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Further Extensions of Sury's Identity

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Abstract

The equation commonly known as Sury's identity is a deceptively simple summation formula that connects the Lucas numbers, Fibonacci numbers, and powers of two. Many authors have given extensions and generalizations over the years; in this paper, we take a different approach that allows us to produce a good number of new summation formulas, all from elementary (but non-trivial) methods.

1 Introduction

The identity

$$2^{n+1}F_{n+1} = \sum_{i=0}^{n} 2^i L_i \tag{1}$$

which connects the Fibonacci numbers, the Lucas numbers, and the powers of 2, is often called Sury's identity thanks to his deceptively simple proof [23] that takes the sum of two geometric series. Fifteen years before Sury's proof, Benjamin and Quinn [5] proved the same identity by counting the tilings of a bracelet with squares and dominoes.

Of course, there are many variations on this identity. Two years after Sury's proof, Martinjak [18] used a telescoping sum to prove that

$$(-1/2)^n F_{n+1} = \sum_{i=0}^{n} (-1/2)^i L_{i+1}, \tag{2}$$

and if we multiply both sides by $(-2)^n$ then this becomes the convolution identity

$$F_{n+1} = \sum_{i=0}^{n} (-2)^{n-i} L_{i+1}. \tag{3}$$

Marques [17] used a quick induction argument to produce a slightly more complicated version of equation (1) with 3^i instead of 2^i, giving us

$$3^{n+1} F_{n+1} = \sum_{i=0}^{n} 3^i (L_i + F_{i+1}), \tag{4}$$

and a clever rearrangement of sums by Edgar [12] produced the more general equation

$$t^{n+1} F_{n+1} = \sum_{i=0}^{n} t^i (L_i + (t-2)F_{i+1}). \tag{5}$$

We note that Edgar's equation (5) will easily give us Sury's equation (1) when $t = 2$, and Marques' equation (4) when $t = 3$. It is a rather pleasant challenge to show that it also gives Martinjak's equation (3) when $t = -1/2$. For two different methods of proving equation (5), with colorful tilings and with generating functions, see Martinjack and Prodinger [19].

Dafnis [8] used a telescoping sum to give a variation on Edgar's equation (5), which in our notation would be

$$(-1)^n F_{n+1} = \sum_{i=0}^{n} (-1)^i t^{n-i} (L_{i+1} + (t-2)F_i). \tag{6}$$

Dafnis went on to prove a similar identity involving the tribonacci numbers.

Abd-Elhameed and Zeyada took Edgar's equation (5) even further by taking up generalized Fibonacci and Lucas sequences, defined as

$$\begin{aligned} U_{n+2}^{a,b} &= aU_{n+1}^{a,b} + bU_n^{a,b} \qquad \text{with } \; U_0^{a,b} = 0, \; U_1^{a,b} = 1, \\ V_{n+2}^{a,b} &= aV_{n+1}^{a,b} + bV_n^{a,b} \qquad \text{with } \; V_0^{a,b} = 2, \; V_1^{a,b} = a. \end{aligned} \tag{7}$$

Their version of Sury's identity (for a, b, t all non-zero) is

$$t^{n+1} U_{n+1}^{a,b} = \frac{1}{a} \sum_{i=0}^{n} t^i (V_i^{a,b} + (at-2)U_{i+1}^{a,b}), \tag{8}$$

and we can clearly see how this is a generalization of Edgar's equation (5). Bhatnagar considered even more geneal sequences by replacing the constants a, b in equation (7) with sequences a_n, b_n. For other approaches, see recent papers by Adegoke and Frontczak [2], Chung, Yao, and Zhou [7], Dafnis, Philippou, and Livieris [9], De Micheli Vitturi [10], Kuhapatanakul and Thongsing [15], Kwong [16], Melham [20], and Philippou and Dafnis [21].

Despite all this, we have discovered a new way to generalize Sury's identity. Every other approach has tried to replace the 2^i in Sury's equation (1) with something like the 3^i in equation (4) and then the t^i as seen in equation (5), which leads to an adjustment in the other term in the summation, from the simple L_i in equation (1) to the slightly more complex $L_i + F_{i+1}$ in equation (4) to the rather complicated $L_i + (t-2)F_{i+1}$ in equation (5), culminating in the general expression $V_i^{a,b} + (at-2)U_{i+1}^{a,b}$ from Abd-Elhameed and Zeyada's equation (8). By contrast, our approach allows us to select the sequence (the Fibonacci numbers, or the Pell numbers, or really any linear recurrence sequence of order 2) inside the summation, and also the initial position, and then produce the particular "t" to make the formula work out nicely.

This is best explained by a demonstration. Recall that Sury and Martinjak began their sums in equations (1) and (2) with L_0 and L_1 respectively, but with our approach we can begin with L_2 or L_3 or even L_4, giving us the new formulas

$$\begin{aligned} F_{n+1} = \sum_{i=0}^{n} (-2)^{n-i} L_{i+1} &= \frac{1}{3} \sum_{i=0}^{n} (-1/3)^{n-i} L_{i+2} \\ &= \frac{1}{4} \sum_{i=0}^{n} (-3/4)^{n-i} L_{i+3} \\ &= \frac{1}{7} \sum_{i=0}^{n} (-4/7)^{n-i} L_{i+4}. \end{aligned}$$

However, our personal favorites are these generalizations of Sury's formula (and again, these seem to be new),

$$\begin{aligned} \sum_{i=0}^{n} (-2)^i L_{3i} &= 2 - (-2)^{n+1} F_{3n}, \\ 4 \sum_{i=0}^{n} 9^i L_{6i} &= 9^{n+1} F_{6n} + 8, \\ 17 \sum_{i=0}^{n} (-38)^i L_{9i} &= 34 - (-38)^{n+1} F_{9n}. \end{aligned} \tag{9}$$

These are similar (in spirit but not in fact) to the formulas we can derive from Theorem 2.1 in Adegoke and Frontczak's recent paper [2], which give us (after some manipulation)

$$\sum_{i=0}^{n}(1/2)^i L_{3i} = F_{3(n+1)}/2^n,$$
$$4\sum_{i=0}^{n}(1/9)^i L_{6i} = F_{6(n+1)}/9^n,$$
$$17\sum_{i=0}^{n}(1/38)^i L_{9i} = F_{9(n+1)}/38^n.$$

We can also produce this next collection of formulas with the Fibonacci and Lucas numbers (which, again, we believe are new):

$$\begin{aligned} F_{2n+2} &= 1\cdot\sum_{i=0}^{n} 1^{n-i}F_{2i+1} = 1\cdot\sum_{i=0}^{n}(-1)^{n-i}L_{2i+1},\\ F_{3n+3} &= 2\cdot\sum_{i=0}^{n} 1^{n-i}F_{3i+1} = 2\cdot\sum_{i=0}^{n}(-3)^{n-i}L_{3i+1},\\ F_{4n+4} &= 3\cdot\sum_{i=0}^{n} 2^{n-i}F_{4i+1} = 3\cdot\sum_{i=0}^{n}(-4)^{n-i}L_{4i+1}. \end{aligned} \tag{10}$$

Moving on to the Pell numbers (which are found at A000129 in the On-Line Encyclopedia of Integer Sequences (OEIS) [22]), we have

$$\begin{aligned} P_{n+1} &= \frac{1}{2}\sum_{i=0}^{n}(-1/2)^{n-i}P_{i+2}\\ &= \frac{1}{5}\sum_{i=0}^{n}(-2/5)^{n-i}P_{i+3}\\ &= \frac{1}{12}\sum_{i=0}^{n}(-5/12)^{n-i}P_{i+4}. \end{aligned} \tag{11}$$

We compare these to equation (23) in Abd-Elhameed and Zeyada's recent paper [1], which gives us

$$(1/3)^{n+1}Q_{n+1} = 1 - \frac{2}{3}\sum_{i=0}^{n}(1/3)^i P_{i-1}, \tag{12}$$

where Q_n represents the nth Pell-Lucas number A001333 as defined later in equation (66).

Finally, if we think of the Pell numbers (with initial values $P_0 = 0$ and $P_1 = 1$ and recurrence $P_n = 2P_{n-1}+P_{n-2}$) as the "silver Fibonacci numbers", then the "bronze Fibonacci numbers" would be the sequence A006190 with same initial values but with recurrence

$B_n = 3B_{n-1} + B_{n-2}$. Here are some interesting (and mostly new) formulas for these bronze Fibonacci numbers, which remind us of the similar identities in equation (10):

$$
\begin{aligned}
B_{2n+2} &= 3 \cdot \sum_{i=0}^{n} 1^{n-i} B_{2i+1}, \\
B_{3n+3} &= 10 \cdot \sum_{i=0}^{n} 3^{n-i} B_{3i+1}, \\
B_{4n+4} &= 33 \cdot \sum_{i=0}^{n} 10^{n-i} B_{4i+1}.
\end{aligned} \tag{13}
$$

There are countless variations on these formulas. Again, what sets our approach apart from all the previous work is that our weighted sums can start at an arbitrary position in our sequence, and our weighted sums avoid the complicated summands as seen in equations (5) and (8). Our proofs rely mostly on induction, along with generalized versions of Cassini's and d'Ocagne's identities.

2 Main results

We begin with a theorem associated with a very particular type of sequence.

Theorem 1. *Let $(A_n)_{n\geq 0}$ be a second-order linear recurrence sequence with initial term $A_0 = 1$ and with recurrence*

$$A_n = c_1 A_{n-1} + c_2 A_{n-2}.$$

If we set $t = (c_1 - A_1)$ then so long as $t \neq 0$, we have

$$A_{n+2} - A_1 A_{n+1} = (A_2 - A_1^2) \sum_{i=0}^{n} t^{n-i} A_i. \tag{14}$$

Proof. We proceed by induction. For the base case $n = 0$, we find that equation (14) simplifies to

$$A_2 - A_1 A_1 = (A_2 - A_1^2) \cdot A_0,$$

and since $A_0 = 1$ we are done.

Next, we assume the identity holds for some fixed n, and we try to prove it for $n + 1$. If we let R represent the right-hand side of equation (14) with $n + 1$ in place of n, we have

$$R = (A_2 - A_1^2) \sum_{i=0}^{n+1} t^{n+1-i} A_i,$$

and if we separate out that last term in the summation and then factor out t then we have

$$R = (A_2 - A_1^2) \left(A_{n+1} + t \sum_{i=0}^{n} t^{n-i} A_i \right).$$

We multiply through to obtain

$$R = (A_2 - A_1^2)A_{n+1} + t \cdot (A_2 - A_1^2) \sum_{i=0}^{n} t^{n-i} A_i,$$

and we use our induction hypothesis on that last term to give us

$$R = (A_2 - A_1^2)A_{n+1} + t \cdot (A_{n+2} - A_1 A_{n+1}).$$

Recalling that $t = c_1 - A_1$, the above expression becomes

$$R = (A_2 - A_1^2)A_{n+1} + (c_1 - A_1) \cdot (A_{n+2} - A_1 A_{n+1}).$$

We see that the $A_1^2 A_{n+1}$ terms cancel, and so we now have

$$R = A_2 A_{n+1} + (c_1 - A_1) \cdot A_{n+2} - c_1 A_1 A_{n+1}. \tag{15}$$

When we replace that leading A_2 term with

$$A_2 = c_1 A_1 + c_2 A_0 = c_1 A_1 + c_2 1 = c_1 A_1 + c_2,$$

then equation (15) becomes

$$R = (c_1 A_1 + c_2)A_{n+1} + (c_1 - A_1) \cdot A_{n+2} - c_1 A_1 A_{n+1}.$$

Further simplification reduces this to

$$R = c_2 A_{n+1} + (c_1 - A_1) \cdot A_{n+2}.$$

Since $c_1 A_{n+2} + c_2 A_{n+1}$ is equal to A_{n+3}, then we have

$$R = A_{n+3} - A_1 A_{n+2},$$

and this is exactly equal to the left-hand side of equation (14) with $n+1$ in place of n, thus completing our induction step. □

With all that taken care of, we are now ready for our next theorem, which covers a much more general collection of sequences.

Theorem 2. *For X_n a second-order recurrence sequence with initial values X_0 and X_1 and with recurrence*

$$X_n = c_1 X_{n-1} + c_2 X_{n-2}, \tag{16}$$

then for any fixed integer k such that X_k and X_{k-1} are both non-zero, we have

$$X_0 X_{n+2} - X_1 X_{n+1} = \frac{X_0 X_2 - X_1^2}{X_k} \sum_{i=0}^{n} \left(-c_2 X_{k-1}/X_k\right)^{n-i} X_{i+k}. \tag{17}$$

We note that we allow k to be negative. Given that we have fixed initial values X_0 and X_1 for our sequence, then we can always define X_{-1} and X_{-2} and so on by the identity

$$X_{n-2} = \frac{1}{c_2}\left(X_n - c_1 X_{n-1}\right),$$

which follows from the definition of our sequence in equation (16). Our requirement of a *second-order* recurrence sequence tells us that $c_2 \neq 0$, and thus X_{-1} and X_{-2} (and so on) are well-defined.

Proof of Theorem 2. For k fixed, we begin by setting

$$A_n = \frac{X_{n+k}}{X_k}. \tag{18}$$

This gives us the new sequence $(A_n)_{n\geq 0}$ with initial term $A_0 = 1$ but with the same recurrence as the sequence $(X_n)_{n\geq 0}$ in the statement of the theorem. Next, for convenience we will set

$$t = c_1 - A_1,$$

which means that

$$t = c_1 - X_{k+1}/X_k = (c_1 X_k - X_{k+1})/X_k = -c_2 X_{k-1}/X_k,$$

and since X_{k-1} is also assumed to be non-zero then $t \neq 0$. This means we can now apply Theorem 1 to the sequence $(A_n)_{n\geq 0}$, giving us

$$A_{n+2} - A_1 A_{n+1} = (A_2 - A_1^2)\sum_{i=0}^{n} t^{n-i} A_i, \tag{19}$$

and it remains to show that we can transform equation (19) into equation (17) in the statement of the theorem.

The first step, of course, is to use equation (18) to re-write equation (19) as

$$\frac{X_{n+k+2}}{X_k} - \frac{X_{k+1}}{X_k}\frac{X_{n+k+1}}{X_k} = \left(\frac{X_{k+2}}{X_k} - \frac{X_{k+1}^2}{X_k^2}\right)\sum_{i=0}^{n} t^{n-i}\frac{X_{k+i}}{X_k}, \tag{20}$$

and if we multiply through by X_k^2 we arrive at

$$X_{n+k+2}X_k - X_{k+1}X_{n+k+1} = \frac{X_{k+2}X_k - X_{k+1}^2}{X_k}\sum_{i=0}^{n} t^{n-i} X_{k+i}. \tag{21}$$

This is almost, but not quite, a match for equation (17). On the left of equation (21) we will show that

$$X_{n+k+2}X_k - X_{k+1}X_{n+k+1} = (-c_2)^k (X_{n+2}X_0 - X_{n+1}X_1), \tag{22}$$

and on the right of equation equation (21) we will show that

$$X_{k+2}X_k - X_{k+1}^2 = (-c_2)^k(X_2X_0 - X_1^2). \tag{23}$$

After canceling $(-c_2)^k$ from both sides, we will obtain a perfect match for equation (17).

So, it remains to establish the identities (22) and (23) for all integer values of k. Equation (22) is just a generalization of d'Ocagne's identity, and it follows immediately from taking the determinant of both sides of the matrix equation

$$\begin{pmatrix} X_{n+k+2} & X_{k+1} \\ X_{n+k+1} & X_k \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ 1 & 0 \end{pmatrix}^k \cdot \begin{pmatrix} X_{n+2} & X_1 \\ X_{n+1} & X_0 \end{pmatrix},$$

which holds for k positive, negative, or zero.

As for equation (23), this is a generalization of Cassini's identity, and we will prove it by induction (twice), once for $k > 0$ and again for $k < 0$. First, take k to be positive. We note that equation (23) is certainly true for $k = 0$, and if we assume it to be true for some fixed k, then at $k + 1$ we can write

$$X_{k+3}X_{k+1} - X_{k+2}^2 = (c_1X_{k+2} + c_2X_{k+1})\, X_{k+1} - X_{k+2}^2,$$

and we re-arrange the right-hand side to get

$$X_{k+3}X_{k+1} - X_{k+2}^2 = c_2X_{k+1}^2 + X_{k+2}\,(c_1X_{k+1} - X_{k+2})\,.$$

Since $X_{k+2} = c_1X_{k+1} + c_2X_k$,then the above equation becomes

$$X_{k+3}X_{k+1} - X_{k+2}^2 = c_2X_{k+1}^2 - c_2X_{k+2}X_k = (-c_2)\left(X_{k+2}X_k - X_{k+1}^2\right),$$

and now our induction hypothesis comes into play for that last expression on the right to give us

$$X_{k+3}X_{k+1} - X_{k+2}^2 = (-c_2)(-c_2)^k(X_2X_0 - X_1^2) = (-c_2)^{k+1}(X_2X_0 - X_1^2),$$

thus establishing equation (23) for k positive.

Finally, take k to be negative. We again note that equation (23) is certainly true for $k = 0$, and if we assume it to be true for some fixed k, then at $k - 1$ we can write

$$X_{k+1}X_{k-1} - X_k^2 = X_{k+1}\left(\frac{1}{c_2}X_{k+1} - \frac{c_1}{c_2}X_k\right) - X_k^2,$$

and we re-arrange the right-hand side to get

$$X_{k+1}X_{k-1} - X_k^2 = \frac{1}{c_2}X_{k+1}^2 - \frac{X_k}{c_2}(c_1X_{k+1} + c_2X_k).$$

Since $X_{k+2} = c_1 X_{k+1} + c_2 X_k$,then the above equation becomes

$$X_{k+1}X_{k-1} - X_k^2 = \frac{1}{c_2}X_{k+1}^2 - \frac{X_k}{c_2}(X_{k+2}) = (-1/c_2)\left(X_{k+2}X_k - X_{k+1}^2\right),$$

and now our induction hypothesis comes into play for that last expression on the right to give us

$$X_{k+1}X_{k-1} - X_k^2 = (-1/c_2)(-c_2)^k(X_2X_0 - X_1^2) = (-c_2)^{k-1}(X_2X_0 - X_1^2),$$

thus establishing equation (23) for k negative. □

3 Examples

We now show how to use Theorems 1 and 2 to produce many of the identities from the Introduction.

Here is an overview of our approach: we begin with a particular sequence (say, the Lucas numbers or the Pell numbers), and then we try out different variations and offsets so that we can apply one of our two theorems. Once we have obtained a generic summation identity from our theorem, we can then attempt various simplifications in order to arrive at an aesthetically-pleasing formula. For example, we might start with the Lucas numbers L_n, but then decide to apply our theorems with either L_{n-1} (giving us Sury's identity in section 3.1 below) or L_n (as seen in section 3.2, with various offsets) or L_{jn} (as done in Theorem 4) or even L_{jn+k}/L_k (which we use to give us equation (54) in Theorem 5).

3.1 Sury's identity

To prove that

$$2^{n+1}F_{n+1} = \sum_{i=0}^{n} 2^i L_i,$$

we will use Theorem 2 with $X_n = L_{n-1}$, which gives us $X_0 = -1$, $X_1 = 2$, and $c_1 = c_2 = 1$. Taking $k = 1$ in Theorem 2, we get

$$-L_{n+1} - 2L_n = \frac{-1-2^2}{2}\sum_{i=0}^{n}(1/2)^{n-i} L_i. \tag{24}$$

After cancelling the negatives, and re-writing $L_{n+1} + 2L_n = L_{n+2} + L_n = 5F_{n+1}$, we have

$$5F_{n+1} = \frac{5}{2}\sum_{i=0}^{n}(1/2)^{n-i} L_i. \tag{25}$$

Finally, we cancel the 5's and we multiply both sides by 2^{n+1}, to give us our desired formula,

$$2^{n+1}F_{n+1} = \sum_{i=0}^{n} 2^i L_i.$$

3.2 Martinjak's convolution and its generalizations

From equation (3) in the introduction we have

$$F_{n+1} = \sum_{i=0}^{n} (-2)^{n-i} L_{i+1},$$

which is part of the family of identities

$$\begin{aligned} F_{n+1} = \sum_{i=0}^{n} (-2)^{n-i} L_{i+1} &= \frac{1}{3} \sum_{i=0}^{n} (-1/3)^{n-i} L_{i+2} \\ &= \frac{1}{4} \sum_{i=0}^{n} (-3/4)^{n-i} L_{i+3} \\ &= \frac{1}{7} \sum_{i=0}^{n} (-4/7)^{n-i} L_{i+4} \end{aligned}$$

from the introduction. The general form is

$$F_{n+1} = \frac{1}{L_k} \sum_{i=0}^{n} (-L_{k-1}/L_k)^{n-i} L_{i+k}, \tag{26}$$

and to produce all of these from Theorem 2 we take $X_n = L_n$ so that we have $X_0 = 2$, $X_1 = 1$, and $c_1 = c_2 = 1$. As a result, equation (17) in Theorem 2 becomes

$$2L_{n+2} - L_{n+1} = \frac{2 \cdot 3 - 1^2}{L_k} \sum_{i=0}^{n} \left(-L_{k-1}/L_k\right)^{n-i} L_{i+k}. \tag{27}$$

Since $2L_{n+2} - L_{n+1} = 5F_{n+1}$, we easily produce equation (26).

3.3 Fibonacci and Lucas j-step numbers

We define the "j-step numbers" of a sequence to be the new sequence made up of every jth term in the original sequence. For example, when we talk about the 3-step Fibonacci numbers then we could be referring to $(F_{3n})_{n\geq 0}$ or perhaps $(F_{3n+1})_{n\geq 0}$ or even $(F_{3n+2})_{n\geq 0}$, depending on our choice of offset. We assume throughout this section that $j \geq 1$, and we note that if $j = 1$ then we get the "regular" Fibonacci (or Lucas) numbers without any skipping.

3.3.1 Fibonacci numbers with zero offset

For our first case, we will consider the j-step Fibonacci numbers such as F_{3n} for $j = 3$ or F_{4n} for $j = 4$, with no offset. Here is our result.

Theorem 3. *For $j \geq 1$ an integer, the j-step Fibonacci numbers satisfy*

$$L_j^{n+1} F_{j(n-1)} + (-1)^{j(n+1)} F_{2j} = \sum_{i=0}^{n} (-1)^{j(n+i)} L_j^i F_{ji}. \tag{28}$$

Proof. We begin with the well-known formula [13, p. 112]

$$F_{a+b} = L_b F_a + (-1)^{b+1} F_{a-b}, \tag{29}$$

and if we replace a with $j(n-1)$ and b with j, we get

$$F_{jn} = L_j F_{j(n-1)} + (-1)^{j+1} F_{j(n-2)}. \tag{30}$$

For $j \geq 1$ fixed, we will use Theorem 2 with $X_n = F_{jn}$, and since Theorem 2 requires both X_k and X_{k-1} to be non-zero, we will set $k = 2$. Thanks to equation (30), we have $c_1 = L_j$ and $c_2 = (-1)^{j+1}$. Since $X_0 = 0$ and $X_1 = F_j$, then the expression on the left-hand side of equation (17) of Theorem 2 is

$$X_0 X_{n+2} - X_1 X_{n+1} = 0 - F_j F_{j(n+1)}, \tag{31}$$

and the expression in front of the sum on the right-hand side of equation (17) is

$$\frac{X_0 X_2 - X_1^2}{X_2} = \frac{0 - F_j^2}{F_{2j}} = \frac{-F_j}{L_j}, \tag{32}$$

where we have used the well-known identity $F_{2j} = F_j L_j$. Likewise, the expression inside the sum in equation (17) is

$$(-c_2 X_1 / X_2)^{n-i} X_{i+2} = ((-1)^j F_j / F_{2j})^{n-i} F_{j(i+2)} = ((-1)^j / L_j)^{n-i} F_{j(i+2)}. \tag{33}$$

When we substitute equations (31), (32), and (33) into equation (17), and cancel the $-F_j$ term from each side, we get

$$F_{j(n+1)} = \frac{1}{L_j} \sum_{i=0}^{n} ((-1)^j / L_j)^{n-i} F_{j(i+2)}.$$

This equation actually looks a bit nicer if we multiply both sides by $(L_j)^{n+1}$ and simplify, giving us

$$L_j^{n+1} F_{j(n+1)} = \sum_{i=0}^{n} (-1)^{j(n+i)} L_j^i F_{j(i+2)}. \tag{34}$$

Further simplification is needed. First, we replace i with $i-2$ in the right-hand summation, giving us

$$L_j^{n+1} F_{j(n+1)} = \sum_{i=2}^{n+2} (-1)^{j(n+i)} L_j^{i-2} F_{ji}. \tag{35}$$

Then, we multiply both sides by L_j^2 to give us

$$L_j^{n+3}F_{j(n+1)} = \sum_{i=2}^{n+2}(-1)^{j(n+i)}L_j^iF_{ji}. \tag{36}$$

Next, we add in the "missing term" for $i = 1$ to both sides, giving us

$$L_j^{n+3}F_{j(n+1)} + (-1)^{j(n+1)}L_jF_j = \sum_{i=1}^{n+2}(-1)^{j(n+i)}L_j^iF_{ji}. \tag{37}$$

Finally, we replace n with $n-2$, we replace L_jF_j with F_{2j}, and we start the summation at $i = 0$ instead of at $i = 1$ to give us

$$L_j^{n+1}F_{j(n-1)} + (-1)^{j(n+1)}F_{2j} = \sum_{i=0}^{n}(-1)^{j(n+i)}L_j^iF_{ji}, \tag{38}$$

which is equation (28) as desired. $\square$

Here are the results for small values of j:

$$\begin{aligned}
&(\text{at } j=2): \qquad 3^{n+1}F_{2(n-1)} + 3 = \sum_{i=1}^{n} 3^iF_{2i},\\
&(\text{at } j=3): \qquad 4^{n+1}F_{3(n-1)} + 8(-1)^{n+1} = (-1)^n\sum_{i=1}^{n}(-4)^iF_{3i},\\
&(\text{at } j=4): \qquad 7^{n+1}F_{4(n-1)} + 21 = \sum_{i=1}^{n} 7^iF_{4i},
\end{aligned}$$

and it is interesting to compare them to this similar collection of formulas from Dresden and Tulskikh [11, p. 2],

$$\begin{aligned}
3 - F_{2(n+2)}/3^n &= \sum_{i=0}^{n}(1/3)^iF_{2i},\\
F_{3(n+2)}/4^n - 8 &= \sum_{i=0}^{n}(1/4)^iF_{3i},\\
21 - F_{4(n+2)}/7^n &= \sum_{i=0}^{n}(1/7)^iF_{4i}.
\end{aligned}$$

3.3.2 Lucas numbers with zero offset

For the j-step Lucas numbers such as L_{2n} for $j = 2$ or L_{3n} for $j = 3$, we will again use Theorem 2 but this time with $X_n = L_{jn}$ and with $k = 1$. Here is our result.

Theorem 4. *For $j \geq 1$ an integer, the j-step Lucas numbers satisfy*

$$2(L_j/2)^{n+1}(F_{jn}/F_j) + 2(-1)^{jn} = \sum_{i=0}^{n} (-1)^{j(n+i)}(L_j/2)^i L_{ji}, \tag{39}$$

Proof. To begin, we will need the formula

$$L_{a+b} = L_b L_a + (-1)^{b+1} L_{a-b}, \tag{40}$$

which follows from adding together two copies of equation (29), the first with $a+1$ in place of a, and the second with $a-1$ in place of a. If we replace a with $j(n-1)$ and b with j in equation (40), we get

$$L_{jn} = L_j L_{j(n-1)} + (-1)^{j+1} L_{j(n-2)}. \tag{41}$$

We set $X_n = L_{jn}$, which means that $X_0 = 2$ and $X_1 = L_j$. Thanks to equation (41) we have $c_1 = L_j$ and $c_2 = (-1)^{j+1}$.

We now turn our attention to the expression on the left-hand side of equation (17) of Theorem 2, which is

$$X_0 X_{n+2} - X_1 X_{n+1} = 2L_{j(n+2)} - L_j L_{j(n+1)}. \tag{42}$$

From equation (41) with $n+2$ in place of n we have

$$L_{j(n+2)} = L_j L_{j(n+1)} + (-1)^{j+1} L_{jn},$$

and from Exercise 55 in [14, p. 118] we have

$$L_{j(n+2)} = 5F_j F_{j(n+1)} - (-1)^{j+1} L_{jn},$$

and so if we add these together we will get

$$2L_{j(n+2)} = L_j L_{j(n+1)} + 5F_j F_{j(n+1)}.$$

When we substitute this into the right-hand side of equation (42) we get

$$X_0 X_{n+2} - X_1 X_{n+1} = 2L_{j(n+2)} - L_j L_{j(n+1)} = 5F_j F_{j(n+1)}. \tag{43}$$

Likewise, the expression in front of the sum on the right-hand side of equation (17) is

$$\frac{X_0 X_2 - X_1^2}{X_1} = \frac{2L_{2j} - L_j^2}{L_j} = \frac{5F_j^2}{L_j}, \tag{44}$$

which follows from equation (43) with $n = 0$.

Now, since we are taking $k = 1$ in this section, then the expression inside the sum in equation (17) is

$$(-c_2 X_0/X_1)^{n-i} X_{i+1} = ((-1)^j \cdot 2/L_j)^{n-i} L_{j(i+1)}. \tag{45}$$

When we substitute equations (43), (44), and (45) into equation (17), and cancel the $5F_j$ term from each side, we get

$$F_{j(n+1)} = \frac{F_j}{L_j} \sum_{i=0}^{n} (2(-1)^j/L_j)^{n-i} L_{j(i+1)}.$$

This equation actually looks a bit nicer if we multiply both sides by $(L_j/2)^{n+1}$ and simplify, giving us

$$(L_j/2)^{n+1} F_{j(n+1)} = \frac{F_j}{2} \sum_{i=0}^{n} (-1)^{j(n+i)} (L_j/2)^i L_{j(i+1)}. \tag{46}$$

We will need to apply some further simplifications. First, we multiply both sides by $2/F_j$ and then replace i with $i-1$ in the right-hand summation, giving us

$$2(L_j/2)^{n+1}(F_{j(n+1)}/F_j) = \sum_{i=1}^{n+1} (-1)^{j(n+i-1)} (L_j/2)^{i-1} L_{ji}. \tag{47}$$

We multiply both sides by $(L_j/2)$ to produce

$$2(L_j/2)^{n+2}(F_{j(n+1)}/F_j) = \sum_{i=1}^{n+1} (-1)^{j(n+i-1)} (L_j/2)^{i} L_{ji}. \tag{48}$$

Next, we add in the "missing term" for $i = 0$ to both sides, giving us

$$2(L_j/2)^{n+2}(F_{j(n+1)}/F_j) + 2(-1)^{j(n+1)} = \sum_{i=0}^{n+1} (-1)^{j(n+i-1)} (L_j/2)^{i} L_{ji}. \tag{49}$$

Finally, we replace n with $n-1$ to get

$$2(L_j/2)^{n+1}(F_{jn}/F_j) + 2(-1)^{jn} = \sum_{i=0}^{n} (-1)^{j(n+i)} (L_j/2)^{i} L_{ji}, \tag{50}$$

and this is equation (39), as desired. $\square$

We note that equation (39) looks rather pleasant when L_j is even; this happens when j is a multiple of 3. For our first such case, we substitute $j = 3$ into equation (39) and multiply both sides by $(-1)^n$ and simplify to get

$$2 - (-2)^{n+1} F_{3n} = \sum_{i=0}^{n} (-2)^i L_{3i}. \tag{51}$$

For the second, we use $j = 6$ and multiply both sides by 4 and simplify to get

$$9^{n+1} F_{6n} + 8 = 4 \sum_{i=0}^{n} 9^i L_{6i}. \tag{52}$$

These are the first two identities from equation (9) in the introduction, and they are quite nice and quite unexpected. (A similar procedure for $j = 9$ or $j = 12$ will give us additional equations.)

3.3.3 Non-zero offsets

The formulas with offsets -1, $+1$, and $+2$, when written down together, seem remarkably similar:

$$\text{(offset } -1\text{):} \qquad F_{j(n+1)} = F_j \sum_{i=0}^{n} (F_{j+1})^{n-i} F_{ji-1} \qquad \text{for } j > 1,$$

$$\text{(offset } +1\text{):} \qquad F_{j(n+1)} = F_j \sum_{i=0}^{n} (F_{j-1})^{n-i} F_{ji+1} \qquad \text{for } j > 1,$$

$$\text{(offset } +2\text{):} \qquad F_{j(n+1)} = F_j \sum_{i=0}^{n} (-F_{j-2})^{n-i} F_{ji+2} \qquad \text{for } j > 2.$$

Here are two additional offset formulas, and there does indeed seem to be a pattern:

$$\text{(offset } +3\text{):} \qquad F_{j(n+1)} = \frac{F_j}{2} \sum_{i=0}^{n} (F_{j-3}/2)^{n-i} F_{ji+3} \qquad \text{for } j > 3,$$

$$\text{(offset } +4\text{):} \qquad F_{j(n+1)} = \frac{F_j}{3} \sum_{i=0}^{n} (-F_{j-4}/3)^{n-i} F_{ji+4} \qquad \text{for } j > 4.$$

Here is the general form:

Theorem 5. *For $j > |k|$ integers, the j-step Fibonacci numbers with offset k satisfy*

$$F_{j(n+1)} = \frac{F_j}{F_k} \sum_{i=0}^{n} ((-1)^{k+1} F_{j-k}/F_k)^{n-i} F_{ji+k} \qquad \textit{for } j > |k| > 0, \tag{53}$$

and for the Lucas numbers we have

$$F_{j(n+1)} = \frac{F_j}{L_k} \sum_{i=0}^{n} ((-1)^{k} L_{j-k}/L_k)^{n-i} L_{ji+k} \qquad \textit{for } j > |k|. \tag{54}$$

We note that if we set $k = 0$ in equation (54) then we obtain Theorem 2.1 from Adegoke and Frontczak's recent paper [2], and if we set $k = 1$ in equations (53) and (54) then we obtain the collection of identities in equation (10).

Proof. To establish equation (53), we will use Theorem 1 with $A_n = F_{jn+k}/F_k$. Thus, we have $A_0 = 1$, $A_1 = F_{j+k}/F_k$, and since equation (29) tells us that

$$F_{jn+k} = L_j F_{j(n-1)+k} + (-1)^{j+1} F_{j(n-2)+k},$$

then we have

$$A_n = L_j A_{n-1} + (-1)^{j+1} A_{n-2}.$$

Thus, for the conditions in the statement of Theorem 1 we have $c_1 = L_j$ and so $t = c_1 - A_1 = L_j - F_{j+k}/F_k$, which gives us

$$t = L_j - F_{j+k}/F_k = \frac{L_j F_k - F_{j+k}}{F_k},$$

and thanks again to equation (29) which tells us that $F_{k+j} = L_j F_k + (-1)^{j+1} F_{k-j}$, this gives us

$$t = (-1)^j F_{k-j}/F_k.$$

Since $k - j < 0$, then we use the well-known fact that $F_{-n} = (-1)^{n+1} F_n$, giving us

$$t = (-1)^j (-1)^{k-j+1} F_{j-k}/F_k = (-1)^{k+1} F_{j-k}/F_k. \tag{55}$$

We are almost ready to use equation (14) from Theorem 1, but we still need to simplify the two expressions

$$A_{n+2} - A_1 A_{n+1} \qquad \text{and} \qquad A_2 - A_1^2$$

that we find in equation (14). From our definition of A_n as F_{jn+k}/F_k, these two expressions are

$$\frac{F_{j(n+2)+k} F_k - F_{j(n+1)+k} F_{j+k}}{F_k^2} \qquad \text{and} \qquad \frac{F_{2j+k} F_k - F_{j+k}^2}{F_k^2} \tag{56}$$

Here we will need to call upon a generalization of Catalan's Identity [13, p. 108] which tells us that

$$F_{a+c} F_{b-c} - F_a F_b = (-1)^{b+c+1} F_{a+c-b} F_c. \tag{57}$$

For the first expression in equation (56), we use equation (57) with $a = j(n+1)+k$, $b = j+k$ and $c = j$ to give us

$$A_{n+2} - A_1 A_{n+1} = \frac{F_{j(n+2)+k} F_k - F_{j(n+1)+k} F_{j+k}}{F_k^2} = \frac{(-1)^{k+1} F_{j(n+1)} F_j}{F_k^2}, \tag{58}$$

For the second expression in equation (56), we simply take $n = 0$ in equation (58) to give us

$$A_2 - A_1^2 = \frac{F_{2j+k} F_k - F_{j+k}^2}{F_k^2} = \frac{(-1)^{k+1} F_j^2}{F_k^2}. \tag{59}$$

We can now use Theorem 1 with $A_n = F_{jn+k}/F_k$, and using our expressions in equations (55), (58), and (59) in equation (14) we get

$$\frac{(-1)^{k+1} F_{j(n+1)} F_j}{F_k^2} = \frac{(-1)^{k+1} F_j^2}{F_k^2} \sum_{i=0}^{n} \left((-1)^{k+1} F_{j-k}/F_k\right)^{n-i} \frac{F_{ji+k}}{F_k}.$$

With a bit of cancellation, this simplifies nicely to give us

$$F_{j(n+1)} = \frac{F_j}{F_k} \sum_{i=0}^{n} \left((-1)^{k+1} F_{j-k}/F_k\right)^{n-i} F_{ji+k},$$

which is equation (53) as desired.

To establish equation (54), we follow most of the same steps, but there are a few interesting new wrinkles. Proceeding as before, we will use Theorem 1 with $A_n = L_{jn+k}/L_k$. Thus, we have $A_0 = 1$, $A_1 = L_{j+k}/L_k$, and since equation (41) tells us that

$$L_{jn} = L_j L_{j(n-1)} + (-1)^{j+1} L_{j(n-2)},$$

then we have

$$A_n = L_j A_{n-1} + (-1)^{j+1} A_{n-2}.$$

Thus, for the conditions in the statement of Theorem 1 we have $c_1 = L_j$ and so $t = c_1 - A_1 = L_j - L_{j+k}/L_k$, which gives us

$$t = L_j - L_{j+k}/L_k = \frac{L_j L_k - L_{j+k}}{L_k},$$

and thanks to equation (40) which tells us that $L_{k+j} = L_j L_k + (-1)^{j+1} L_{k-j}$, this gives us

$$t = (-1)^j L_{k-j}/L_k.$$

Since $k - j < 0$, then we use the well-known fact that $L_{-n} = (-1)^n L_n$, giving us

$$t = (-1)^j (-1)^{k-j} L_{j-k}/L_k = (-1)^k L_{j-k}/L_k. \tag{60}$$

We are almost ready to use equation (14) from Theorem 1, but we still need to simplify the two expressions

$$A_{n+2} - A_1 A_{n+1} \qquad \text{and} \qquad A_2 - A_1^2$$

that we find in equation (14). From our definition of A_n as L_{jn+k}/L_k, these two expressions are

$$\frac{L_{j(n+2)+k} L_k - L_{j(n+1)+k} L_{j+k}}{L_k^2} \qquad \text{and} \qquad \frac{L_{2j+k} L_k - L_{j+k}^2}{L_k^2} \tag{61}$$

Here we will need to produce a variant of Catalan's Identity but with Lucas numbers instead of Fibonacci numbers. If we take two versions of equation (57) with $a+1$ and $a-1$ in place of a, we have

$$F_{a+1+c} F_{b-c} - F_{a+1} F_b = (-1)^{b+c+1} F_{a+1+c-b} F_c,$$
$$F_{a-1+c} F_{b-c} - F_{a-1} F_b = (-1)^{b+c+1} F_{a-1+c-b} F_c,$$

and if we add these together we find that

$$L_{a+c} F_{b-c} - L_a F_b = (-1)^{b+c+1} L_{a+c-b} F_c. \tag{62}$$

We now repeat the process, but this time with equation (62) and this time with $b+1$ and $b-1$ in place of b, giving us

$$L_{a+c} F_{b+1-c} - L_a F_{b+1} = (-1)^{b+c} L_{a+c-b-1} F_c,$$
$$L_{a+c} F_{b-1-c} - L_a F_{b-1} = (-1)^{b+c} L_{a+c-b+1} F_c,$$

and if we add these together we find that

$$L_{a+c}L_{b-c} - L_aL_b = 5(-1)^{b+c}F_{a+c-b}F_c. \tag{63}$$

This is not quite as beautiful as the Fibonacci version in equation (57), but it will certainly meet our needs.

If we turn now to the first expression in equation (61), we use equation (63) with $a = j(n+1)+k$, $b = j+k$ and $c = j$ to give us

$$A_{n+2} - A_1A_{n+1} = \frac{L_{j(n+2)+k}L_k - L_{j(n+1)+k}L_{j+k}}{L_k^2} = \frac{5(-1)^kF_{j(n+1)}F_j}{L_k^2}, \tag{64}$$

For the second expression in equation (61), we simply take $n = 0$ in equation (64) to give us

$$A_2 - A_1^2 = \frac{L_{2j+k}L_k - L_{j+k}^2}{L_k^2} = \frac{5(-1)^kF_j^2}{L_k^2}. \tag{65}$$

We can now use Theorem 1 with $A_n = L_{jn+k}/L_k$, and using our expressions in equations (60), (64), and (65) in equation (14) we get

$$\frac{5(-1)^kF_{j(n+1)}F_j}{L_k^2} = \frac{5(-1)^kF_j^2}{L_k^2}\sum_{i=0}^{n}\left((-1)^kL_{j-k}/L_k\right)^{n-i}\frac{L_{ji+k}}{L_k}.$$

With a bit of cancellation, this simplifies nicely to give us

$$F_{j(n+1)} = \frac{F_j}{L_k}\sum_{i=0}^{n}\left((-1)^kL_{j-k}/L_k\right)^{n-i}L_{ji+k},$$

which is equation (54) as desired. □

3.4 Pell and Pell-Lucas numbers

We recall [14, p. 23] that the Pell and Pell-Lucas numbers P_n and Q_n are defined as

$$\begin{aligned} P_n &= 2P_{n-1} + P_{n-2} \qquad \text{with } \; P_0 = 0,\; P_1 = 1, \\ Q_n &= 2Q_{n-1} + Q_{n-2} \qquad \text{with } \; Q_0 = 1,\; Q_1 = 1. \end{aligned} \tag{66}$$

These are the sequences A000129 and A001333, respectively. If we apply Theorem 2 with $X_n = P_n$ and with $k \neq 0, 1$, then since $P_0P_{n+2} - P_1P_{n+1} = -P_{n+1}$ and $P_0P_2 - P_1^2 = -1$, then equation (17) gives us

$$P_{n+1} = \frac{1}{P_k}\sum_{i=0}^{n}(-P_{k-1}/P_k)^{n-i}P_{i+k}.$$

If we take $k = 2, 3$, and 4, we will produce the identities in equation (11). However, it is interesting to note that with $k = -1$ we produce the equation

$$P_{n+1} = \sum_{i=0}^{n} 2^{n-i} P_{i-1}, \tag{67}$$

which is quite different from equation (12). However, it is equivalent to the almost-identical formula in [11, page 3],

$$P_{n+2} = 2^{n+1} + \sum_{i=0}^{n} 2^{n-i} P_i.$$

To see this, we multiply both sides of equation (67) by 2, we replace $i - 1$ with i, we pull out the nth term, we replace $2P_{n+1}$ with $P_{n+2} - P_n$, and we absorb P_n into the sum.

3.5 The bronze Fibonacci numbers

As mentioned in the introduction, we define the "bronze Fibonacci numbers" as $B_0 = 0$, $B_1 = 1$, and

$$B_n = 3B_{n-1} + B_{n-2}.$$

This gives us the sequence A006190. To prove the first identity in equation (13), we will use Theorem 1 with $A_n = B_{2n+1}$, and so $A_0 = B_1 = 1$ and $A_1 = B_3 = 10$. Starting with

$$B_n = 3B_{n-1} + B_{n-2},$$

then it is easy to show that

$$B_{2n+1} = 11B_{2n-1} - B_{2n-3}$$

which means that

$$A_n = 11A_{n-1} - A_{n-2}.$$

So, using Theorem 1 with $t = c_1 - A_1 = 11 - 10 = 1$, we have

$$A_{n+2} - 10A_{n+1} = (A_2 - A_1^2) \sum_{i=0}^{n} 1^{n-1} A_i. \tag{68}$$

Now, $A_{n+2} - 10A_{n+1}$ simplifies as follows:

$$\begin{aligned} A_{n+2} - 10A_{n+1} &= (11A_{n+1} - A_n) - 10A_{n+1} \\ &= A_{n+1} - A_n \\ &= B_{2n+3} - B_{2n+1} \\ &= 3B_{2n+2}. \end{aligned}$$

Furthermore, $A_2 - A_1^2$ is equal to $B_5 - B_3^2$, and since $B_3 = 10$ and $B_5 = 109$, then putting everything together we see that equation (68) becomes

$$3B_{2n+2} = 9\sum_{i=0}^{n} 1^{n-1} B_{2i+1}, \tag{69}$$

and this simplifies nicely to give us the first identity in equation (13). The other two identities have similar proofs.

4 Conclusion

As we mentioned in the introduction, there are countless other identities that we can prove: we simply select a sequence, choose a starting value, and then apply our theorem. To take a random example, if we let a_n be the sequence with initial values $a_0 = 0$ and $a_1 = 1$ and with recurrence $a_n = 4a_{n-1} + 3a_{n-2}$, then we can use Theorem 2 to quickly produce the following identities for this sequence (which is sequence A015530 in the OEIS):

$$a_{n+1} = \frac{1}{4}\sum_{i=0}^{n}(-3/4)^{n-i}a_{i+2} = \frac{1}{19}\sum_{i=0}^{n}(-12/19)^{n-i}a_{i+3}.$$

We invite the reader to try their hand at developing new such identities. It would also be reasonable (and quite interesting) to try adapting the methods of this paper to the Fibonacci and Lucas *polynomials*, as seen in Adegoke and Frontczak's recent article [2]. The possibilities are unlimited.

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References

[1] W. M. Abd-Elhameed and N. A. Zeyada, New identities involving generalized Fibonacci and generalized Lucas numbers, *Indian J. Pure Appl. Math.*, **49** (2018), 527–537.

[2] Kunle Adegoke and Robert Frontczak, Fibonacci sums and divisibility properties, *Electron. J. Math.* **6** (2023) 82–92.

[3] Brother U. Alfred, On the ordering of the Fibonacci sequence, *Fibonacci Quart.* **1** (1963) 43–46.

[4] Arthur T. Benjamin and Jennifer J. Quinn, *Proofs that Really Count*, Mathematical Association of America, 2003.

[5] Arthur T. Benjamin and Jennifer J. Quinn, Fibonacci and Lucas identities through colored tilings, *Util. Math.* **56** (1999), 137–142.

[6] Gaurav Bhatnagar, Analogues of a Fibonacci-Lucas identity, *Fibonacci Quart.* **54** (2016), 166–171.

[7] Chan-Liang Chung, Jialing Yao, and Kanglun Zhou, Extensions of Sury's relation involving Fibonacci k-step and Lucas k-step polynomials, *INTEGERS* **23** (2023), article #A38.

[8] Spiros D. Dafnis, On the relation between Fibonacci and Lucas numbers, *Fibonacci Quart.* **58** (2020), 111–114.

[9] Spiros D. Dafnis, Andreas N. Philippou, and Ioannis E. Livieris, An identity relating Fibonacci and Lucas numbers of order k, *Electron. Notes in Discrete Math.* **70** (2018), 37–42.

[10] Petra Marija De Micheli Vitturi, A tiling involution for the Sury's identity, *Math. Commun.* **30** (2025), 145–152.

[11] Greg Dresden and Michael Tulskikh, Convolutions of sequences with similar linear recurrence formulas, *J. Integer Sequences* **25** (2022), Article 22.3.3.

[12] Tom Edgar, Extending some Fibonacci-Lucas relations, *Fibonacci Quart.* **54** (2016), 79.

[13] Thomas Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, 2018.

[14] Thomas Koshy, *Pell and Pell-Lucas Numbers with Applications*, Springer, 2014.

[15] K. Kuhapatanakul and K. Thongsing, Generalizations of the Fibonacci-Lucas relations, *Amer. Math. Monthly* **126** (2019), 81.

[16] Harris Kwong, An alternate proof of Sury's Fibonacci-Lucas relation, *Amer. Math. Monthly* **121** (2014), 514.

[17] Diego Marques, A new Fibonacci-Lucas relation, *Amer. Math. Monthly* **122** (2015), 683.

[18] Ivica Martinjak, Two extensions of the Sury's identity, *Amer. Math. Monthly* **123** (2016), 919.

[19] Ivica Martinjak and Helmut Prodinger, Complementary families of the Fibonacci-Lucas relations, *INTEGERS* **19** (2019), article #A2.

[20] R. S. Melham, Closed formulas for finite sums of weighted fractional generalized Fibonacci products, *Fibonacci Quart.* **56** (2018), 167–176.

[21] Andreas N. Philippou and Spiros D. Dafnis, A simple proof of an identity generalizing Fibonacci-Lucas identities, *Fibonacci Quart.* **56** (2018), 334–336.

[22] N. J. A. Sloane et al., The On-Line Encyclopedia of Integer Sequences, 2026. Available at https://oeis.org.

[23] B. Sury, A polynomial parent to a Fibonacci-Lucas relation, *Amer. Math. Monthly* **121** (2014), 236.

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