal time exactly in the regime of abundant torches, and the answer turns out to be a partial sum of a floor function, giving a scheduling interpretation of several sequences of such partial sums.

1 Introduction

The bridge and torch problem is an interesting scheduling puzzle: a group of people must cross a single bridge at night with one torch. At most two people can be on the bridge at any time, the torch must be carried on every crossing, and a pair moves at the speed of its slower member. Assign each person a crossing time. The goal is to move everyone from the left side to the right side in minimum total time (Figure 1).

This problem has appeared in many forms over the years with various anecdotes attached to it. There are numerous names such as the bridge-crossing puzzle, the four men problem, the flashlight puzzle etc. The oldest reference of this classic problem is a puzzle book by Levmore and Cook [4]. The well-known version with four people whose crossing times are $1, 2, 5, 10$ units is perhaps the most common [6]. Beyond being a fun puzzle, the bridge and torch problem is also a neat example of an optimization task with constraints and trade-offs. With only a few simple rules, it forces careful thought about order and timing. It has also often been used to introduce ideas from computer science, such as dynamic programming.

The general problem is well understood. Rote [6] solved the capacity-two problem for arbitrary crossing times $0 < t_1 < \cdots < t_N$, and his analysis ends with an optimal recursive procedure. Jianu, Jianu and Popescu [3] later gave a combinatorial proof of the same result. For arbitrary capacity c, Backhouse [1] treats the capacity-c torch problem with the capacity and the crossing times among the inputs, and obtains an algorithm whose number of primitive computations is proportional to the square of the number of people; Backhouse and Truong [2] revisit the problem by dynamic programming. Consequently, *computing* the optimum for a given instance is not the issue here. What can still be of interest, and what we pursue, is the shape of the answer for the particular family of crossing times $t_i = i$: for such instances one may hope for a short closed formula, and for a description of the structure of the schedules that attain it.

Our contributions are as follows.

- In Section 2, we revisit the known formula for $T(n)$ by specializing Rote's procedure. We then consider a secondary objective, and show that the least number of trips made by the fastest person in a time-optimal schedule is $2\lceil \frac{n}{2} \rceil - 1$ (Proposition 6).
- In Section 3, we treat the capacity-three case with crossing times $\{1, \ldots, n\}$ and determine $T_3(n)$ exactly for $n \geq 7$. The upper bound is a block construction. The lower bound is proved by exhibiting a system of weights which is verified one move at a time, so that no reasoning about the order of moves, the states visited, or the pattern of returns is required.

- In Section 4, we move beyond the simple linear 'left bank to right bank' model to explore the problem on star graphs $\mathcal{S}(k)$ for k leaves, allowing for multiple torches. The exact answer we obtain concerns the regime in which torches are abundant enough that no torch ever has to be returned, where the problem becomes an accessible batching problem. Moreover, the answer is a partial sum of a floor function, which gives those partial sums a scheduling interpretation.

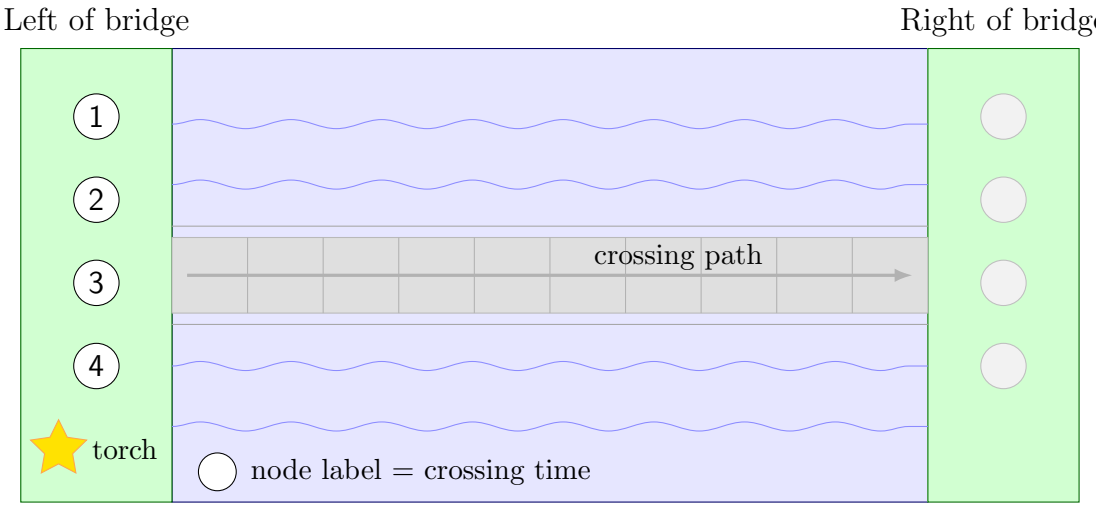

Figure 1: Visualizing the bridge and torch problem

2 The capacity-two case

Throughout this section, the people are $\{1,\ldots,n\}$ and person i has crossing time i. A state is a pair (L,λ) where $L\subseteq\{1,\ldots,n\}$ is the set currently on the left bank and $\lambda\in\{\mathsf{L},\mathsf{R}\}$ is the position of the torch. The initial and terminal states are $s=(\{1,\ldots,n\},\mathsf{L})$ and $t=(\emptyset,\mathsf{R})$ respectively. A move sends a non-empty set A of at most c people, all on the side of the torch, to the other side; its cost is $\max A$, and the torch changes sides. A schedule is a sequence of moves from s to t, and its cost is the sum of the costs of its moves. Since the torch changes sides at every move, a schedule alternates between left-to-right moves (forward moves) and right-to-left moves (returns), begins with a forward move and ends with a forward move; if it has f forward moves, then it has exactly $f-1$ returns.

Theorem 1. *Let there be n persons on one side of a bridge of capacity $c=2$ with crossing times $\{1,\ldots,n\}$, and let $T(n)$ denote the optimal crossing time. Then*

$$T(n)=\frac{n^2}{4}+3n-5+\frac{(-1)^n-1}{8}\quad\text{for all positive integers } n\geq 2. \tag{1}$$

The terms $T(1),T(2),T(3),T(4),\ldots=1,2,6,11,\ldots$ appear in sequence A078476. It denotes the set of times taken to get n people from one side of a bridge to the other where

(a) the only *flashlight* must be carried when crossing; (b) only one or two people may cross at the same time; (c) a pair crosses at the speed of the slowest member; and (d) the k^{th} person's speed requires k units to cross the bridge, where $1 \leq k \leq n$. This sequence is a special case of A318271, which denotes the optimal crossing time for the bridge and torch problem given that the crossing times for the group's members are given by the n^{th} partition of the triangle in which the n^{th} row lists juxtaposed lexicographically ordered partitions of n.

We emphasize that Theorem 1 is a known formula, which we revisit rather than establish for the first time. The expression (1), including its parity correction, appears in A078476, whose comments dated 2013 record the equivalent formulas for even and odd n; and, as the proof below shows, the recurrence behind it is an immediate specialization of Rote's optimal recursive procedure [6]. Our reason for including a proof is that the specialization is short and elementary, and that the same summation technique is used again in Section 3.

Also, note that equivalently, for all $n \geq 2$,

$$T(n) = \left\lfloor \frac{n^2 + 12n - 20}{4} \right\rfloor. \tag{2}$$

Before proving Theorem 1, we state Lemma 2, which provides the general formula of a sequence that is defined by separate formulae for even and odd terms.

Lemma 2. *Let $f, g : \mathbb{N} \to \mathbb{R}$ and let $\{a_n\}_{n\geq 0}$ be the sequence defined by $a_{2m} = f(m)$ and $a_{2m+1} = g(m)$ for all $m \in \mathbb{N}$. Then, for every $n \geq 0$,*

$$a_n = \frac{1 + (-1)^n}{2} f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \frac{1 - (-1)^n}{2} g\left(\left\lfloor \frac{n}{2} \right\rfloor\right).$$

Proof. Both f and g are evaluated at $\lfloor \frac{n}{2} \rfloor \in \mathbb{N}$, so the right side is defined for every $n \geq 0$. If $n = 2m$ is even, then $\lfloor \frac{n}{2} \rfloor = m$ and the two coefficients are 1 and 0, so the right-hand side equals $f(m) = a_n$. If $n = 2m + 1$ is odd, then $\lfloor \frac{n}{2} \rfloor = m$ and the coefficients are 0 and 1, so the right-hand side equals $g(m) = a_n$. □

We now give a proof of Theorem 1.

Proof. Rote's analysis of the capacity-two problem with arbitrary crossing times $t_1 < t_2 < \cdots < t_N$ ends with an optimal recursive procedure [6]: there is an optimal schedule in which the two slowest people $N - 1$ and N are transported either together, by the moves

$$\{1, 2\} \to \mathsf{R}, \quad 1 \to \mathsf{L}, \quad \{N - 1, N\} \to \mathsf{R}, \quad 2 \to \mathsf{L},$$

with cost $t_1 + 2t_2 + t_N$, or separately, by the moves

$$\{1, N\} \to \mathsf{R}, \quad 1 \to \mathsf{L}, \quad \{1, N - 1\} \to \mathsf{R}, \quad 1 \to \mathsf{L},$$

with cost $2t_1 + t_{N-1} + t_N$. In either case, the remaining instance is the one with crossing times $t_1, \ldots, t_{N-2}$, and the optimum is obtained by taking the cheaper of the two options and solving that instance optimally.

In particular, for $t_i = i$ and $N = k$, the two incremental costs are $t_1 + 2t_2 + t_k = k + 5$ and $2t_1 + t_{k-1} + t_k = 2k + 1$. Since $k + 5 \leq 2k + 1$ when $k \geq 4$, the cheaper option for every $k \geq 4$ is to transport the two slowest people together, and therefore

$$T(k) = T(k-2) + k + 5 \quad \text{for all } k \geq 4. \tag{3}$$

For the base cases, one can directly check that $T(2) = 2$ and $T(3) = 6$. We then solve the recurrence relation. If $n = 2m$ with $m \geq 1$, then

$$T(2m) = T(2) + \sum_{j=2}^{m} (2j + 5) = m^2 + 6m - 5 = \frac{n^2}{4} + 3n - 5.$$

On the other hand, if $n = 2m + 1$ with $m \geq 1$, then

$$T(2m+1) = T(3) + \sum_{j=2}^{m} ((2j + 1) + 5) = m^2 + 7m - 2 = \frac{n^2}{4} + 3n - 5 - \frac{1}{4}.$$

Applying Lemma 2 with $f(m) = m^2 + 6m - 5$ and $g(m) = m^2 + 7m - 2$ combines the parity cases into the single formula (1), valid for all $n \geq 2$. □

Remark 3. At $k = 4$ the two incremental costs coincide, $k + 5 = 2k + 1 = 9$, which is the source of the non-uniqueness recorded below.

Remark 4. Optimal schedules are not unique. The shuttle schedule in Figure 2 attains $T(4) = 11$, but its five-person analogue costs $17 > T(5) = 16$. The schedule $\{1, 4\} \to \mathsf{R},\ 1 \to \mathsf{L},\ \{1, 3\} \to \mathsf{R},\ 1 \to \mathsf{L},\ \{1, 2\} \to \mathsf{R}$ is also optimal, so arguments about when the two slowest people first arrive together fail. Hence, we rely on Rote's procedure.

Step	Direction	People	Time	Cumulative
1	L to R	1, 2	2	2
2	R to L	1	1	3
3	L to R	1, 3	3	6
4	R to L	1	1	7
5	L to R	1, 4	4	11

Figure 2: One optimal schedule for $T(4) = 11$.

As mentioned in Remark 4, optimal schedules are not unique. As such, it is convenient to also consider another objective where we seek to minimize the number of trips made by person 1.

Definition 5. For $n \geq 1$, let

$$a(n) = \min \{\#\{\text{trips made by person 1 in } \sigma\} : \sigma \text{ is a schedule of total time } T(n)\}.$$

The restriction to schedules of minimum total time is essential. Without it, the quantity is uninteresting because person 1 may cross once and then remain on the right bank while person 2 carries the torch back and forth, so the unrestricted minimum is 1 for every n.

Proposition 6. *For all $n \geq 1$, we have $a(n) = 2\left\lceil \frac{n}{2} \right\rceil - 1$.*

We will prove Proposition 6 later.

Lemma 7. *Let $n \geq 3$ and let σ be any schedule for $\{1, \ldots, n\}$ with capacity $c = 2$. Suppose person 1 takes part in a forward moves, m of the returns carry two people, and s of the forward moves carry one person. Then,*

$$\operatorname{cost}(\sigma) \geq T(n) + \left\lceil \frac{n}{2} \right\rceil - a + 4m + 3s.$$

Proof. Let f denote the number of forward moves of σ. Since the torch changes sides at every move and σ begins and ends with a forward move, there are $f - 1$ returns.

Each forward move carries one or two people, and s of them carry one, so the forward moves carry $2f - s$ people in total. Each return carries one or two people, and m of them carry two, so the returns carry $(f - 1) + m$ people in total. The two totals differ by n, the number of people who end up across, so

$$2f - s = n + (f - 1) + m \quad \text{which implies} \quad f = n - 1 + m + s. \tag{4}$$

Person 1 starts on the left, ends on the right, and changes side at every move it joins, so its moves alternate and it makes a forward moves and $a-1$ returns. The other $(f-1)-(a-1)$ returns carry nobody faster than person 2, so each of them costs at least 2, while each return containing person 1 costs at least 1. Hence, the returns cost at least

$$(a - 1) + 2(f - a) = 2f - a - 1.$$

Next, at most s forward moves are cheaper than 2, since a forward move costing 1 can only be person 1 traveling alone. By (4), the number of forward moves costing at least 2 is therefore at least $f - s = n - 1 + m \geq \left\lceil \frac{n}{2} \right\rceil$ since $n \geq 3$.

Next, we bound the $\left\lceil \frac{n}{2} \right\rceil$ most expensive forward moves. Fix $j \leq \left\lceil \frac{n}{2} \right\rceil$ and consider the $2(j-1)+1$ slowest people, namely those with labels at least $n - 2(j-1)$. Each of them goes forward at least once, and a forward move carries at most two of them, so their forward moves cannot all fit into $j - 1$ moves: at least j forward moves cost at least $n - 2(j-1)$. Hence, the j^{th} most expensive forward move costs at least $n - 2(j-1)$, and also at least 2 by the previous paragraph. As such, the $\left\lceil \frac{n}{2} \right\rceil$ most expensive forward moves cost at least

$$S := \sum_{j=1}^{\left\lceil \frac{n}{2} \right\rceil} \max\{n - 2(j-1), 2\} = \begin{cases} \frac{n^2}{4} + \frac{n}{2}, & \text{if } n \text{ even;} \\ \frac{(n+1)^2}{4} + 1, & \text{if } n \text{ odd.} \end{cases}$$

The two values arise because for even n, the terms are $n, n-2, \ldots, 2$, while for odd n, they are $n, n-2, \ldots, 3$ followed by a final term 2 in place of 1. Each of the remaining $f - \left\lceil \frac{n}{2} \right\rceil$ forward moves costs at least 2, except for at most s of them which cost at least 1, so all the forward moves together cost at least $S + 2\left(f - \left\lceil \frac{n}{2} \right\rceil\right) - s$.

Hence,

$$\begin{aligned}\operatorname{cost}(\sigma) &\geq S + 2\left(f - \left\lceil \frac{n}{2} \right\rceil\right) - s + 2f - a - 1 \\ &= S + 4f - 2\left\lceil \frac{n}{2} \right\rceil - s - a - 1 \\ &= S + 4n + 4m + 3s - 2\left\lceil \frac{n}{2} \right\rceil - a - 5.\end{aligned}$$

If n is even, then $\left\lceil \frac{n}{2} \right\rceil = \frac{n}{2}$ and $S = \frac{n^2}{4} + \frac{n}{2}$, so the right side is

$$\frac{n^2}{4} + 3n - 5 + \left(\frac{n}{2} - a\right) + 4m + 3s = T(n) + \left(\left\lceil \frac{n}{2} \right\rceil - a\right) + 4m + 3s.$$

If n is odd, then $\left\lceil \frac{n}{2} \right\rceil = \frac{n+1}{2}$ and $S = \frac{(n+1)^2}{4} + 1$, and $T(n) = \frac{n^2-1}{4} + 3n - 5$. Since $\frac{(n+1)^2}{4} - \frac{n^2-1}{4} = \frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil$, the right side is again $T(n) + \left(\left\lceil \frac{n}{2} \right\rceil - a\right) + 4m + 3s$. This is the claimed inequality. □

We can now prove Proposition 6.

Proof. For the upper bound, we exhibit a schedule of total time $T(n)$ in which person 1 makes exactly $2\left\lceil \frac{n}{2} \right\rceil - 1$ trips. By the proof of Theorem 1, iterating the cheaper option of Rote's procedure is optimal, and it consists of blocks of the form

$$\{1,2\} \to \mathsf{R}, \quad 1 \to \mathsf{L}, \quad \{k-1,k\} \to \mathsf{R}, \quad 2 \to \mathsf{L}.$$

If $n \geq 4$ is even, then there are $\frac{n}{2} - 1$ such blocks, followed by a single move $\{1,2\} \to \mathsf{R}$; if $n \geq 5$ is odd, then there are $\frac{n-3}{2}$ such blocks, followed by the moves $\{1,2\} \to \mathsf{R}$, $1 \to \mathsf{L}$, $\{1,3\} \to \mathsf{R}$. Person 1 makes exactly two trips per block, and one further trip in the even case, respectively three further trips in the odd case, giving $2\left(\frac{n}{2} - 1\right) + 1 = n - 1$ trips for even $n \geq 4$ and $2 \cdot \frac{n-3}{2} + 3 = n$ for odd $n \geq 5$; both equal $2\left\lceil \frac{n}{2} \right\rceil - 1$. The cases $n \leq 3$ are immediate.

For the lower bound, note first that for $n \leq 2$, the claim is trivial, since $2\left\lceil \frac{n}{2} \right\rceil - 1 = 1$ and person 1 must cross at least once. Suppose $n \geq 3$ and let σ be any schedule such that $\operatorname{cost}(\sigma) = T(n)$. By Lemma 7,

$$0 = \operatorname{cost}(\sigma) - T(n) \geq \left\lceil \frac{n}{2} \right\rceil - a + 4m + 3s,$$

so $a \geq \left\lceil \frac{n}{2} \right\rceil + 4m + 3s \geq \left\lceil \frac{n}{2} \right\rceil$. Person 1 makes a forward trips and $a - 1$ returns, hence $2a - 1 \geq 2\left\lceil \frac{n}{2} \right\rceil - 1$ trips in total. As σ was an arbitrary time-optimal schedule, $a(n) \geq 2\left\lceil \frac{n}{2} \right\rceil - 1$, and with the upper bound, this gives equality. □

We can visualize the set of moves taken to obtain $T(n)$. We give two examples for $T(5)$ and $T(8)$ as shown in Figures 3 and 4 respectively. Here, each non-loop edge denotes a forward move made by the two vertices it joins, where vertex i represents person i with crossing time i; a loop at vertex i denotes a solo return by person i. Unlike Rote's visualization in [6], ours incorporates loops for vertices 1 and 2 so that Proposition 6 becomes more apparent. Moreover, we have colored vertices 1 and 2 as they have more interaction than the others. We must be careful about what the loops contribute. Under the usual graph-theoretic convention, a loop adds 2 to the degree of its vertex, whereas a solo return is a single trip. Accordingly, we do not use the degree of vertex 1, but its *trip-incidence count*

$$\tau_1 = e_1 + \ell_1,$$

where e_1 is the number of non-loop edges incident with vertex 1 and ℓ_1 is the number of loops at vertex 1; its usual degree is $e_1 + 2\ell_1$. With this convention, $a(n) = \tau_1$ for the displayed schedules: in Figure 3 we have $e_1 = 3$ and $\ell_1 = 2$, so $\tau_1 = 5 = a(5)$, and in Figure 4, we have $e_1 = 4$ and $\ell_1 = 3$, so $\tau_1 = 7 = a(8)$.

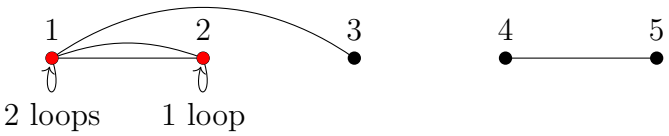

Figure 3: Visualizing the set of moves to obtain $T(5)$

Figure 4: Visualizing the set of moves to obtain $T(8)$

3 The capacity-three case

Let c denote the capacity of the bridge, meaning that at most c persons may be on the bridge in a single move. The classical bridge and torch puzzle corresponds to the case $c = 2$, which was discussed in Section 2. In this section, we study the bridge and torch problem for capacity $c = 3$ with crossing times $\{1, \ldots, n\}$, while retaining the single-torch constraint. We write $T_3(n)$ for the optimal total time.

Proposition 8. *$T_3(1) = 1$, $T_3(2) = 2$, $T_3(3) = 3$, $T_3(4) = 7$, $T_3(5) = 9$, $T_3(6) = 14$.*

Proof. By complete search over the finite state space. □

Lemma 9. *There exists an optimal schedule in which every return is made by a single person, and that person is the fastest person then present on the right bank.*

Proof. See Sniedovich [8], whose analysis exhibits properties possessed by at least one optimal policy. □

Remark 10. Lemma 9 must not be strengthened to the assertion that in *every* optimal schedule, returns are made by the fastest person on the right. For example, if $n = 6$ and $c = 3$, the schedule

$$\{1,2,3\} \to \mathsf{R}, \quad 2 \to \mathsf{L}, \quad \{4,5,6\} \to \mathsf{R}, \quad 1 \to \mathsf{L}, \quad \{1,2\} \to \mathsf{R}$$

has total time $3 + 2 + 6 + 1 + 2 = 14 = T_3(6)$ and is therefore optimal, yet its first return is made by person 2 while person 1 is also on the right bank.

3.1 An upper bound

Proposition 11. *For all $k \geq 9$, we have $T_3(k) \leq T_3(k-6) + 2k + 6$.*

Proof. We construct a sequence of moves from the state $S_k = (\{1,\ldots,k\}, \mathsf{L})$ to the state $S_{k-6} = (\{1,\ldots,k-6\}, \mathsf{L})$, namely the six moves in Figure 5.

Step	Direction	People	Time	Cumulative
1	L to R	1, 2, 3	3	3
2	R to L	1	1	4
3	L to R	$k-2$, $k-1$, k	k	$k+4$
4	R to L	2	2	$k+6$
5	L to R	$k-5$, $k-4$, $k-3$	$k-3$	$2k+3$
6	R to L	3	3	$2k+6$

Figure 5: A sequence of moves from S_k to S_{k-6}, valid for $k \geq 9$.

Each move respects the capacity constraint, and for $k \geq 9$, each move takes people who are on the side of the torch. The hypothesis $k \geq 9$ is needed: step 5 requires $k-5, k-4, k-3$ all to be on the left bank at that moment, and after steps 1–4 the left bank is $\{1,2\} \cup \{4,\ldots,k-3\}$, so $k - 5 \geq 4$ is required. For $k = 7$ and $k = 8$ the sequence is not legal, since person 3 has already crossed and would appear again in step 5; those two values are not used below. After these moves, exactly the six people $\{k-5, k-4, k-3, k-2, k-1, k\}$ are on the right bank, while $\{1,\ldots,k-6\}$ remain on the left bank and the torch is on the left; that is, we have reached S_{k-6}. The total cost of these six moves is $3 + 1 + k + 2 + (k-3) + 3 = 2k + 6$. Appending an optimal schedule for the remaining instance, which is the $(k-6)$-person instance with crossing times $1,\ldots,k-6$, gives a feasible schedule of total cost $(2k+6) + T_3(k-6)$, whence the stated upper bound. □

Only the range $k \geq 13$ is used below, so the restriction to $k \geq 9$ costs nothing. Iterating Proposition 11 from the values

$$T_3(7) = 17, \quad T_3(8) = 22, \quad T_3(9) = 26, \quad T_3(10) = 32, \quad T_3(11) = 37, \quad T_3(12) = 43, \quad (5)$$

which are obtained by exhaustive search on the finite state space, yields the following explicit bound.

Theorem 12. *For all $n \geq 7$,*

$$T_3(n) \leq U(n) := \frac{n^2}{6} + 2n - \frac{181}{36} + \frac{(-1)^n}{4} - \frac{2}{9}\cos\left(\frac{2n\pi}{3}\right). \quad (6)$$

Proof. Define $E(n) = U(n) - \frac{n^2}{6} - 2n$, so E is the sum of a period-2 and a period-3 term and hence is 6-periodic. Note that $U(n)$ satisfies the recurrence relation $U(n) = U(n-6) + 2n + 6$ for all n. A direct evaluation shows that $U(n)$ agrees with the six values in (5) for $n = 7, \ldots, 12$:

n	7	8	9	10	11	12
$U(n) - \frac{n^2}{6} - 2n$	$-\frac{31}{6}$	$-\frac{14}{3}$	$-\frac{11}{2}$	$-\frac{14}{3}$	$-\frac{31}{6}$	-5

and these are the values of $T_3(n) - \frac{n^2}{6} - 2n$ for $n = 7, \ldots, 12$. The claim now follows by induction on n: it holds for $7 \leq n \leq 12$, and if it holds for $n - 6$ with $n \geq 13$, then by Proposition 11,

$$T_3(n) \leq T_3(n-6) + 2n + 6 \leq U(n-6) + 2n + 6 = U(n). \qquad \square$$

3.2 A lower bound

The difficulty in bounding $T_3(n)$ from below is that expensive crossings may be interleaved with the transport of other people, and no particular state need be visited. We now record a reformulation in which the order is discarded entirely, so that the quantity to be bounded is the minimum of an explicit weight over a set defined by two counting conditions. The idea of replacing a sequence of crossings by the bag of its forward trips is due to Backhouse [1], whose central theorem transforms the problem of determining an optimal sequence into that of determining an optimal *bag* of forward trips.

Definition 13 (admissible bag)**.** We say that a finite multiset F of non-empty subsets of $\{1, \ldots, n\}$ is an admissible bag if the following hold:

(i) $|T| \leq 3$ for every $T \in F$ since at most three people cross in one trip;

(ii) every $i \in \{1, \ldots, n\}$ belongs to at least one $T \in F$;

(iii) $\sum_{T \in F} (|T| - 1) = n - 1$.

Its weight is

$$w(F) = \sum_{T \in F} \left(\max T + \sum_{i \in T} i \right) - \frac{n(n+1)}{2},$$

and we write $B_3(n)$ for the minimum of $w(F)$ over all admissible bags F.

Proposition 14. *Let σ be a schedule for capacity 3 in which every return carries exactly one person, and let $F(\sigma)$ be the multiset of its forward trips. Then $F(\sigma)$ is an admissible bag and the total cost of σ equals $w(F(\sigma))$. Consequently, $T_3(n) \geq B_3(n)$.*

Proof. For every $i \in \{1, \ldots, n\}$, let d_i denote the number of forward trips of σ containing person i. As person i starts on the left and ends on the right, and every trip containing i changes the side of i, then person i makes $d_i - 1$ returns. Since each return carries one person, then

$$\text{the total cost of the returns is } \sum_i (d_i - 1)\, i \quad \text{and}$$
$$\text{the total cost of the forward trips is } \sum_{T \in F(\sigma)} \max T.$$

Let $f = |F(\sigma)|$ denote the number of forward trips of σ. Then, σ has $f - 1$ returns, so $\sum_i (d_i - 1) = f - 1$. Also, note that $\sum_i d_i = \sum_{T \in F(\sigma)} |T|$, which implies

$$\sum_{T \in F(\sigma)} (|T| - 1) = \sum_i d_i - f = f - 1 + \sum_i 1 - f = n - 1,$$

which is condition (iii) of Definition 13. Note that condition (i) holds trivially since this refers to the capacity constraint; condition (ii) holds too because every person crosses forward at least once. Next, we note the identity

$$\sum_i d_i i = \sum_{T \in F(\sigma)} \sum_{i \in T} i,$$

so

$$\text{cost}(\sigma) = \sum_{T \in F(\sigma)} \max T + \sum_i (d_i - 1)\, i = \sum_{T \in F(\sigma)} \left(\max T + \sum_{i \in T} i \right) - \frac{n(n+1)}{2} = w(F(\sigma)).$$

By Lemma 9, there is an optimal schedule in which every return is made by a single person. Applying the identity to it gives $T_3(n) = w(F) \geq B_3(n)$ for that bag F. □

When every return carries one person, the cost of a schedule depends only on its bag of forward trips, and both the cost and the constraints of Definition 13 are sums over the trips. This suggests bounding each trip separately and adding. We look for a price $y_i \geq 0$ for each

person and a price λ for each unit of progress $|T| - 1$, such that no trip is worth more than it costs. That is,

$$\sum_{i \in T} y_i + \lambda \left(|T| - 1 \right) \leq \max T + \sum_{i \in T} i. \tag{7}$$

Summing over the trips of an admissible bag and using conditions (ii) and (iii) of Definition 13 yields

$$w(F) \geq \sum_i y_i + \lambda(n-1) - \frac{n(n+1)}{2}. \tag{8}$$

In the language of linear programming, the prices are said to form a *feasible dual solution*. To make the bound as strong as possible, the trips used by the construction of Proposition 11 should be tight. That is to say, the trips must satisfy (7) with equality. We determine the prices in three steps.

First, for $T = \{1, 2, 3\}$, the right side of (7) is $3 + (1 + 2 + 3) = 9$ and $|T| - 1 = 2$, so $y_1 + y_2 + y_3 + 2\lambda \leq 9$. Since $y_i \geq 0$, the largest admissible value is $\lambda = \frac{9}{2}$, and it forces $y_1 = y_2 = y_3 = 0$.

Second, for a consecutive triple $T = \{c-2, c-1, c\}$, the right side is $c + (3c - 3) = 4c - 3$, so tightness with $\lambda = \frac{9}{2}$ means

$$y_{c-2} + y_{c-1} + y_c = 4c - 3 - 9 = 4c - 12.$$

Trying $y_i = \alpha i + \beta$, the left side is $3\alpha c - 3\alpha + 3\beta$. Comparing coefficients gives $\alpha = \frac{4}{3}$ and $\beta = -\frac{8}{3}$, so

$$y_i = \frac{4}{3}(i - 2).$$

Third, this formula is too large for some small i, because trips containing person 1 are cheap. For $T = \{1, 2, c\}$ with $c \geq 3$, the right side is $c + (3 + c) = 2c + 3$, so $y_1 = y_2 = 0$ gives

$$y_c \leq 2c + 3 - 9 = 2c - 6.$$

At $c = 4$ this requires $y_4 \leq 2$, whereas $\frac{4}{3}(4 - 2) = \frac{8}{3}$, so we take $y_4 = 2$. At $c = 5$ it requires $y_5 \leq 4$, which the formula meets with equality, so $y_5 = 4$. For $T = \{1, 5, 6\}$, the right side is $6 + 12 = 18$, so

$$y_5 + y_6 \leq 18 - 9 = 9,$$

and with $y_5 = 4$ this requires $y_6 \leq 5$, whereas $\frac{4}{3}(6 - 2) = \frac{16}{3}$, so we take $y_6 = 5$. For $i \geq 7$, the linear formula satisfies every constraint, as Lemma 15 shows.

Lemma 15 (local inequality)**.** *For $i \geq 1$, define the sequence (y_i) as follows: $y_1 = y_2 = y_3 = 0$, $y_4 = 2$, $y_5 = 4$, $y_6 = 5$, and for $i \geq 7$, $y_i = \frac{1}{3}(4i - 8)$. Then, for every non-empty $T \subseteq \{1, 2, 3, \ldots\}$ with $|T| \leq 3$,*

$$\sum_{i \in T} y_i + \frac{9}{2}\left(|T| - 1 \right) \leq \max T + \sum_{i \in T} i.$$

Proof. Clearly $y_i \geq 0$ for all i. Moreover, we use the bounds

$$y_i \leq \tfrac{4}{3}(i-2) \quad \text{for } i \geq 2, \tag{9}$$
$$y_i \leq 2i - 6 \quad \text{for } i \geq 3, \tag{10}$$
$$y_i \leq 2i - 7 \quad \text{for } i \geq 6. \tag{11}$$

For $i \leq 6$, these are checked directly. For $i \geq 7$, (9) holds with equality, and (10) and (11) reduce to $i \geq 5$ and $i \geq \frac{13}{2}$.

Write $T = \{a\}$, $\{a, b\}$ or $\{a, b, c\}$ with $a < b < c$. The claim is then

$$y_a \leq 2a, \qquad y_a + y_b \leq a + 2b - \tfrac{9}{2}, \qquad y_a + y_b + y_c \leq a + b + 2c - 9.$$

If $a \geq 2$, then by (9), it suffices to show that $4(a-2) \leq 6a$, $2a \leq 4b+5$, and $a+b \leq 2c-3$. All three hold because $a < b < c$ are integers.

If $a = 1$, then $y_1 = 0$ and the claim becomes

$$0 \leq 2, \qquad y_b \leq 2b - \tfrac{7}{2}, \qquad y_b + y_c \leq b + 2c - 8.$$

The second inequality holds for $b = 2$ since $0 \leq \frac{1}{2}$, and for $b \geq 3$ by (10). For the third, there are three cases.

- If $b \leq 4$, combine $y_b \leq b - 2$ with (10) for c.
- If $b = 5$, combine $y_5 = b - 1$ with (11) for $c \geq 6$.
- If $b \geq 6$, then (9) reduces the claim to $b + 8 \leq 2c$, which follows from $c \geq b + 1$. □

Lemma 16 (return inequality)**.** *Let the sequence (y_i) be as defined in Lemma 15. Then, for every non-empty $R \subseteq \{1, 2, 3, \ldots\}$ with $|R| \leq 3$,*

$$g(R) := \frac{9}{2}(|R| - 1) + \max R + \sum_{i \in R} (y_i - i) \geq 0.$$

Proof. We use the fact that

$$y_i - i \geq -3 \quad \text{for every } i \geq 1. \tag{12}$$

If $R = \{a\}$, then $g(R) = a + (y_a - a) = y_a \geq 0$. If $R = \{a, b\}$ with $a < b$, then

$$g(R) = \frac{9}{2} + b + (y_a - a) + (y_b - b) = \frac{9}{2} + (y_a - a) + y_b \geq \frac{9}{2} - 3 + 0 = \frac{3}{2}.$$

If $R = \{a, b, c\}$ with $a < b < c$, then

$$g(R) = 9 + c + (y_a - a) + (y_b - b) + (y_c - c) = 9 + (y_a - a) + (y_b - b) + y_c \geq 9 - 3 - 3 + 0 = 3.$$

In each case, we used the facts that $y_i \geq 0$ and (12). □

We can now bound the cost of an arbitrary schedule, with no assumption whatever about the form of its returns.

Proposition 17. *Let $n \geq 7$ and let σ be any schedule for $\{1, \ldots, n\}$ with capacity $c = 3$. Then,*

$$\operatorname{cost}(\sigma) \geq V(n) := \frac{n^2}{6} + 2n - \frac{11}{2}.$$

Proof. Let σ have forward trips $T_1, \ldots, T_f$ and returns $R_1, \ldots, R_{f-1}$, and for each person i, let d_i and b_i be the numbers of forward trips and of returns containing i. Each trip containing i changes the side of i, and i starts on the left and ends on the right, so

$$d_i = b_i + 1 \quad \text{for every } i. \tag{13}$$

Since

$$\sum_i d_i = \sum_u |T_u| \quad \text{and} \quad \sum_i b_i = \sum_j |R_j|,$$

then (13) gives

$$\sum_u |T_u| = n + \sum_j |R_j|.$$

Next, using the fact that

$$\sum_j |R_j| = \sum_j (|R_j| - 1) + (f - 1),$$

we have

$$\sum_u (|T_u| - 1) = n - 1 + \sum_j (|R_j| - 1). \tag{14}$$

Applying Lemma 15 to each forward trip,

$$\max T_u + \sum_{i \in T_u} i \geq \sum_{i \in T_u} y_i + \frac{9}{2} (|T_u| - 1).$$

Summing over all u yields

$$\sum_u \left(\max T_u + \sum_{i \in T_u} i \right) \geq \sum_u \sum_{i \in T_u} y_i + \frac{9}{2} \sum_u (|T_u| - 1).$$

Hence,

$$\sum_u \max T_u + \sum_i d_i i \geq \sum_i d_i y_i + \frac{9}{2} \sum_u (|T_u| - 1).$$

By (13), we have

$$\sum_i d_i i = \frac{n(n+1)}{2} + \sum_i b_i i \quad \text{and} \quad \sum_i d_i y_i = \sum_i y_i + \sum_i b_i y_i,$$

so

$$\sum_u \max T_u \geq \left(\sum_i y_i + \frac{9}{2}(n-1) - \frac{n(n+1)}{2} \right) + \frac{9}{2} \sum_j \left(|R_j| - 1 \right) + \sum_i b_i \left(y_i - i \right).$$

Adding $\sum_j \max R_j$ to both sides and using $\sum_i b_i \left(y_i - i \right) = \sum_j \sum_{i \in R_j} \left(y_i - i \right)$,

$$\operatorname{cost}(\sigma) = \sum_u \max T_u + \sum_j \max R_j \geq \left(\sum_i y_i + \frac{9}{2}(n-1) - \frac{n(n+1)}{2} \right) + \sum_j g\left(R_j \right),$$

with g as in Lemma 16. Note that $g(R_j) \geq 0$ by that lemma, so it remains to evaluate the bracket. For $n \geq 7$,

$$\sum_{i=1}^{n} y_i = (0+0+0+2+4+5) + \sum_{i=7}^{n} \frac{4i-8}{3} = \frac{2}{3}n^2 - 2n - 1.$$

Hence,

$$\sum_{i=1}^{n} y_i + \frac{9}{2}(n-1) - \frac{n(n+1)}{2} = \frac{n^2}{6} + 2n - \frac{11}{2}, \tag{15}$$

which is $V(n)$. □

Corollary 18. *For every $n \geq 7$, $B_3(n) \geq V(n)$.*

Proof. Lemma 15 states that the sequence (y_i) defined there, together with $\lambda = \frac{9}{2}$, satisfies (7) for every non-empty T with $|T| \leq 3$. Hence, (8) holds with $\lambda = \frac{9}{2}$, that is,

$$w(F) \geq \sum_{i=1}^{n} y_i + \frac{9}{2}(n-1) - \frac{n(n+1)}{2}$$

for every admissible bag F. By (15), the right side equals $V(n)$ for $n \geq 7$. Taking the minimum over all admissible bags gives $B_3(n) \geq V(n)$. □

The weights are not *quite* sharp: $V(n)$ falls below $U(n)$, but always by less than 1. To close the gap, we invoke Lemma 19.

Lemma 19. *For every $n \geq 7$ we have $U(n) \in \mathbb{Z}$ and $0 \leq U(n) - V(n) < 1$.*

Proof. As noted in the proof of Theorem 12, $U(n) = U(n-6) + 2n + 6$ for every n, and $U(7), \ldots, U(12)$ are the integers $17, 22, 26, 32, 37, 43$. By induction on n, $U(n) \in \mathbb{Z}$ for all $n \geq 7$. Next, $U(n) - V(n) = E(n) + \frac{11}{2}$ where $E(n) = U(n) - \frac{n^2}{6} - 2n$ is 6-periodic with the six values $-\frac{31}{6}, -\frac{14}{3}, -\frac{11}{2}, -\frac{14}{3}, -\frac{31}{6}, -5$ listed in the proof of Theorem 12. Hence, $U(n) - V(n)$ takes only the values $\frac{1}{3}, \frac{5}{6}, 0, \frac{5}{6}, \frac{1}{3}, \frac{1}{2}$, all of which lie in $[0, 1)$. □

Theorem 20. *For all $n \geq 7$,*

$$T_3(n) = \frac{n^2}{6} + 2n - \frac{181}{36} + \frac{(-1)^n}{4} - \frac{2}{9}\cos\left(\frac{2n\pi}{3}\right).$$

Proof. By Proposition 17, every schedule costs at least $V(n)$, so $T_3(n) \geq V(n)$. Since $T_3(n) \in \mathbb{Z}$, then $T_3(n) \geq \lceil V(n) \rceil$. By Lemma 19, $U(n) \in \mathbb{Z}$ such that $U(n) - 1 < V(n) \leq U(n)$, so $\lceil V(n) \rceil = U(n)$. Hence, $T_3(n) \geq U(n)$. The reverse inequality is Theorem 12. □

Corollary 21. $T_3(n) = T_3(n-6) + 2n + 6$ *for all* $n \geq 13$.

Proof. Immediate from Theorem 20 and the identity $U(n) = U(n-6) + 2n + 6$. □

Example 22 ($c = 3$)**.** The values of $T_3(n)$ for small n are as follows; those for $n \leq 6$ are from Proposition 8 and those for $n \geq 7$ are given by Theorem 20.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$T_3(n)$	1	2	3	7	9	14	17	22	26	32	37	43	49	56	62	70	77	85	93	102

These are the terms of A392834.

Example 23 ($c = 4$)**.** When $c = 4$, by complete search one generates the sequence

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$T_4(n)$	1	2	3	4	8	10	12	17	20	23	28	32	36	42	47	52	58	64	70	77

Note that $T_c(n)$ is most interesting for small capacities c. Indeed, as c grows relative to n, the constraint becomes weak: if $c \geq n$, everyone can cross in a single move, so $T_c(n) = n$. For $c \geq 4$ two questions should be distinguished. Computing the optimum for a given instance is settled by the general methods of [1, 2]. What may remain interesting is whether a particularly short closed formula, or a recurrence of bounded order, exists for the consecutive crossing times $t_i = i$; we conjecture that it does for each small c, in analogy with the period-6 quadratic of Theorem 20. We note that the method of Section 3 is not special to $c = 3$: the bag formulation and the local inequality both make sense for any capacity, and the task for a given c is to find a system of weights whose value matches the construction.

4 A generalization to star graphs

As we have seen, the bridge and torch problem usually involves a single bridge with limited capacity. The task is to decide who crosses together, who returns with the torch, and in what order, to finish as soon as possible. Now, consider a setting with more than one bridge. At a node, some travelers go in different directions. The goal is to schedule movements so that everyone reaches a destination quickly.

Here, we will discuss some applications to star graphs (Definition 24). In such a graph, travelers start at the root and must reach the leaves.

Definition 24 (star graph)**.** Fix an integer $k \geq 1$. The star graph with k edges, denoted by $\mathcal{S}(k)$, is the graph with vertex set $V = \{o\} \cup \{u_1, \ldots, u_k\}$ where o is the center and $u_1, \ldots, u_k$ are the leaves. We define the edge set to be $E = \{\{o, u_i\} : i = 1, \ldots, k\}$. Equivalently, $\mathcal{S}(k)$ is the unique graph on $k+1$ vertices in which $\deg(o) = k$ and $\deg(u_i) = 1$ for all $i = 1, \ldots, k$, up to graph isomorphism.

Essentially, one can think of a star graph as a hub-and-spoke system where there is a single root node and multiple child nodes connected only to the root node. We will work on star graphs up to rotation. For example, Figure 6 shows a star graph with root/center o and edges $\{o, u_i\}$ where $i = 1, \ldots, 7$.

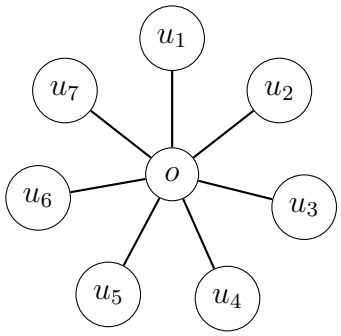

Figure 6: The star graph $\mathcal{S}(7)$ with 7 edges

We now formally define the problem for star graphs. Since several modeling choices are possible, we state our assumptions in full.

Definition 25 (the star model)**.** Let $\mathcal{S}(k)$ have center o and leaves $u_1, \ldots, u_k$, let the n persons have crossing times $1 < 2 < \cdots < n$, and let $t \geq 1$ torches be given. Initially, all n persons and all t torches are at the center o. A torch-movement consists of one or two persons together with one torch, all currently at the same vertex, traversing a single edge; the persons carry the torch, and the duration of the movement is the larger crossing time among its participants. A parallel step is a collection of torch-movements executed simultaneously subject to:

- *disjoint people*: no person participates in more than one movement in that step;

- *disjoint star edges*: no star edge is traversed by more than one movement in that step;
- *disjoint torches*: no torch is used by more than one movement in that step.

The duration of a parallel step is the maximum duration among its movements, and steps are executed one after another, each step finishing before the next begins. The objective is to, in minimum total duration, reach a configuration in which every person is at some leaf. Any person may finish at any leaf, several persons may finish at the same leaf, and torches may be left at the center or at any leaf. We write $T(n,k,t)$ for the minimum total duration.

Theorem 26 (bridge and torch problem for star graphs)**.** *In the model of Definition 25, let* $m=\min(k,t)$ *and* $s=\left\lceil \frac{n}{2m} \right\rceil$. *Then, for* $n\geq 2$,

$$sn-ms(s-1)\leq T(n,k,t)\leq T(n)=\frac{n^2}{4}+3n-5+\frac{(-1)^n-1}{8}, \tag{16}$$

and the lower bound is attained whenever $t\geq \left\lceil \frac{n}{2} \right\rceil$. *Moreover, writing* $n=2mq+r$ *with* $0\leq r<2m$,

$$sn-ms(s-1)=\frac{n^2}{4m}+\frac{n}{2}+\frac{r(2m-r)}{4m}\geq \frac{n^2}{4m}+\frac{n}{2}. \tag{17}$$

Before discussing the proof, we look at some important cases. First, if $m\geq \left\lceil \frac{n}{2} \right\rceil$, then $s=1$ and $T(n,k,t)=n$. Indeed, this holds because everybody leaves the center in a single parallel step. On the other hand, if $k=1$ and $t\geq \left\lceil \frac{n}{2} \right\rceil$, then $m=1$ and $s=\left\lceil \frac{n}{2} \right\rceil$, which implies

$$T(n,1,t)=\left\lfloor \frac{(n+1)^2}{4} \right\rfloor.$$

By considering the expression in (2), we see that for all $n\geq 2$,

$$T(n,1,t)=\left\lfloor \frac{(n+1)^2}{4} \right\rfloor \leq \left\lfloor \frac{n^2+12n-20}{4} \right\rfloor = T(n).$$

This makes sense because the conventional bridge and torch problem only allows one torch, so that all the returns which the conventional problem must pay for are avoided here.

In the general setting where $t=\left\lceil \frac{n}{2} \right\rceil$, and $k\geq 2$ and n vary, a query of the values in Figure 7 on the OEIS [5] returns sequences of partial sums of floor functions. One checks directly from Theorem 26 that

$$T\left(n,k,\left\lceil \tfrac{n}{2} \right\rceil\right)=\sum_{j=0}^{n+2k-1}\left\lfloor \frac{j}{2k} \right\rfloor \quad \text{for all } n\geq 1 \text{ and } k\geq 2, \tag{18}$$

which is what identifies the rows of Figure 7 with the sequences listed there. What the star model contributes is therefore a scheduling interpretation of these partial sums: $\sum_{j=0}^{n+2k-1}\left\lfloor \frac{j}{2k} \right\rfloor$

is the least total time needed to move n people of speeds $1, \ldots, n$ from the center of $\mathcal{S}(k)$ to its leaves when torches are abundant.

We stress that the closed forms themselves are not new. For $k = 2$, combining (18) with Theorem 26 gives

$$\sum_{j=0}^{n} \left\lfloor \frac{j}{4} \right\rfloor = \left\lceil \frac{n-3}{4} \right\rceil \left(n - 3 - 2 \left(\left\lceil \frac{n-3}{4} \right\rceil - 1 \right) \right) \quad \text{where } n \geq 3,$$

but since $\left\lceil \frac{n-3}{4} \right\rceil = \left\lfloor \frac{n}{4} \right\rfloor$ for every n, the right side is $\left\lfloor \frac{n}{4} \right\rfloor \left(n - 1 - 2 \left\lfloor \frac{n}{4} \right\rfloor \right)$, which is one of the formulas already recorded in A130519 (as is the shorter $\left\lfloor \frac{(n-1)^2}{8} \right\rfloor$). The same applies for the other values of k. As such, the derivation above should be read as a new route to known identities and as an interpretation of them, rather than as a source of new ones.

k	Sequence values for $n = 1, \ldots, 20$	Related seq.
2	1, 2, 3, 4, 6, 8, 10, 12, 15, 18, 21, 24, 28, 32, 36, 40, 45, 50, 55, 60	A130519
3	1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 16, 18, 21, 24, 27, 30, 33, 36, 40, 44	A174709
4	1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14, 16, 18, 20, 22, 24, 27, 30, 33, 36	A118729
5	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30	A131242
6	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 18, 20, 22, 24, 26, 28	A221912

Figure 7: Values of $T(n, k, \lceil \frac{n}{2} \rceil)$ for varying $n = 1, \ldots, 20$ and $k = 2, \ldots, 6$. The related sequence in each row is the sequence of partial sums of $\lfloor j/2k \rfloor$, in accordance with (18).

We now give a proof of Theorem 26.

Proof. Let $m = \min(k, t)$. In any parallel step, the movements leaving the center use distinct edges and distinct torches, so there are at most $\min(k, t) = m$ of them. Since each carries at most two people, at most $2m$ people depart from the center in any one step. Call a step outward if it contains at least one movement leaving the center.

For the lower bound, we do not assume anything about the chronological order in which people leave the center. List the durations of the outward steps in decreasing order as $d_1 \geq d_2 \geq \cdots$. Fix j with $1 \leq j \leq s$ and set

$$h_j = n - 2m(j-1) \geq 1,$$

and the inequality holds because $s = \lceil \frac{n}{2m} \rceil$ gives $2m(s-1) < n$. Consider the $2m(j-1)+1$ people with labels at least h_j. Each of them starts at the center and finishes at a leaf, so each has at least one departure from the center. As at most $2m$ such departures occur in any one step, the departures of these people occupy at least

$$\left\lceil \frac{2m(j-1)+1}{2m} \right\rceil = j$$

distinct steps. Each of those steps contains a person with label at least h_j, so its duration is at least h_j. Therefore, at least j outward steps have duration at least h_j, which is to say $d_j \geq h_j$. In particular, at least s outward steps exist, and

$$T(n,k,t) \geq \sum_{j=1}^{s} d_j \geq \sum_{j=1}^{s} (n - 2m(j-1)) = sn - 2m\sum_{j=1}^{s}(j-1) = sn - ms(s-1).$$

Next, write $n = 2mq + r$ with $0 \leq r < 2m$. If $r = 0$, then $s = q$ and

$$sn - ms(s-1) = mq^2 + mq = \frac{n^2}{4m} + \frac{n}{2},$$

which is in agreement with (17) since $r(2m-r) = 0$. If $r > 0$, then $s = q + 1$ and

$$sn - ms(s-1) = (q+1)(2mq+r) - m(q+1)q = mq^2 + qr + mq + r,$$

while

$$\frac{n^2}{4m} + \frac{n}{2} + \frac{r(2m-r)}{4m} = mq^2 + qr + \frac{r^2}{4m} + mq + \frac{r}{2} + \frac{r}{2} - \frac{r^2}{4m} = mq^2 + qr + mq + r,$$

as required. Since $0 \leq r < 2m$, we have $r(2m-r) \geq 0$, which yields (17).

As for the upper bound, since every person is permitted to finish at the same leaf, we may simply fix one leaf u_1, one torch, and ignore all other edges and torches. What remains is exactly the capacity-two bridge and torch problem of Section 2 along the edge $\{o, u_1\}$, so any optimal schedule for that problem is feasible here and $T(n,k,t) \leq T(n)$ for $n \geq 2$, with $T(n)$ as in (1).

Lastly, we shall prove that equality holds when $t \geq \lceil \frac{n}{2} \rceil$. Assume $t \geq \lceil \frac{n}{2} \rceil$ and let $s = \lceil \frac{n}{2m} \rceil$ as above. Partition $\{1, \ldots, n\}$ into s consecutive blocks in decreasing order of labels, namely

$$B_j = \{n - 2mj + 1, \ldots, n - 2m(j-1)\} \cap \{1, \ldots, n\} \quad \text{where } j = 1, \ldots, s,$$

each of size at most $2m$. In step j, split B_j into pairs together with at most one singleton, and send these groups outward along distinct edges, each group carrying a torch not used before. This is a legal move because $|B_j| \leq 2m$ gives at most m movements in the step, and $m \leq k$ edges are available, and the total number of torches used over the whole schedule is $\sum_{j=1}^{s} \lceil |B_j|/2 \rceil = \lceil \frac{n}{2} \rceil \leq t$, so a torch is always available and no torch ever has to be returned to the center. The largest label in B_j is $n - 2m(j-1)$, so the duration of step j is $n - 2m(j-1)$ and the total duration is

$$\sum_{j=1}^{s} (n - 2m(j-1)) = sn - ms(s-1).$$

Together with the lower bound, this gives equality, so $T(n,k,t) = sn - ms(s-1)$. □

5 Acknowledgments

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