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Congruence Properties for the Delannoy Triangle Coefficients

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Abstract

Let $\binom{n}{k}_{[2]}$ denote the entry in the nth row and kth column of the Delannoy triangle. In the present paper, we extend Wolstenholme's theorem to Delannoy triangle coefficients: for any prime $p \geq 5$, we determine $\binom{2p}{p}_{[2]}$, $\binom{2p-1}{p-1}_{[2]}$ and, more generally, $\binom{np-1}{p-1}_{[2]}$ modulo p^3 in terms of the Fermat quotient $q_p(2)$. We also establish, for any odd prime p, the congruence $\sum_{0\leq k\leq n\leq p-1}\binom{n}{k}_{[2]} \equiv \left(\frac{2}{p}\right) \pmod{p}$, where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol.

1 Introduction

Let $\binom{n}{k}_{[2]}$ be the element (Delannoy coefficient, A008288) in the nth row and kth column of the Delannoy triangle, see for instance [1]. This triangle can be obtained by the recurrence relation

$$\binom{n}{k}_{[2]} = \binom{n-1}{k}_{[2]} + \binom{n-1}{k-1}_{[2]} + \binom{n-2}{k-1}_{[2]}, \tag{1}$$

where $\binom{n}{0}_{[2]} = \binom{n}{n}_{[2]} = 1$. We use the convention $\binom{n}{k}_{[2]} = 0$ for $k \notin \{0, \ldots, n\}$.

The Delannoy triangle is the triangular presentation of the classical Delannoy square array: writing $D(a, b)$ for the number of lattice paths from $(0, 0)$ to (a, b) using only the steps $(1, 0)$, $(0, 1)$ and $(1, 1)$ (see [10]), one has

$$\binom{n}{k}_{[2]} = D(k,\, n-k). \tag{2}$$

In particular, $\binom{2n}{n}_{[2]} = D(n, n) = D_n$ is the nth central Delannoy number (A001850).

Moreover, Barry [3] has proved that the coefficients of the Delannoy triangle satisfy the following two identities

$$\binom{n}{k}_{[2]} = \sum_{j=0}^{k} \binom{k}{j} \binom{n-j}{k} \tag{3}$$

and

$$\binom{n}{k}_{[2]} = \sum_{j=0}^{k} \binom{k}{j} \binom{n-k}{j} 2^j. \tag{4}$$

Many great mathematicians have explored the congruence properties of binomial coefficients since early times. Let p be an odd prime number.

In 1876, Hermite (see [14]) showed that for odd n, the following congruence holds:

$$\sum_{\substack{1 \le m \le n \\ m \equiv 0 \pmod{p-1}}} \binom{n}{m} \equiv 0 \pmod{p}. \tag{5}$$

Later, Glaisher (see [14]) generalized it: for every given prime p and integers $1 \le j, k \le p-1$, established for $n \equiv k \pmod{p-1}$,

$$\sum_{\substack{1 \le m \le n \\ m \equiv j \pmod{p-1}}} \binom{n}{m} \equiv \binom{k}{j} \pmod{p}. \tag{6}$$

In 1862, Wolstenholme [26] established the well-known congruence for binomial coefficients, namely for all prime $p \ge 5$,

$$\binom{2p-1}{p-1} = \frac{1}{2}\binom{2p}{p} \equiv 1 \pmod{p^3}. \tag{7}$$

In 1895, Morley [17] obtained that

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3}, \tag{8}$$

for any prime number $p \geq 5$. Later, in 1900, Glaisher [12] proved that

$$\binom{np-1}{p-1} \equiv 1 - p^3 \frac{n(n-1)}{3} B_{p-3} \pmod{p^4}, \tag{9}$$

where $n \geq 1$ is an integer, and B_n is the n–th Bernoulli number.

Recall that the bi^{2}nomial coefficients (see [5, 4]) are those coefficients denoted by $\binom{n}{k}_2$, and defined by:

$$\left(1 + x + x^2\right)^n = \sum_{j=0}^{2n} \binom{n}{j}_2 x^j.$$

The exploration of congruence properties associated with bi^{2}nomial coefficients is currently expanding. Apagodu and Liu [2] established for all prime $p \geq 5$ that

$$\sum_{n=0}^{p-1} \sum_{k=0}^{p-1} \binom{n}{k}_2 \equiv \frac{1}{2}\left((-1)^{\frac{p-1}{2}} + 1\right) \pmod{p} \tag{10}$$

and

$$\binom{2p}{p}_2 \equiv 2 + \frac{2p^2}{3}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}, \tag{11}$$

where $B_m(t)$ is the Bernoulli polynomial of order m, see for instance [6].

Elkhiri and Mihoubi [11] showed that

$$\binom{np-1}{p-1}_2 \equiv \begin{cases} 1 + npq_p(3) \pmod{p^2}, & \text{if } p \equiv 1 \pmod 3, \\ -1 - npq_p(3) \pmod{p^2}, & \text{if } p \equiv 2 \pmod 3, \end{cases} \tag{12}$$

and

$$\binom{np-1}{(p-1)/2}_2 \equiv \begin{cases} 1 + np\left(2q_p(2) + \frac{1}{2}q_p(3)\right) \pmod{p^2}, & \text{if } p \equiv 1 \pmod 6, \\ -\frac{1}{2}pnq_p(3) \pmod{p^2}, & \text{if } p \equiv 5 \pmod 6, \end{cases} \tag{13}$$

where $q_p(x) := \dfrac{x^{p-1} - 1}{p}$ is called the Fermat quotient and x is coprime with p.

Other congruences involving the bi^{s}nomial coefficients can be found in the work of Belbachir and Otmani [7, 8], where the authors establish results analogous to Wolstenholme's congruence, as well as those of Morley and Glaisher.

Arithmetic properties of Delannoy numbers themselves have also received considerable attention. Guo and Zeng [15] obtained congruences for sums involving central Delannoy

numbers, and Sun [22, 23, 24] established numerous congruences and supercongruences for central Delannoy numbers, Delannoy polynomials and generalized central trinomial coefficients. In contrast with these works, which mainly concern the central column $D_n = \binom{2n}{n}_{[2]}$ of the triangle, the results of the present paper deal with the entries $\binom{n}{k}_{[2]} = D(k, n-k)$ of the whole Delannoy triangle.

Our aim is to prove some congruences involving the Delannoy triangle coefficients. Motivated by the previous results, we begin by establishing congruences similar to Wolstenholme's congruence, followed by proving a double sum congruence similar to the identity of Apagodu and Liu (10).

2 Wolstenholme-type congruences for Delannoy triangle coefficients

The first result consists of the following two congruences, which are similar to Wolstenholme's theorem.

Theorem 1. *For all prime $p \geq 5$, we have*

$$\binom{2p}{p}_{[2]} \equiv 3 + 2pq_p(2) - p^2 q_p(2)^2 \pmod{p^3}, \tag{14}$$

$$\binom{2p-1}{p-1}_{[2]} \equiv 1 + 2pq_p(2) - p^2 q_p(2)^2 \pmod{p^3}, \tag{15}$$

where $q_p(2) = \big(2^{p-1} - 1\big)/p$ is the Fermat quotient.

Remark 2. By (2), we have $\binom{2p}{p}_{[2]} = D_p$, so that congruence (14) determines the central Delannoy number D_p modulo p^3; in particular, it refines the immediate consequence $D_p \equiv 3 \pmod{p}$. The known arithmetic results of Sun on Delannoy numbers [22, 23, 24] concern sums of products $\sum_{k=0}^{p-1} D_k(x) s_{k+1}(x)$ and sums such as $\sum_{k=1}^{p-1} D_k/k^2$, rather than an individual coefficient D_p; to the best of our knowledge, the explicit residues modulo p^3 given in Theorems 1 and 4, expressed in terms of the Fermat quotient $q_p(2)$, do not appear in the literature. The proofs, however, rely on the known congruences for harmonic-type sums recalled in Lemmas 6 and 7, so that the novelty lies in the resulting formulas rather than in the method.

Remark 3. Congruence (14) fails for $p = 3$: indeed, $\binom{6}{3}_{[2]} = D_3 = 63 \equiv 9 \pmod{27}$, while the right-hand side of (14) equals $3 + 6 - 9 \equiv 0 \pmod{27}$, since $q_3(2) = 1$. On the other hand, congruence (15) does hold for $p = 3$, as $\binom{5}{2}_{[2]} = 25 \equiv 1 + 6 - 9 \pmod{27}$, but our proof requires $p \geq 5$.

This second congruence is inspired by the Glaisher congruence (9) and (12).

Theorem 4. *Let $p \geq 5$ be a prime and n a positive integer. Then*

$$\binom{np-1}{p-1}_{[2]} \equiv 1 + (n-1)pq_p(2)\big(2 - pq_p(2)\big) \pmod{p^3}, \tag{16}$$

where $q_p(2)$ is the Fermat quotient.

For the proofs we need some results.

Lemma 5. *Let s be an integer and let p be an odd prime. Then, for each $k = 1, 2, \ldots, p-1$,*

$$\binom{sp-1}{k} \equiv (-1)^k\big(1 - spH_k + \frac{s^2p^2}{2}\big(H_k^2 - H_{k,2}\big)\big) \pmod{p^3}, \tag{17}$$

where

$$H_n := \sum_{1\leq k\leq n} \frac{1}{k} \quad \textit{and } H_{n,2} := \sum_{1\leq k\leq n} \frac{1}{k^2} \quad \textit{for } n \in \mathbb{N}^*, \textit{ with } H_0 = H_{0,2} = 0,$$

are the harmonic numbers (of order one and two, respectively).

Proof. For a fixed $1 \leq k \leq p-1$ we have

$$\begin{aligned}(-1)^k\binom{sp-1}{k} &= \prod_{i=1}^{k}\Big(1 - \frac{sp}{i}\Big) \equiv 1 - \sum_{i=1}^{k}\frac{sp}{i} + \sum_{1\leq i<j\leq k}\frac{s^2p^2}{ij} \pmod{p^3}\\ &= 1 - spH_k + \frac{s^2p^2}{2}\bigg(\Big(\sum_{i=1}^{k}\frac{1}{i}\Big)^2 - \sum_{i=1}^{k}\frac{1}{i^2}\bigg)\\ &= 1 - spH_k + \frac{s^2p^2}{2}\big(H_k^2 - H_{k,2}\big) \pmod{p^3},\end{aligned}$$

whence, we have (17). □

Lemma 6. *[21, Theorem 4.1]. If $p > 3$ is a prime, then*

$$\sum_{k=1}^{p-1}\frac{2^k}{k} \equiv -2q_p(2) \pmod{p^2}, \tag{18}$$

$$\sum_{k=1}^{p-1}\frac{2^k}{k^2} \equiv -q_p(2)^2 \pmod{p}. \tag{19}$$

Lemma 7. *[16]. Let $p \geq 5$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{H_k 2^k}{k} \equiv -q_p(2)^2 \pmod{p}, \tag{20}$$

where H_k is the harmonic number.

Remark 8. Meštrović's result [16] is stated for primes $p > 5$. The remaining case $p = 5$ of (20) is checked directly:

$$\sum_{k=1}^{4} \frac{H_k 2^k}{k} = 2 + 3 + \frac{44}{9} + \frac{25}{3} \equiv 2 + 3 + 1 + 0 \equiv 1 \equiv -q_5(2)^2 \pmod 5,$$

since $q_5(2) = 3$.

Thus, we now proceed with the proof of the first Theorem 1.

Proof of Theorem 1. From Lemma 5, we have

$$\binom{p}{k} = \frac{p}{k}\binom{p-1}{k-1} \equiv p\frac{(-1)^{k-1}}{k} \pmod{p^2},\ k \in \{1, \ldots, p-1\}.$$

Then,

$$\binom{p}{k}^2 \equiv \frac{p^2}{k^2} \pmod{p^3},\ k \in \{1, \ldots, p-1\}.$$

It follows from (4) that

$$\binom{2p}{p}_{[2]} = \sum_{k=0}^{p} \binom{p}{k}^2 2^k \equiv 1 + 2^p + p^2 \sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod{p^3}.$$

Substituting (19) (Lemma 6) and using the fact that $2^{p-1} = 1 + pq_p(2)$, we get (14).

Also from (4), we have $\binom{2p-1}{p-1}_{[2]} = \sum_{k=0}^{p-1} \binom{p}{k}\binom{p-1}{k} 2^k$, and from Lemma 5, we get

$$\binom{p}{k}\binom{p-1}{k} \equiv -\frac{p}{k} + \frac{p^2}{k}\big(H_k + H_{k-1}\big) \pmod{p^3}, \tag{21}$$

whence taking $H_{k-1} = H_k - \frac{1}{k}$ and inserting the congruences (18)–(19) and (20) from Lemmas 6 and 7, we obtain (15). □

Now, we prove the second Theorem 4.

Proof of Theorem 4. We have from the explicit formula (4), using the absorption identity $\binom{p(n-1)}{j} = \frac{(n-1)p}{j}\binom{p(n-1)-1}{j-1}$ for $j \geq 1$ and separating the term $j = 0$,

$$\begin{aligned}\binom{np-1}{p-1}_{[2]} &= \sum_{j=0}^{p-1} \binom{p-1}{j}\binom{p(n-1)}{j} 2^j \\ &= 1 + (n-1)p \sum_{j=1}^{p-1} \binom{p-1}{j}\binom{p(n-1)-1}{j-1}\frac{2^j}{j}.\end{aligned}$$

Then from Lemma 5 we obtain

$$\begin{aligned}\binom{np-1}{p-1}_{[2]} &\equiv 1+(n-1)p\sum_{j=1}^{p-1}\Big(-1+(n-1)pH_{j-1}+pH_j\Big)\frac{2^j}{j} \pmod{p^3}\\ &= 1-(n-1)p\sum_{j=1}^{p-1}\frac{2^j}{j}+(n-1)^2p^2\sum_{j=1}^{p-1}\frac{H_{j-1}2^j}{j}+(n-1)p^2\sum_{j=1}^{p-1}\frac{H_j2^j}{j}.\end{aligned}$$

Combining Lemma 7 and Lemma 6 with $H_{j-1}=H_j-\frac{1}{j}$, which gives in particular $\sum_{j=1}^{p-1}\frac{H_{j-1}2^j}{j}\equiv 0 \pmod{p}$, we arrive at the desired result. □

3 Congruence on double sum of Delannoy coefficients

Our next result is motivated by the congruence (10) in the following theorem:

Theorem 9. *Let p be an odd prime. Then*

$$\sum_{0\le k\le n\le p-1}\binom{n}{k}_{[2]}\equiv\left(\frac{2}{p}\right) \pmod{p}, \tag{22}$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol.

Remark 10. As kindly pointed out by the anonymous referee, Theorem 9 admits an alternative short proof via Pell numbers. Indeed, the row sums of the Delannoy triangle are the Pell numbers,

$$\sum_{k=0}^{n}\binom{n}{k}_{[2]}=P_{n+1},\qquad \text{where } P_0=0,\ P_1=1 \text{ and } P_{r+1}=2P_r+P_{r-1},$$

see Mu and Zheng [18] (for the Pell sequence, see Bicknell [9]). Hence

$$\sum_{0\le k\le n\le p-1}\binom{n}{k}_{[2]}=\sum_{r=1}^{p}P_r=\frac{P_p+P_{p+1}-1}{2},$$

and the classical Lucas-sequence congruences $P_p\equiv\left(\frac{2}{p}\right)$ and $P_{p+1}\equiv 1+\left(\frac{2}{p}\right) \pmod{p}$ yield the result at once. The proof given below, based on generalized central trinomial coefficients, is of a different nature and may be of independent interest.

To establish this theorem, we require specific definitions and lemmas.

The central Delannoy numbers (see [10]) (A001850, OEIS [19]) are defined by

$$D_n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k},\quad n\in\mathbb{N}.$$

More generally, the Delannoy polynomials (see [22]) are defined by

$$D_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k, \quad n \in \mathbb{N},$$

so that $D_n = D_n(1)$. They satisfy $D_n(x) = P_n(2x+1)$, where P_n denotes the nth Legendre polynomial, see [22].

Given $b, c \in \mathbb{Z}$, Sun, [23], defined the generalized central trinomial coefficients by

$$\begin{aligned} T_n(b,c) := \left[x^n\right]\left(x^2+bx+c\right)^n &= \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k \\ &= \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{k}\binom{n}{k} b^{n-2k}c^k. \end{aligned}$$

In particular, $D_n(x) = T_n(2x+1,\, x^2+x)$, see [23].

Lemma 11. *Let p be a prime number and let j and k be integers with $0 \le j \le k \le p-2$. Then*

$$\binom{p-j}{k+1} + \binom{k+j}{k+1}(-1)^k \equiv 0 \pmod{p}. \tag{23}$$

Proof. By using the general binomial coefficient, see [20, Equation 1.2, p. 2], and since $k+1 \le p-1$, so that $(k+1)!$ is invertible modulo p, we obtain

$$\binom{p-j}{k+1} \equiv \binom{-j}{k+1} \pmod{p}.$$

And from the binomial identity (-1)–transformation:

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k},$$

see in [20, Equation 1.6, p. 3], then the proof of Lemma 11 is obtained. □

Lemma 12. *[23, Theorem 1.2(i)] Let p be an odd prime and let $b, c \in \mathbb{Z}$. For any integer $m \not\equiv 0 \pmod p$, we have*

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)}{m^k} \equiv \left(\frac{(m-b)^2-4c}{p}\right) \pmod{p}. \tag{24}$$

Lemma 13. *For every integer $k \ge 0$, we have*

$$\sum_{j=0}^{k} \binom{k}{j}\binom{k+j}{k+1} = \frac{D_{k+1} - 3D_k}{4}. \tag{25}$$

Proof. For $k=0$ both sides vanish, so let $k \geq 1$. Since $\binom{k+j}{k+1} = \frac{j}{k+1}\binom{k+j}{j}$, the left-hand side of (25) equals

$$\frac{1}{k+1}\sum_{j=0}^{k} j\binom{k}{j}\binom{k+j}{j} = \frac{D_k'(1)}{k+1}.$$

From $D_k(x) = P_k(2x+1)$ and the classical Legendre relation $(x^2-1)P_k'(x) = k\big(xP_k(x) - P_{k-1}(x)\big)$, see, e.g., [25], we get

$$D_k'(1) = 2P_k'(3) = \frac{k\big(3D_k - D_{k-1}\big)}{4}.$$

Moreover, the three-term Legendre recurrence $(k+1)P_{k+1}(x) = (2k+1)xP_k(x) - kP_{k-1}(x)$, evaluated at $x=3$, gives $(k+1)D_{k+1} = 3(2k+1)D_k - kD_{k-1}$, whence

$$k\big(3D_k - D_{k-1}\big) = (k+1)\big(D_{k+1} - 3D_k\big),$$

and the result follows. The sequence $\big(\frac{D_{k+1}-3D_k}{4}\big)_{k\geq 0}$ is A050151 in the OEIS [19]. □

Now, we are able to proceed to the proof of Theorem 9.

Proof of Theorem 9. We have from (3)

$$\begin{aligned}
\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}_{[2]} &= \sum_{n=0}^{p-1}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\binom{n-j}{k} \\
&= \sum_{k=0}^{p-1}\sum_{n=k}^{p-1}\sum_{j=0}^{k}\binom{k}{j}\binom{n-j}{k} \\
&= \sum_{k=0}^{p-1}\sum_{j=0}^{p-1}\sum_{n=\max(j,k)}^{p-1}\binom{k}{j}\binom{n-j}{k} \\
&= \sum_{k=0}^{p-1}\sum_{j=0}^{p-1}\binom{k}{j}\sum_{n=k+j}^{p-1}\binom{n-j}{k} \\
&= \sum_{k=0}^{p-1}\sum_{j=0}^{p-1}\binom{k}{j}\sum_{n=0}^{p-1-k-j}\binom{n+k}{k},
\end{aligned}$$

where we used the fact that $\binom{n-j}{k} = 0$ for $\max(j,k) \leq n < k+j$.

In [13, Identity (1.48)], we have $\sum_{k=0}^{n}\binom{k+s}{r} = \binom{n+s+1}{r+1} - \binom{s}{r+1}$, for $s, r, n \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}_{[2]} &= \sum_{k=0}^{p-1}\sum_{j=0}^{k}\binom{p-j}{k+1}\binom{k}{j} \\
&= \sum_{k=0}^{p-2}\sum_{j=0}^{k}\binom{p-j}{k+1}\binom{k}{j} + \sum_{j=0}^{p-1}\binom{p-j}{p}\binom{p-1}{j}.
\end{aligned}$$

In the last sum, $\binom{p-j}{p} = 0$ for $1 \le j \le p-1$, so that it reduces to its term $j = 0$, namely $\binom{p}{p}\binom{p-1}{0} = 1$. Applying Lemma 11 to the first sum (its indices satisfy $0 \le j \le k \le p-2$), and then Lemma 13, we obtain

$$\begin{aligned}\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}_{[2]} &\equiv 1 + \sum_{k=0}^{p-2}\sum_{j=0}^{k}(-1)^{k+1}\binom{k+j}{k+1}\binom{k}{j} \pmod p \\ &= 1 + \sum_{k=0}^{p-2}(-1)^{k+1}\frac{D_{k+1} - 3D_k}{4}.\end{aligned}$$

We have from [22, Remark 1.2], $D_n(-x-1) = (-1)^n D_n(x)$.

For $x = 1$, then $(-1)^n D_n = D_n(-2)$ and since $D_n(x) = T_n(2x+1, x^2+x)$, thus $D_n(-2) = T_n(-3, 2)$.

Applying Lemma 12 with $m = 1$, $b = -3$ and $c = 2$, for which $(m-b)^2 - 4c = 16 - 8 = 8$ and $\left(\frac{8}{p}\right) = \left(\frac{2}{p}\right)$, we obtain

$$\sum_{k=0}^{p-1} T_k(-3, 2) \equiv \left(\frac{2}{p}\right) \pmod p \text{ then } \sum_{k=0}^{p-1}(-1)^k D_k \equiv \left(\frac{2}{p}\right) \pmod p.$$

Then, after reindexing the first sum,

$$\begin{aligned}\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}_{[2]} &\equiv 1 + \sum_{k=0}^{p-2}(-1)^{k+1}\frac{D_{k+1} - 3D_k}{4} \pmod p \\ &\equiv \frac{3}{4} + \left(\frac{2}{p}\right) - \frac{3}{4}(-1)^{p-1}D_{p-1} \pmod p \\ &\equiv \left(\frac{2}{p}\right) \pmod p,\end{aligned}$$

since $D_{p-1} \equiv 1 \pmod p$. Indeed, in $D_{p-1} = \sum_{j=0}^{p-1}\binom{p-1}{j}\binom{p-1+j}{j}$, every term with $j \ge 1$ is divisible by p, because the product $(p-1+1)\cdots(p-1+j)$ contains the factor p while $j!$ is prime to p. □

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