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# Total Positivity of Toeplitz Matrices Involving Generalized Hyper-Fibonacci Numbers

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**Abstract**

We study the total positivity of Toeplitz matrices built from generalized hyper-Fibonacci sequences of a fixed generation. Using this approach, we prove that each sequence becomes log-concave beyond a certain point for every generation. We also present several notable special cases that illustrate the scope of our results.

# 1 Introduction

Let $M$ be an infinite real matrix. If all minors of $M$ have positive determinants, we say that $M$ is *totally positive*, or simply TP. If all minors of $M$ of every order $\leq r$ for some integer $r$ are positive, we say that $M$ is *totally positive of order* $r$, or simply $\mathrm{TP}_r$.

Several studies in the literature have addressed the total positivity of matrices associated with various classes of recurrent sequences. In particular, many combinatorial triangular matrices have been shown to possess the total positivity property. For example, Wang and Wang [30] investigated the total positivity of the Catalan triangle. Additionally, Ahmia and Belbachir [1] proved that the generalized Pascal triangle is $\mathrm{TP}_2$. For additional results concerning triangular matrices and Riordan arrays, we refer the reader to [8, 9, 10].

A *Toeplitz* matrix $T = [t_{i,j}]$ is a (finite or infinite) matrix whose entries satisfy $t_{i+1,j+1} = t_{i,j}$. In the finite case, that is

$$T = \begin{pmatrix} t_0 & t_1 & \cdots & t_n \\ t_{-1} & t_0 & \cdots & t_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{-n} & t_{-n+1} & \cdots & t_0 \end{pmatrix}.$$

A *Hankel* matrix $H = [h_{i,j}]$ is a (finite or infinite) matrix whose entries satisfy $t_{i+1,j-1} = t_{i,j}$. In the finite case, that is

$$H = \begin{pmatrix} h_0 & h_1 & \cdots & h_n \\ h_1 & h_2 & \cdots & h_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_n & h_{n+1} & \cdots & h_{2n} \end{pmatrix}.$$

Recall that a sequence $(b_n)_{n\geq 0}$ of positive numbers is *log-concave* (resp., *log-convex*) if $b_{i+1}^2 \geq b_i b_{i+2}$ for all $i \geq 0$ (resp., $b_{i+1}^2 \leq b_i b_{i+2}$ for all $i \geq 0$). The $\mathrm{TP}_2$ property of Hankel and Toeplitz matrices is closely connected to the log-convexity and log-concavity, respectively, of the associated sequences. The properties of log-concavity and log-convexity have been widely studied in the literature due to their importance in combinatorics. Recent advances have introduced novel techniques for establishing such properties; see, for instance, [12, 20, 22, 30]. These expansions complement the classical results given by Stanley [27].

The total positivity of Toeplitz and Hankel matrices is among the important properties of these matrices; such properties can find applications in a number of fields, including chemistry, electrical networks, game theory, differential equations, stochastic processes, orthogonal polynomials, combinatorics, quantum groups, algebraic geometry, symmetric functions, and representation theory; see [8, 16, 17, 23, 24] for some applications.

Let $p \geq 1$, $a \geq 0$, and $b \geq 0$ be integers. We consider the generalized Fibonacci sequence $(W_{p,n})_{n\geq 0}$ defined recursively as follows:

$$W_{p,n} = pW_{p,n-1} + W_{p,n-2}\ (n \geq 2), \quad W_{p,0} = a,\ W_{p,1} = b. \tag{1}$$

*Remark* 1. Some particular cases of the above sequence are

1. For $(a, b) = (0, 1)$, we obtain the $p$-Fibonacci numbers $F_{p,n}$ defined by Falcón [15].

2. For $(a, b) = (2, p)$, we obtain the $p$-Lucas numbers $L_{p,n}$ defined by Falcón and Plaza [14].

Furthermore, for $p = 1$, the sequences $(F_{1,n})_{n\geq 0} = (F_n)_{n\geq 0}$ and $(L_{1,n})_{n\geq 0} = (L_n)_{n\geq 0}$ correspond to the classical Fibonacci and Lucas sequences, respectively. We refer to [14, 15, 19, 21, 28, 29] for some properties and identities involving the $p$-Fibonacci and $p$-Lucas numbers.

Let $r \geq 0$ be a fixed integer. Dil and Mező [11] introduced the hyper-Fibonacci numbers $F_n^{(r)}$ and the hyper-Lucas numbers $L_n^{(r)}$ as follows:

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)},\ F_0^{(r)} = 0,\ F_1^{(r)} = 1,\ F_n^{(0)} = F_n,$$

$$L_n^{(r)} = \sum_{k=0}^{n} L_k^{(r-1)},\ L_0^{(r)} = 2,\ L_1^{(r)} = 2r+1,\ L_n^{(0)} = L_n.$$

Let $p \geq 1$ and $r \geq 1$ be two integers. The generalized hyper-Fibonacci numbers $W_{p,n}^{(r)}$ associated with the generalized Fibonacci numbers $W_{p,n}$ are defined as follows [3, 6]:

$$W_{p,n}^{(r)} = \sum_{k=0}^{n} p^{n-k} W_{p,k}^{(r-1)}, \quad W_{p,n}^{(0)} = W_{p,n}, \quad W_{p,0}^{(r)} = a, \quad W_{p,1}^{(r)} = par + b. \tag{2}$$

*Remark* 2. Some particular cases of the generalized hyper-Fibonacci numbers are

1. For $(a, b) = (0, 1)$, we get the hyper $p$-Fibonacci numbers $F_{p,n}^{(r)}$; see [3, 4, 7].

2. For $(a, b) = (2, p)$, we get the hyper $p$-Lucas numbers $L_{p,n}^{(r)}$; see [5].

Furthermore, for $p = 1$, we have $F_{1,n}^{(r)} = F_n^{(r)}$ and $L_{1,n}^{(r)} = L_n^{(r)}$. Also, for $p = 2$, the numbers $F_{2,n}^{(r)} = P_n^{(r)}$ and $L_{2,n}^{(r)} = Q_n^{(r)}$ correspond to the $n^{\text{th}}$ hyper-Pell and hyper-Pell-Lucas numbers, respectively; see [2].

Došlić et al. [13] studied the total positivity of a class of Toeplitz matrices composed of the hyper-Fibonacci sequence $(F_n^{(r)})_{n\geq 0}$. A similar study but for the hyper-Lucas sequence $(L_n^{(r)})_{n\geq 0}$ has been studied by Rezig and Ahmia [25]. The aim of our work is to extend these results to the generalized hyper-Fibonacci numbers $(W_{p,n}^{(r)})_{n\geq 0}$. To do so, we define the following $m \times m$ Toeplitz matrix:

$$T_{m,n}^{(r)} := \begin{pmatrix} W_{p,n}^{(r)} & W_{p,n-1}^{(r)} & \cdots & W_{p,n-m+1}^{(r)} \\ W_{p,n+1}^{(r)} & W_{p,n}^{(r)} & \cdots & W_{p,n-m+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ W_{p,n+m-1}^{(r)} & W_{p,n+m-2}^{(r)} & \cdots & W_{p,n}^{(r)} \end{pmatrix}.$$

In Section 2, we present several identities and auxiliary results that we use in this work. In Section 3, we establish the positivity of the determinant $\det\big(T_{m,n}^{(r)}\big)$ under the condition $m \leq r+1$. This, in turn, implies the log-concavity of the sequences $(W_{p,n}^{(r)})_{n\geq 0}$ for all positive integers $r$ and sufficiently large $n$. Finally, in Section 4, we examine the total positivity of the Toeplitz matrix $T_{r+2,n}^{(r)}$.

Throughout this paper, we assume that $p \geq 1$, $a \geq 0$, and $b > 0$ are integers.

## 2 Preliminary results

In this section, we present some of the results we need for the rest of the paper.

Initial minors play a fundamental role in the study of total positivity of matrices. Let $I$ and $J$ denote the row set and column set, respectively. A *minor* $b_{I,J}$ is called *initial* if both $I$ and $J$ consist of consecutive indices and $I \cup J$ contain 1. Consequently, each matrix entry is the lower-right corner of exactly one initial minor. The result established by Gasca and Peña [18] provides a useful method for testing the total positivity of matrices:

**Theorem 3.** *[18] A matrix is totally positive if and only if all its initial minors are positive.*

Numerous identities involving generalized Fibonacci numbers have been established in the literature. Let $\Delta = a^2 + pab - b^2$, $\alpha = \frac{p+\sqrt{p^2+4}}{2}$, and $\beta = \frac{p-\sqrt{p^2+4}}{2}$. Then we have

**Lemma 4.** *Let $m \geq 0$ and $n \geq 0$ be two integers. Then the following identities hold.*

$$W_{p,n} = A\alpha^n + B\beta^n, \text{ where } A = \frac{b - a\beta}{\sqrt{p^2+4}},\ B = \frac{a\alpha - b}{\sqrt{p^2+4}}, \tag{3}$$

$$F_{p,m-2}W_{p,n-2} - F_{p,m-3}W_{p,n-1} = (-1)^{m-1}W_{n-m+1}, \tag{4}$$

$$W_{p,n}W_{p,n+2} - W_{n+1}^2 = (-1)^n\Delta, \tag{5}$$

$$\lim_{n\to+\infty} \frac{W_{p,n+1}}{W_{p,n}} = \alpha. \tag{6}$$

**Lemma 5.** *[3] Let $r \geq 1$ be an integer. The following identities hold*

$$W_{p,n}^{(r)} = pW_{p,n-1}^{(r)} + W_{p,n}^{(r-1)} \quad (n \geq 1), \tag{7}$$

$$W_{p,n-r}^{(r)} = W_{p,n+r} - S_r \quad (n \geq r), \tag{8}$$

*where* $S_r = p^n \sum_{k=0}^{r-1} \binom{n-k-1}{r-k-1} \frac{W_{p,2k+1}}{p^{r-1}}$.

*Remark* 6. In (8), the numbers $S_r$ can be expressed as $S_r = p^n f_r(n)$, where $f_r(n)$ is a polynomial in $n$.

Let $r \geq 0$ and $n \geq 0$ be two integers. We consider the following Hankel matrix of order $r+2$:

$$H_{r,n} = \begin{pmatrix} W_{p,n}^{(r)} & W_{p,n+1}^{(r)} & \cdots & W_{p,n+r+1}^{(r)} \\ W_{p,n+1}^{(r)} & W_{p,n+2}^{(r)} & \cdots & W_{p,n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ W_{p,n+r+1}^{(r)} & W_{p,n+r+2}^{(r)} & \cdots & W_{p,n+2r+2}^{(r)} \end{pmatrix}.$$

**Proposition 7.** *[3] Let $r \geq 0$ and $n \geq 0$ be integers. We have*

$$\det(H_{r,n}) = (-1)^{n+1+\lfloor \frac{r+3}{2} \rfloor} p^{nr+r^2} b^r \Delta. \tag{9}$$

# 3 Positivity of Toeplitz determinants

In this section, we discuss the positivity of determinants of some classes of Toeplitz matrices where the entries are generalized hyper-Fibonacci numbers of the $r^{\text{th}}$ generation.

Let $T_{m,n}^{(r)}$ denote the Toeplitz matrix of order $m$ consisting of generalized hyper-Fibonacci numbers of the $r^{\text{th}}$ generation

$$T_{m,n}^{(r)} = \begin{pmatrix} W_{p,n}^{(r)} & W_{p,n-1}^{(r)} & \cdots & W_{p,n-m+1}^{(r)} \\ W_{p,n+1}^{(r)} & W_{p,n}^{(r)} & \cdots & W_{p,n+-m+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ W_{p,n+m-1}^{(r)} & W_{p,n+m-2}^{(r)} & \cdots & W_{p,n}^{(r)} \end{pmatrix},$$

with the condition $m \leq r+1$.

**Theorem 8.** *Let $m \geq 1$ be an integer. There exists a positive integer $n_m$ such that $\det\left(T_{m,n}^{(m-1)}\right) > 0$ for all $n \geq n_m$.*

*Proof.* For $m = 2$, using (7), we obtain

$$\det\left(T_{2,n}^{(1)}\right) := \begin{vmatrix} W_{p,n}^{(1)} & W_{p,n-1}^{(1)} \\ W_{p,n+1}^{(1)} & W_{p,n}^{(1)} \end{vmatrix} = \begin{vmatrix} W_{p,n} & W_{p,n-1}^{(1)} \\ W_{p,n+1} & W_{p,n}^{(1)} \end{vmatrix} = \begin{vmatrix} W_{p,n} & W_{p,n-1}^{(1)} \\ W_{p,n-1} & W_{p,n} \end{vmatrix}.$$

Thus, by using (8), we have

$$\det\big(T_{2,n}^{(1)}\big) = \begin{vmatrix} W_{p,n} & W_{p,n+1} - S_1 \\ W_{p,n-1} & W_{p,n} \end{vmatrix}.$$

According to (5), the last determinant is positive starting from a certain value $n_2$.

In the rest of the proof, we assume that $m \geq 3$. Using (7) and some elementary transformations on columns, we get

$$\begin{aligned}\det\big(T_{m,n}^{(m-1)}\big) &:= \begin{vmatrix} W_{p,n}^{(m-1)} & W_{p,n-1}^{(m-1)} & \cdots & W_{p,n-m+1}^{(m-1)} \\ W_{p,n+1}^{(m-1)} & W_{p,n}^{(m-1)} & \cdots & W_{p,n-m+2}^{(m-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_{p,n+m-1}^{(m-1)} & W_{p,n+m-2}^{(m-1)} & \cdots & W_{p,n}^{(m-1)} \end{vmatrix} \\ &= \begin{vmatrix} W_{p,n} & W_{p,n-1}^{(1)} & W_{p,n-2}^{(2)} & \cdots & W_{p,n-m+1}^{(m-1)} \\ W_{p,n+1} & W_{p,n}^{(1)} & W_{p,n-1}^{(2)} & \cdots & W_{p,n-m+2}^{(m-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W_{p,n+m-1} & W_{p,n+m-2}^{(1)} & W_{p,n+m-3}^{(2)} & \cdots & W_{p,n}^{(m-1)} \end{vmatrix}.\end{aligned}$$

Again, using (7) and some elementary transformations on rows, we get

$$\det\big(T_{m,n}^{(m-1)}\big) = \begin{vmatrix} W_{p,n} & W_{p,n-1}^{(1)} & W_{p,n-2}^{(2)} & \cdots & W_{p,n-m+1}^{(m-1)} \\ W_{p,n-1} & W_{p,n} & W_{p,n-1}^{(1)} & \cdots & W_{p,n-m+2}^{(m-2)} \\ \vdots & \vdots & \vdots & & \vdots \\ W_{p,n-m+2} & W_{p,n-m+3} & W_{p,n-m+4} & \cdots & W_{n-1}^{(1)} \\ W_{p,n-m+1} & W_{p,n-m+2} & W_{p,n-m+3} & \cdots & W_{p,n} \end{vmatrix}.$$

Now, using (8), we write

$$\det\big(T_{m,n}^{(m-1)}\big) = \begin{vmatrix} W_{p,n} & W_{p,n+1} - S_1 & W_{p,n+2} - S_2 & \cdots & W_{p,n+m-1} - S_{m-1} \\ W_{p,n-1} & W_{p,n} & W_{p,n+1} - S_1 & \cdots & W_{p,n+m-2} - S_{m-2} \\ \vdots & \vdots & \vdots & & \vdots \\ W_{p,n-m+2} & W_{p,n+m+3} & W_{p,n-m+3} & \cdots & W_{p,n+1} - S_1 \\ W_{p,n-m+1} & W_{p,n-m+2} & W_{p,n-m+3} & \cdots & W_{p,n} \end{vmatrix}.$$

Using the recurrence relation (1) on columns, we obtain

$$\det\big(T_{m,n}^{(m-1)}\big) = \begin{vmatrix} S_2 - pS_1 & S_3 - pS_2 - S_1 & \cdots & W_{p,n+m-2} - S_{m-2} & W_{p,n+m-1} - S_{m-1} \\ S_1 & S_2 - pS_1 & \cdots & W_{p,n+m-3} - S_{m-3} & W_{p,n+m-2} - S_{m-2} \\ 0 & S_1 & \cdots & W_{p,n+m-4} - S_{m-4} & W_{p,n+m-3} - S_{m-3} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & W_{p,n} & W_{p,n+1} - S_1 \\ 0 & 0 & \cdots & W_{p,n-1} & W_{p,n} \end{vmatrix}.$$

Again, using (1) on rows, we obtain

$$\det\big(T_{m,n}^{(m-1)}\big) = \begin{vmatrix} S_2-2pS_1 & S_3-2pS_2+(p^2-2)S_1 & \cdots & -S_{m-2}+pS_{m-3}+S_{m-4} & -S_{m-1}+pS_{m-2}+S_{m-3} \\ S_1 & S_2-2pS_1 & \cdots & -S_{m-3}+pS_{m-4}+S_{m-5} & -S_{m-2}+pS_{m-3}+S_{m-4} \\ 0 & S_1 & \cdots & -S_{m-4}+pS_{m-5}+S_{m-6} & -S_{m-3}+pS_{m-4}+S_{m-5} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & W_{p,n} & W_{p,n+1}-S_1 \\ 0 & 0 & \cdots & W_{p,n-1} & W_{p,n} \end{vmatrix}$$
$$= \det(\Omega).$$

Let $a_{ij}$ denote the elements of the first $m-2$ column of $\Omega$ and let $b_{i,j}$ denote the elements of the last two columns, i.e., $\Omega = \{a_{i,j}, b_{i,m-1}, b_{i,m}\}_{\substack{1\le i\le m \\ 1\le j\le m-2}}$.

The elements $a_{i,j}$ of $\Omega$ satisfy $a_{i,j} = a_{i+1,j+1},\ (1 \le i \le m-1,\ 1 \le j \le m-3)$, where

$$\begin{cases} a_{1,1} = S_2 - 2pS_1, \\ a_{1,2} = S_3 - 2pS_2 + (p^2-2)S_1, \\ a_{1,3} = S_4 - 2pS_3 + (p^2-2)S_2 + 2pS_1, \\ a_{1,j} = S_{j+1} - 2pS_j + (p^2-2)S_{j-1} + 2pS_{j-2} + S_{j-3}, \quad 4 \le j \le m-2; \\ a_{2,1} = S_1, \\ a_{i,1} = 0, \quad 3 \le i \le m. \end{cases}$$

The elements $b_{i,j}$ of $\Omega$ satisfy $b_{i,m-1} = b_{i+1,m},\ (1 \le i \le m-3)$, where

$$\begin{cases} b_{i,m-1} = -S_{m-i-1} + pS_{m-i-2} + S_{m-i-3}, \quad 1 \le i \le m-4; \\ b_{m-3,m-1} = -S_2 + pS_1, \\ b_{m-2,m-1} = -S_1, \\ b_{m-1,m-1} = W_{p,n}, \\ b_{m,m-1} = W_{p,n-1}, \\ b_{1,m} = -S_{m-1} + pS_{m-2} + S_{m-3}, \\ b_{m-1,m} = W_{p,n+1} - S_1, \\ b_{m,m} = W_{p,n}. \end{cases}$$

Now we use the following column transformations in the last determinant:

$$b_{i,m-1} \longrightarrow b'_{i,m-1} := b_{i,m-1} + \sum_{j=1}^{m-3} a_{i,j}, \quad (1 \le i \le m),$$

$$b_{i,m} \longrightarrow b'_{i,m} := b_{i,m} + \sum_{j=1}^{m-2} a_{i,j}, \quad (1 \le i \le m),$$

we obtain

$$\det\left(T_{m,n}^{(m-1)}\right) = \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m-2} & b'_{1,m-1} & b'_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m-2} & b'_{2,m-1} & b'_{2,m} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-2,n-2} & b'_{m-2,m-1} & b'_{m-2,m} \\ 0 & 0 & \cdots & S_1 & W_{p,n} & W_{p,n+1} \\ 0 & 0 & \cdots & 0 & W_{p,n-1} & W_{p,n} \end{vmatrix}, \tag{10}$$

where the last two columns are

$$\begin{pmatrix} (p^2-1)S_1 + p^2\sum\limits_{j=2}^{m-6} S_j + (p^2-1)S_{m-5} + (p^2-2p)S_{m-4} + (1-p)S_{m-3} & (p^2-1)S_1 + p^2\sum\limits_{j=2}^{m-5} S_j + (p^2-1)S_{m-4} + (p^2-2p)S_{m-3} + (1-p)S_{m-2} \\ p^2\sum\limits_{j=1}^{m-7} S_j + (p^2-1)S_{m-6} + (p^2-2p)S_{m-5} + (1-p)S_{m-4} & p^2\sum\limits_{j=1}^{m-6} S_j + (p^2-1)S_{m-5} + (p^2-2p)S_{m-4} + (1-p)S_{m-3} \\ \vdots & \vdots \\ p^2(S_1+S_2) + (p^2-1)S_3 + (p^2-2p)S_4 + (1-p)S_5 & p^2(S_1+S_2+S_3) + (p^2-1)S_4 + (p^2-2p)S_5 + (1-p)S_6 \\ p^2S_1 + (p^2-1)S_2 + (p^2-2p)S_3 + (1-p)S_4 & p^2(S_1+S_2) + (p^2-1)S_3 + (p^2-2p)S_4 + (1-p)S_5 \\ (p^2-1)S_1 + (p^2-2p)S_2 + (1-p)S_3 & p^2S_1 + (p^2-1)S_2 + (p^2-2p)S_3 + (1-p)S_4 \\ (p^2-2p)S_1 + (1-p)S_2 & (p^2-1)S_1 + (p^2-2p)S_2 + (1-p)S_3 \\ (1-p)S_1 & (p^2-2p)S_1 + (1-p)S_2 \\ 0 & (1-p)S_1 \\ W_{p,n} & W_{p,n+1} \\ W_{p,n-1} & W_{p,n} \end{pmatrix}.$$

Furthermore, we perform the following column transformations on (10):

$$b'_{i,m-1} \longrightarrow b''_{i,m-1} := b'_{i,m-1} + \sum_{j=1}^{m-4}(F_{p,m-j-2} - 1)a_{i,j}, \quad (1 \leq i \leq m),$$

$$b'_{i,m} \longrightarrow b''_{i,m} := b'_{i,m} + \sum_{j=1}^{m-3}(F_{p,m-j-1} - 1)a_{i,j}, \quad (1 \leq i \leq m),$$

where $F_{p,n}$ is the $n^{\text{th}}$ $p$-Fibonacci number, we get

$$\det\left(T_{m,n}^{(m-1)}\right) = \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m-2} & -S_1F_{p,m-2} & -S_1F_{p,m-1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m-2} & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{m-2,m-2} & 0 & 0 \\ 0 & 0 & \cdots & S_1 & W_{p,n} & W_{p,n+1} \\ 0 & 0 & \cdots & 0 & W_{p,n-1} & W_{p,n} \end{vmatrix}.$$

Next, we apply (5) with the following computational techniques on two last rows:

$$
\begin{aligned}
\det\bigl(T_{m,n}^{(m-1)}\bigr) &= \frac{1}{W_{p,n-1}}\begin{vmatrix} & \vdots & \vdots & \vdots \\ \cdots & S_1 & W_{p,n} & W_{p,n+1} \\ \cdots & 0 & W_{p,n-1}^2 & W_{p,n}W_{p,n-1}\end{vmatrix} \\
&= \frac{1}{W_{p,n-1}}\begin{vmatrix} & \vdots & \vdots & \vdots \\ \cdots & S_1 & W_{p,n} & W_{p,n+1} \\ \cdots & -S_1W_{p,n-2} & (-1)^{n-1}\Delta & p(-1)^{n-1}\Delta\end{vmatrix} \\
&= \frac{(-1)^{n-1}}{\Delta W_{p,n-1}}\begin{vmatrix} & \vdots & \vdots & \vdots \\ \cdots & (-1)^{n-1}\Delta S_1 & (-1)^{n-1}\Delta W_{p,n} & (-1)^{n-1}\Delta W_{p,n+1} \\ \cdots & -S_1W_{p,n-2} & (-1)^{n-1}\Delta & p(-1)^{n-1}\Delta\end{vmatrix} \\
&= \frac{(-1)^{n-1}}{\Delta}\begin{vmatrix} a_{1,1} & \cdots & a_{1,m-2} & -S_1F_{p,m-2} & -S_1F_{p,m-1} \\ a_{2,1} & \cdots & a_{2,m-2} & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & S_1W_{p,n-1} & 0 & (-1)^{n-1}\Delta \\ 0 & \cdots & -S_1W_{p,n-2} & (-1)^{n-1}\Delta & (-1)^{n-1}p\Delta\end{vmatrix}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\det\bigl(T_{m,n}^{(m-1)}\bigr) &= \frac{(-1)^{n}}{\Delta}\begin{vmatrix} a_{1,1} & \cdots & a_{1,m-2} & -S_1F_{p,m-3} & -S_1F_{p,m-2} \\ a_{2,1} & \cdots & a_{2,m-2} & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & S_1W_{p,n-1} & (-1)^{n-1}\Delta & 0 \\ 0 & \cdots & -S_1W_{p,n-2} & 0 & (-1)^{n-1}\Delta\end{vmatrix} \\
&= \frac{(-1)^{n}}{\Delta}\begin{vmatrix} a_{1,1} & \cdots & A & -S_1F_{p,m-3} & -S_1F_{p,m-2} \\ a_{2,1} & \cdots & a_{2,m-2} & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & (-1)^{n-1}\Delta & 0 \\ 0 & \cdots & 0 & 0 & (-1)^{n-1}\Delta\end{vmatrix} \\
&= (-1)^{n}\Delta\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m-3} & A \\ S_1 & a_{2,2} & \cdots & a_{2,m-3} & a_{2,m-2} \\ 0 & S_1 & \cdots & a_{3,m-3} & a_{3,m-2} \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & S_1 & a_{m-2,m-2}\end{vmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
A &= a_{1,m-2} + \frac{(-1)^nS_1^2}{\Delta}(F_{p,m-2}W_{p,n-2} - F_{p,m-3}W_{p,n-1}) \\
&= a_{1,m-2} + \frac{(-1)^nS_1^2}{\Delta}(-1)^{m-1}W_{p,n-m+1} \quad \text{(using (4))}. \qquad (11)
\end{aligned}
$$

The last determinant can be rewritten as follows:

$$\det\bigl(T_{m,n}^{(m-1)}\bigr) = (-1)^n \Delta \left( (-1)^{m-1} S_1^{m-3} A + \sum_{j=1}^{m-3} (-1)^{i+1} a_{1,j} M_{1,j} \right), \tag{12}$$

where $M_{1,j}$ denotes the determinant obtained from $\det\bigl(T_{m,n}^{(m-1)}\bigr)$ by omitting the first row and $j^{\text{th}}$ column $(1 \le j \le m-3)$. Using Remark 6, we get $a_{i,j} = p^n f_{i,j}(n)$ where $f_{i,j}(n)$ is a polynomial in $n$. This implies that $M_{1,j} = p^{n(m-3)} f_j(n)$ where $f_j(n)$ is a polynomial on $n$ $(1 \le j \le m-3)$. Therefore, we have

$$\begin{aligned} \det\bigl(T_{m,n}^{(m-1)}\bigr) &= (-1)^n \Delta((-1)^{m-1} S_1^{m-3} A + p^{n(m-2)} f(n)) \quad (S_1 := bp^n) \\ &= p^{n(m-2)} b^{m-1} (p^n W_{p,n-m+1} + g(n)) \quad (\text{using } (11)), \end{aligned}$$

where $f(n)$ and $g(n)$ are polynomials in $n$. The existence of a positive integer $n_m$ such that $W_{p,n-m+1} > P(n)$ for all $n > n_m$, where $P(n)$ is a polynomial of any degree, completes the proof. □

**Corollary 9.** *Let $r \ge 1$ be an integer. For every integer $m$ such that $1 \le m \le r+1$, there exists a positive integer $n_m$ such that $\det\bigl(T_{m,n}^{(r)}\bigr) > 0$ for all $n \ge n_m$.*

*Proof.* We use induction on $r$. The case $r = m-1$ is given in Theorem 8. Assume that the claim is true for all $m-1 \le t \le r-1$; we prove that

$$\det\bigl(T_{m,n}^{(r)}\bigr) = \begin{vmatrix} W_{p,n}^{(r)} & W_{p,n-1}^{(r)} & \cdots & W_{p,n-m+1}^{(r)} \\ W_{p,n+1}^{(r)} & W_{p,n}^{(r)} & \cdots & W_{p,n-m+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ W_{p,n+m-1}^{(r)} & W_{p,n+m-2}^{(r)} & \cdots & W_{p,n}^{(r)} \end{vmatrix} > 0.$$

By performing row transformations using relation (7), we obtain

$$\begin{aligned} \det\bigl(T_{m,n}^{(r)}\bigr) &= \begin{vmatrix} W_{p,n}^{(r)} & W_{p,n-1}^{(r)} & \cdots & W_{p,n-m+1}^{(r)} \\ W_{p,n+1}^{(r-1)} & W_{p,n}^{(r-1)} & \cdots & W_{p,n-m+2}^{(r-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_{p,n+m-1}^{(r-1)} & W_{p,n+m-2}^{(r-1)} & \cdots & W_{p,n}^{(r-1)} \end{vmatrix} \\ &= \sum_{j=1}^{m} (-1)^{j+1} \frac{W_{p,n-j+1}^{(r)}}{W_{p,n-j+1}^{(r-1)}} W_{p,n-j+1}^{(r-1)} M_{1,j}, \end{aligned}$$

where $M_{1,j}$ denotes the determinant obtained from $\det\bigl(T_{m,n}^{(r)}\bigr)$ by omitting the first row and $j^{\text{th}}$ column $(1 \le j \le m)$. We define a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{m} (-1)^{j+1} x_j W_{p,n-j+1}^{(r-1)} M_{1,j}.$$

Let $\phi := 1+p\alpha$, where $\alpha = \frac{p+\sqrt{p^2+4}}{2}$. We have $f(\phi,\phi,\ldots,\phi) = \phi\sum_{j=1}^{m}(-1)^{j+1}W_{p,n-j+1}^{(r-1)}M_{1,j} = \phi\det\big(T_{m,n}^{(r-1)}\big) > 0$. Since $f$ is continuous, there is a neighborhood

$$I = (\phi-\epsilon_1,\phi+\epsilon_1)\times(\phi-\epsilon_2,\phi+\epsilon_2)\times\cdots\times(\phi-\epsilon_n,\phi+\epsilon_n)$$

of the point $(\phi,\phi,\ldots,\phi)$ such that $f$ is positive on $I$. Using (8), by dividing both sides by $W_n^{(r-1)}$, and passing to limit $n\longrightarrow\infty$, one readily obtains

$$\lim_{n\to\infty}\frac{W_{p,n}^{(r)}}{W_{p,n}^{(r-1)}} = \lim_{n\to\infty}\frac{W_{p,n+2r}}{W_{p,n+2r-2}} = \phi.$$

This implies that, for sufficiently large $n$, the coefficient $x_j = \frac{W_{p,n-j+1}^{(r)}}{W_{p,n-j+1}^{(r-1)}}$ falls into $(\phi-\epsilon_j,\phi+\epsilon_j)$ for all $1\le j\le n$. Therefore, there exists a positive integer $n_m$ such that

$$\det\big(T_{m,n}^{(r)}\big) = f\left(\frac{W_{p,n}^{(r)}}{W_{p,n}^{(r-1)}},\frac{W_{p,n-1}^{(r)}}{W_{p,n-1}^{(r-1)}},\ldots,\frac{W_{p,n-m+1}^{(r)}}{W_{p,n-m+1}^{(r-1)}}\right) > 0$$

for all $n\ge n_m$. The proof is thus completed. $\square$

If we take $m=2$ in Corollary 9, we obtain

**Corollary 10.** *Let $r\ge 1$ be an integer. The sequence $(W_{p,n}^{(r)})_n$ is log-concave for large enough $n$.*

**Corollary 11.** *Let $r\ge 1$ be an integer. The sequences $(F_{p,n}^{(r)})_n$ and $(L_{p,n}^{(r)})_n$ are log-concave for large enough $n$.*

For $p=1$, we obtain the log-concavity of the sequences $(F_n^{(r)})_n$ and $(L_n^{(r)})_n$. This recovers the main result established by Zheng et al. [31].

# 4 Total positivity of Toeplitz matrices

Let $T_{r+2,n}^{(r)}$ denote the Toeplitz matrix of order $r+2$ consisting of generalized hyper-Fibonacci numbers of the $r^{\text{th}}$ generation; that is

$$T_{r+2,n}^{(r)} = \begin{pmatrix} W_{p,n}^{(r)} & W_{p,n-1}^{(r)} & \cdots & W_{p,n-r-1}^{(r)} \\ W_{p,n+1}^{(r)} & W_{p,n}^{(r)} & \cdots & W_{p,n-r}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ W_{p,n+r+1}^{(r)} & W_{p,n+r}^{(r)} & \cdots & W_{p,n}^{(r)} \end{pmatrix}.$$

Let $\Delta := a^2+pab-b^2$. Then we have the following main theorem.

**Theorem 12.** *Let $r \geq 1$ be an integer.*

1. *If $\Delta > 0$, then there exists a positive integer $n_r$ such that the matrix $T^{(r)}_{r+2,2n}$ is totally positive for every $n \geq n_r$.*

2. *If $\Delta < 0$, then there exists a positive integer $n'_r$ such that the matrix $T^{(r)}_{r+2,2n+1}$ is totally positive for every $n \geq n'_r$.*

*Proof.* The $2r+1$ initial minors of order two of $T^{(r)}_{r+2,n}$ have the form $T^{(r)}_{2,n_2}$, where $n_2 \geq n-r$. Using Corollary 9, there exists a positive integer $q_2$ such that these initial minors are positive for $n_2 \geq q_2$. Similarly, according to Corollary 9, there exist positive integers $q_3, q_4, \ldots, q_{r+1}$ such that the initial minors of order $3, 4, \ldots, r+1$, which are of the form $T^{(r)}_{3,n_3}, T^{(r)}_{4,n_4}, \ldots, T^{(r)}_{r+1,n_{r+1}}$, are positive for $n_3 \geq q_3$, $n_4 \geq q_4, \ldots, n_{r+1} \geq q_{r+1}$. It remains to discuss the positivity of $\det\left(T^{(r)}_{r+2,n}\right)$. By reversing the order of columns of $T^{(r)}_{r+2,n}$, we obtain

$$\begin{aligned}
\det\left(T^{(r)}_{r+2,n}\right) &= (-1)^{\lfloor\frac{r+2}{2}\rfloor} \begin{vmatrix} W^{(r)}_{p,n-r-1} & W^{(r)}_{p,n-r} & \cdots & W^{(r)}_{p,n} \\ W^{(r)}_{p,n-r} & W^{(r)}_{p,n-r+1} & \cdots & W^{(r)}_{p,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ W^{(r)}_{p,n} & W^{(r)}_{p,n+1} & \cdots & W^{(r)}_{p,n+r+1} \end{vmatrix} \\
&= (-1)^{\lfloor\frac{r+2}{2}\rfloor} \det(H_{r,n-r-1}) \quad \text{(using Proposition 7)} \\
&= (-1)^n p^{(n-1)r} b^r \Delta.
\end{aligned}$$

□

The following corollary follows directly from Theorem 12.

**Corollary 13.** *Let $r \geq 1$ be a positive integer. There exists a positive integer $n_r$ such that the matrix $T^{(r)}_{r+2,n}$ is $TP_{r+1}$ for all $n \geq n_r$.*

Furthermore, the following corollaries follow directly from Theorem 12 and Corollary 13.

**Corollary 14.** *Let $r \geq 1$ be a positive integer. There exist positive integers $n_r$ and $n'_r$ such that*

1. *The matrix*

$$\mathcal{F}_{r+2,2n+1} = \begin{pmatrix} F^{(r)}_{p,2n+1} & F^{(r)}_{p,2n} & \cdots & F^{(r)}_{p,2n-r} \\ F^{(r)}_{p,2n+2} & F^{(r)}_{p,2n+1} & \cdots & F^{(r)}_{p,2n-r+1} \\ \vdots & \vdots & \ddots & \vdots \\ F^{(r)}_{p,2n+r+2} & F^{(r)}_{p,2n+r+1} & \cdots & F^{(r)}_{p,2n+1} \end{pmatrix}$$

   *is totally positive for every $n \geq n_r$.*

*2. The matrix*

$$\mathcal{F}_{r+2,n} = \begin{pmatrix} F_{p,n}^{(r)} & F_{p,n-1}^{(r)} & \cdots & F_{p,n-r-1}^{(r)} \\ F_{p,n+1}^{(r)} & F_{p,n}^{(r)} & \cdots & F_{p,n-r}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{p,n+r+1}^{(r)} & F_{p,n+r}^{(r)} & \cdots & F_{p,n}^{(r)} \end{pmatrix}$$

*is* $TP_{r+1}$ *for every* $n \geq n'_r$.

For $p = 1$, we obtain the main results given by Došlić et al. [13].

**Corollary 15.** *Let* $r \geq 1$ *be an integer. There exist positive integers* $n_r$ *and* $n'_r$ *such that:*

*1. The matrix*

$$\mathcal{L}_{r+2,2n} = \begin{pmatrix} L_{p,2n}^{(r)} & L_{p,2n-1}^{(r)} & \cdots & L_{p,2n-r-1}^{(r)} \\ L_{p,2n+1}^{(r)} & L_{p,2n}^{(r)} & \cdots & L_{p,2n-r}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ L_{p,2n+r+1}^{(r)} & L_{p,2n+r}^{(r)} & \cdots & L_{p,2n}^{(r)} \end{pmatrix}$$

*is totally positive for every* $n \geq n_r$.

*2. The matrix*

$$\mathcal{L}_{r+2,n} = \begin{pmatrix} L_{p,n}^{(r)} & L_{p,n-1}^{(r)} & \cdots & L_{p,n-r-1}^{(r)} \\ L_{p,n+1}^{(r)} & L_{p,n}^{(r)} & \cdots & L_{p,n-r}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ L_{p,n+r+1}^{(r)} & L_{p,n+r}^{(r)} & \cdots & L_{p,n}^{(r)} \end{pmatrix}$$

*is* $TP_{r+1}$ *for every* $n \geq n'_r$.

For $p = 1$, we obtain the main results given by Rezig and Ahmia [25].

# 5 Final remarks

This work relies on specific choices of coefficients, initial conditions, and parity assumptions. A complete characterization of total positivity without these constraints remains open and will likely require new techniques.

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