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Meta-Automatic Sequences

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Abstract

Nested (or *meta-Fibonacci*) recurrences, such as the recurrence used to define Hofstadter's Q-sequence, along with the *digit-based* recurrences that underlie automatic sequences are of interest from both number-theoretic and combinatorial points of view. In this direction, Allouche and Shallit showed how the frequency sequence of a variant of the Q-sequence is 2-automatic. This inspires us to introduce what may be seen as a natural combination of the recurrences for meta-Fibonacci and automatic sequences, by introducing the concept of a *meta-automatic sequence*. We exhibit two binary meta-automatic sequences $\mathcal{M}_1$ and $\mathcal{M}_2$ whose defining recurrences do not satisfy the Allouche-Shallit automaticity criterion directly, and this is formalized in our paper. For each of these integer sequences $\mathcal{M}_1$ and $\mathcal{M}_2$, we prove explicit DFAO evaluations, together with 4-uniform morphisms, and we also consider the factor complexities of these sequences.

1 Introduction

A *nested recurrence* is a recurrence in which a term $a(n)$ is defined using values of a at indices that themselves depend on earlier values of the sequence. The classical example is Hofstadter's Q-sequence [20, p. 137], indexed in the On-Line Encyclopedia of Integer Sequences [30] as A005185, defined by $Q(1) = Q(2) = 1$ and

$$Q(n) = Q(n - Q(n - 1)) + Q(n - Q(n - 2)) \tag{1}$$

for $n > 2$. Recurrences as in (1) are also known as *meta-Fibonacci recurrences*. Other examples of integer sequences satisfying nested recurrences include the sequence A244477 studied by Golomb [18]. Many problems concerning Hofstadter's Q-sequence and its generalizations remain open to this day. Notably, it is not known whether or not $Q(n)$ is even *defined* for all $n > 0$.

In contrast to nested recurrences, *digit-based recurrences* (also called *divide-and-conquer recurrences*) define $a(k^t n + j)$ for $j \in \{0, 1, \ldots, k^t - 1\}$ as a function of previously computed values $a(k^{t'} n' + j')$ with $t' < t$ and $j' \in \{0, 1, \ldots, k^{t'} - 1\}$. Digit-based recurrences can be used to define *automatic sequences* [3, 9], which appear in many areas of number theory, computer science, and combinatorics.

Allouche and Shallit [4] showed that the frequency sequence of a variant of Hofstadter's Q-sequence satisfies a digit-based recurrence, and, moreover, is 2-automatic. They also proved an automaticity criterion [4, Theorem 2.2] reviewed in Section 1.2 below. This inspires us to introduce the concept of a *meta-automatic sequence*, which may, informally, be thought of as "combining" the concepts of meta-Fibonacci and automatic sequences. This is formalized below. For example, in contrast to a rule such as $a(4n + 2) = a(2n)$ used to define an automatic sequence, we consider rules such as

$$a(4n + 2) = a(2n + 1 - a(n)), \tag{2}$$

in which the argument of a on the right-hand side depends on the value $a(n)$. A recurrence of the form suggested in (2) would not, in general, satisfy the Allouche-Shallit automaticity criterion, since the argument on the right is not of the required polynomial form.

The idea of combining automatic and meta-Fibonacci recurrences is motivated by past work on meta-Fibonacci sequences, in addition to the above referenced work by Allouche and Shallit [4]. We point to references on the Hofstadter Q-sequence [16, 17, 19, 27], on variants and generalizations of the Hofstadter Q-sequence [1, 2, 5, 12, 14, 32], and on meta-Fibonacci sequences [7, 8, 10, 11, 13, 15, 21, 22, 23, 28, 31].

1.1 Outline

A formal definition of a meta-automatic sequence is given in Section 2. A meta-automatic recurrence may, after simplification, reduce to a digit-based recurrence, in which case the Allouche-Shallit criterion applies directly. We formalize, in this paper, how meta-automatic sequences are not necessarily reducible in this way.

A binary sequence $a : \mathbb{N}_0 \to \{0,1\}$ is said to be *balanced* if

$$a(2n) + a(2n+1) = 1 \qquad (n \geq 0). \tag{3}$$

The binary sequences under consideration in this paper are, for the most part, balanced. In a related way, much of this paper makes use of the additive operation underlying the field $\mathbb{F}_2 = \{0,1\}$, letting $\oplus$ denote this operation, which may be referred to as *XOR*, with $0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1 \oplus 0 = 1$. For balanced binary sequences, Lemma 5 gives the identity

$$a(2n + 1 - a(n)) = a(2n + 1) \oplus a(n).$$

We prove that balanced, binary, meta-automatic sequences are, for the cases covered in Lemma 6, ultimately periodic. To obtain non-periodic examples, we therefore work with base-4 recurrences. We consider:

- The integer sequence $\mathcal{M}_1$ (A392736), defined via a base-4 recurrence in which one of the rules in the recurrence is nested in the sense outlined above. After applying Lemma 5 below, the pair $(\mathcal{M}_1(n), b(n))$, for $b(n)$ defined in (8) below, yields a 4-state DFAO (Section 3).
- The integer sequence $\mathcal{M}_2$ (A391614), defined via a base-4 recurrence in which *both* of the rules in the recurrence are nested in the sense outlined above. After applying Lemma 5, the pair $\big(\mathcal{M}_2(n), \mathcal{M}_2(2n + \mathcal{M}_2(n))\big)$ obeys an affine rule over $\mathbb{F}_2^2$ with linear part $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. The sequence satisfies $\mathcal{M}_2(n) = \mathbf{t}(q(n))$, where q is an explicit bit-masking operator and $\mathbf{t}$ is the Thue-Morse sequence (Theorem 26).

Although $\mathcal{M}_1$ and $\mathcal{M}_2$ both have 4-state DFAOs, we have that $\mathcal{M}_2$ admits the evaluation $\mathcal{M}_2(n) = \mathbf{t}(q(n))$, whereas $\mathcal{M}_1$ is described by means of the pair $(\mathcal{M}_1(n), b(n))$. Their factor-complexity functions, studied in Section 5, also differ. For each of the sequences $\mathcal{M}_1$ and $\mathcal{M}_2$, we give a base-4 DFAO (MSB-first) and an explicit 4-uniform morphism, and we also investigate the factor complexity functions for $\mathcal{M}_1$ and $\mathcal{M}_2$ in Section 5.

1.2 Preliminaries

For background on automatic sequences, we refer to the texts of Allouche and Shallit [3] and of Shallit [29]. We proceed with the following definition, which may be used, as below, to define automatic sequences.

Definition 1. Let $U = (U(n))_{n \geq 0}$ denote an infinite sequence. The *k-kernel* of $(U(n))_{n \geq 0}$ [3, p. 185] is

$$K_k(U) = \left\{ \left(U_{k^i n + j}\right)_{n \geq 0} : i \geq 0 \text{ and } 0 \leq j < k^i \right\}.$$

For $k \geq 2$, a *k-automatic sequence* is a sequence $U = (U(n))_{n\geq 0}$ such that $K_k(U)$ is finite. Equivalently, the sequence U is k-automatic if there exists a DFAO such that, upon inputting the base-k representation of n, this DFAO outputs $U(n)$. See Allouche and Shallit's text [3] for details. A prototypical instance of an automatic sequence is the *Thue-Morse sequence* $\mathbf{t} : \mathbb{N}_0 \to \{0, 1\}$ defined so that $\mathbf{t}(0) = 0$ and

$$\mathbf{t}(2n) = \mathbf{t}(n), \qquad \mathbf{t}(2n + 1) = 1 - \mathbf{t}(n) \qquad (n \geq 0).$$

Equivalently, $\mathbf{t}(n) = s_2(n) \bmod 2$, where $s_2(n)$ is the sum of the binary digits of n, noting that the balanced condition in (3) holds. We introduce, in this paper, a variant of an automaticity criterion due to Allouche and Shallit [4], which we now recall.

Let $U = (U(n))_{n\geq 0}$ denote a sequence, and let α denote an integer. Adopting notation from Allouche and Shallit's work [4], for $n \geq -\alpha$, we write U^α in place of the sequence given by

$$U^\alpha := U(n + \alpha).$$

Similarly, letting q and i and j be positive integers, we write $U_{q,i,j}$ in place of the sequence defined by

$$U_{q,i,j}(n) = U\left(q^i n + j\right) \tag{4}$$

for $n \geq 0$. Moreover, for integers $q \geq 2$ and $t \geq 1$, we write $s(q, t) = \frac{1-q^{t+1}}{1-q}$ in place of the number of pairs (i, j) of integers i and j such that $0 \leq i \leq t$ and $0 \leq j \leq q^i - 1$. Fix an ordering of these pairs (i, j), with $(0, 0)$ appearing first, and write the corresponding sequences as $U_1 = U$, $U_2, \ldots, U_{s(q,t)}$. The choice of ordering plays no role in what follows. We also let $m(q, t) = q^{t+1} - 1$.

Theorem 2. *(Allouche and Shallit, 2012) Let the terms of $(U(n))_{n\geq 0}$ be in a finite set $\mathcal{A}$, and let $q \geq 2$ denote an integer. Then U is q-automatic if there are nonnegative integers t, a, b, and n_0 together with a family $\{f_j : j = 0, 1, \ldots, m(q, t)\}$ of functions from $\mathcal{A}^{a+b+s(q,t)}$ to $\mathcal{A}$ such that*

$$U\big(q^{t+1}n + j\big) = f_j\big(U^{-a}(n), \ldots, U^{-1}(n), U^0(n), U^1(n), \ldots, U^b(n), U_2(n), U_3(n), \ldots, U_{s(q,t)}(n)\big)$$

for all $n \geq n_0$ and for all $j \in \{0, 1, \ldots, m(q, t)\}$ [4].

2 Meta-automatic sequences

Let $q > 0$, $i_0 > 0$, and $j_0 \geq 0$ be integers. For some $v \geq 0$, and for indices $k \in \{1, 2, \ldots, v\}$, let c_k, $i_k > 0$, and $j_k \geq 0$ be integers, with $i_k < i_0$ for all k. By analogy with (4), and assuming that the codomain of U is contained in $\mathbb{N}_0$, we write

$$U_{q,i_0,j_0,c_1,i_1,j_1,\ldots,c_v,i_v,j_v}(n) = U\Big(q^{i_0}n + j_0 + c_1U(q^{i_1}n + j_1) + \cdots + c_vU(q^{i_v}n + j_v)\Big).$$

The following definition may be thought of as formalizing the intuition behind recurrences as in (2).

Definition 3. Suppose that each term of $(U(n))_{n\geq 0}$ is in a finite subset $\mathcal{A}$ of $\mathbb{N}_0$, and let $q \geq 2$ be an integer. The sequence U is *q-meta-automatic* if there exist nonnegative integers t, a, b, n_0 together with a family $\{f_j : j = 0, 1, \ldots, m(q,t)\}$ of functions to $\mathcal{A}$, with domains as below, such that, for all $n \geq n_0$ and all $j \in \{0, 1, \ldots, m(q,t)\}$, the value $U(q^{t+1}n + j)$ equals f_j evaluated at the tuple

$$\Big(U^{-a}(n),\, \ldots,\, U^{-1}(n),\, U^0(n),\, U^1(n),\, \ldots,\, U^b(n),\\ U_{q,i_0^{(1)},j_0^{(1)},c_1^{(1)},i_1^{(1)},j_1^{(1)},\ldots,c_{v_1}^{(1)},i_{v_1}^{(1)},j_{v_1}^{(1)}}(n),\, \ldots,\\ U_{q,i_0^{(w)},j_0^{(w)},c_1^{(w)},i_1^{(w)},j_1^{(w)},\ldots,c_{v_w}^{(w)},i_{v_w}^{(w)},j_{v_w}^{(w)}}(n)\Big)$$

for some $w \geq 0$ and integers $i_{k_1}^{(k_2)} \leq t$, $j_{k_1}^{(k_2)}$, and $c_{k_1}^{(k_2)}$, where the domain of f_j is $\mathcal{A}^{a+b+w+1}$.

Example 4. Define $(\mathcal{T}(n))_{n\geq 0}$ by $\mathcal{T}(0) = 0$ and

$$\mathcal{T}(2n) = \mathcal{T}(n - \mathcal{T}(n-1)) \text{ for } n \geq 1, \text{ and}$$
$$\mathcal{T}(2n+1) = 1 - \mathcal{T}(n) \text{ for } n \geq 0.$$

Then $(\mathcal{T}(n))_{n\geq 0}$ is 2-meta-automatic in the sense of Definition 3. For $n \geq 1$, this sequence agrees with the OEIS sequence A039982, which was defined without nested recurrences.

Lemma 5. *If a is balanced, then*

$$a(2n + a(n)) = a(2n) \oplus a(n) \quad \text{and} \quad a(2n + 1 - a(n)) = a(2n+1) \oplus a(n),$$

for all $n \geq 0$.

Proof. The first identity follows from (3) by case analysis on $a(n) \in \{0, 1\}$, and similarly for the second. □

2.1 Base-4 recurrences

We now record a result that motivates the use of base-4 recurrences in the sequel. We let

$$a(2n) = a(n + a(n)) \qquad \text{or} \qquad a(2n) = a(n + 1 - a(n)),$$

and we let $a(2n+1) = 1 - a(2n)$. In both cases, we obtain ultimate periodicity, as we now show.

Lemma 6. *Let $a : \mathbb{N}_0 \to \{0, 1\}$ be balanced.*

(a) If $a(2n) = a(n + a(n))$ for all $n \geq 0$, then $a(0) = 0$ and a is ultimately 2-periodic, with

$$a(2n) = 0, \qquad a(2n+1) = 1 \qquad (n \geq 2).$$

(b) If $a(2n) = a(n + 1 - a(n))$ for all $n \geq 0$, then $a(0) = 1$ and a is ultimately 2-periodic, with

$$a(2n) = 1, \qquad a(2n+1) = 0 \qquad (n \geq 2).$$

Proof.

(a) Taking $n = 0$ gives $a(0) = a(a(0))$. If $a(0) = 1$, then balance gives us that $a(1) = 0$, and hence $a(0) = a(1) = 0$, a contradiction. Thus $a(0) = 0$ and $a(1) = 1$.

Write $b_m = a(2m)$, so that balance gives that $a(2m+1) = 1 - b_m$. Using $a(2n) = a(n + a(n))$ with $n = 2m$, we obtain

$$b_{2m} = a(4m) = a(2m + a(2m)) = a(2m + b_m) = \begin{cases} a(2m) = b_m, & \text{if } b_m = 0; \\ a(2m+1) = 1 - b_m = 0, & \text{if } b_m = 1. \end{cases}$$

So we have that $b_{2m} = 0$ for all $m \geq 0$.

Now, using $a(2n) = a(n + a(n))$ with $n = 2m + 1$ and $a(2m+1) = 1 - b_m$, we get

$$b_{2m+1} = a(4m+2) = a(2m+1+a(2m+1)) = a(2m+2-b_m) = \begin{cases} a(2m+2) = b_{m+1}, & \text{if } b_m = 0; \\ a(2m+1) = 0, & \text{if } b_m = 1. \end{cases}$$

We claim that $b_m = 0$ for all $m \geq 2$. This holds for $m = 2$, since $b_2 = 0$, and for $m = 3$ because the above identity with $m = 1$ gives $b_3 = b_2$ if $b_1 = 0$, and $b_3 = 0$ if $b_1 = 1$; in either case $b_3 = 0$ since $b_2 = 0$. Now let $m \geq 4$. If m is even then $b_m = b_{2r} = 0$. If m is odd, write $m = 2r + 1$ with $r \geq 1$. If $b_r = 1$ then $b_m = 0$; if $b_r = 0$ then $b_m = b_{r+1}$, and $r + 1 < m$, so by induction $b_{r+1} = 0$. Hence $b_m = 0$ for all $m \geq 2$, i.e., $a(2n) = 0$ for all $n \geq 2$. The balance condition in (3) then gives $a(2n+1) = 1$ for all $n \geq 2$.

(b) This follows from (a) by complementing: if a satisfies (b) then $\bar{a}(n) = 1 - a(n)$ satisfies (a), and vice versa.

□

Lemma 6 leads us to consider base-4 recurrences such that the balance identity in (3) is preserved.

2.2 A non-nested case

Fix $a(0) = 0$ and $a(1) = 1$, and define

$$a(4n) = F(n), \quad a(4n+1) = 1 - F(n), a(4n+2) = G(n), \quad a(4n+3) = 1 - G(n), \qquad (5)$$

so that $a(2n)+a(2n+1) = 1$ holds automatically. The sequences studied in this paper arise as specializations of (5). We first consider a sequence $(\mathcal{Q}(n))_{n\geq 0}$ whose recurrence is not nested, but which is used in the sequel.

Definition 7. The *Thue-Morse Quarto* sequence $\mathcal{Q} : \mathbb{N}_0 \to \{0, 1\}$ (A298952, where we use the convention $\mathcal{Q}(0) = 0$ rather than $a(0) = 1$ as in the OEIS entry) is defined by $\mathcal{Q}(0) = 0$, $\mathcal{Q}(1) = 1$, and

$$\begin{aligned} \mathcal{Q}(4n) &= \mathcal{Q}(n), & \mathcal{Q}(4n+1) &= 1 - \mathcal{Q}(n), \\ \mathcal{Q}(4n+2) &= \mathcal{Q}(2n), & \mathcal{Q}(4n+3) &= 1 - \mathcal{Q}(2n). \end{aligned}$$

The sequence $\mathcal{Q}$ is 4-automatic. It is computed by the 3-state DFAO with states in $\{A, B, C\}$, initial state A, and output $\tau(A) = \tau(B) = 0$, $\tau(C) = 1$, shown in Figure 1. This DFAO corresponds to the 4-uniform morphism

$$A \mapsto ACBC, \quad B \mapsto BCCB, \quad C \mapsto CBBC,$$

along with a coding such that $\tau(A) = \tau(B) = 0$ and $\tau(C) = 1$.

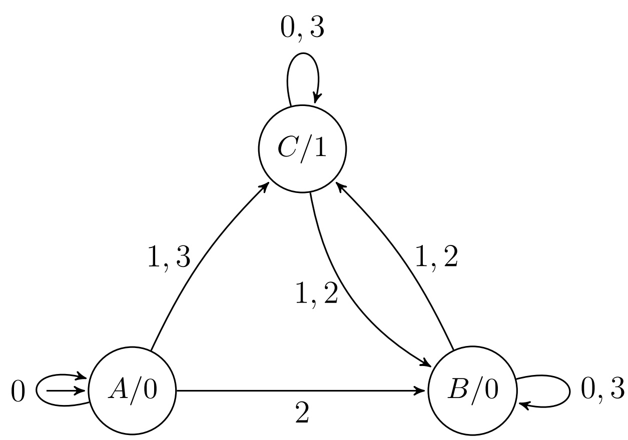

Figure 1: DFAO for $\mathcal{Q}$ (MSB-first). State labels are *state/output*. The initial state is A.

Proposition 8. *Define* $d : \mathbb{N}_0 \to \{0, 1\}$ *so that* $d(0) = 0$ *and* $d(n) = \lfloor \log_2 n \rfloor \bmod 2$ *for* $n \geq 1$*. Then*

$$\mathcal{Q}(n) = \mathbf{t}(n) \oplus d(n) \qquad (n \geq 0).$$

Proof. The cases $n = 0, 1$ are immediate. For $n \geq 1$ and $r \in \{0, 1, 2, 3\}$, since $\lfloor \log_2(4n+r) \rfloor = 2 + \lfloor \log_2 n \rfloor$, we have $d(4n + r) = d(n)$ for all r. We also use $\mathbf{t}(4n + r) = \mathbf{t}(n) \oplus \mathbf{t}(r)$, and we proceed by induction.

Case $r = 0$: $\mathcal{Q}(4n) = \mathcal{Q}(n) = \mathbf{t}(n) \oplus d(n)$ and $\mathbf{t}(4n) \oplus d(4n) = \mathbf{t}(n) \oplus d(n)$.

Case $r = 1$: $\mathcal{Q}(4n+1) = 1 \oplus \mathcal{Q}(n) = 1 \oplus \mathbf{t}(n) \oplus d(n)$ and $\mathbf{t}(4n+1) \oplus d(4n+1) = (1 \oplus \mathbf{t}(n)) \oplus d(n)$.

Case $r = 2$: $\mathcal{Q}(4n+2) = \mathcal{Q}(2n)$. By induction, $\mathcal{Q}(2n) = \mathbf{t}(2n) \oplus d(2n) = \mathbf{t}(n) \oplus (d(n) \oplus 1) = 1 \oplus \mathbf{t}(n) \oplus d(n)$. Also $\mathbf{t}(4n+2) = \mathbf{t}(2n+1) = 1 \oplus \mathbf{t}(n)$, so $\mathbf{t}(4n+2) \oplus d(4n+2) = (1 \oplus \mathbf{t}(n)) \oplus d(n)$.

Case $r = 3$: $\mathcal{Q}(4n+3) = 1 \oplus \mathcal{Q}(2n) = \mathbf{t}(n) \oplus d(n)$, and $\mathbf{t}(4n+3) \oplus d(4n+3) = \mathbf{t}(n) \oplus d(n)$. □

3 The sequence $\mathcal{M}_1$

Definition 9. Define $\mathcal{M}_1(0) = 0$ and $\mathcal{M}_1(1) = 1$. For $n \geq 1$, set

$$\mathcal{M}_1(4n) = \mathcal{M}_1(n), \qquad \mathcal{M}_1(4n + 1) = 1 - \mathcal{M}_1(n). \tag{6}$$

For $n \geq 0$, set

$$\mathcal{M}_1(4n + 2) = \mathcal{M}_1(2n + 1 - \mathcal{M}_1(n)), \qquad \mathcal{M}_1(4n + 3) = 1 - \mathcal{M}_1(2n + 1 - \mathcal{M}_1(n)). \tag{7}$$

Lemma 10. *The sequence $\mathcal{M}_1(n)$ is well-defined for all $n \geq 0$ and satisfies $\mathcal{M}_1(2n) + \mathcal{M}_1(2n + 1) = 1$.*

Proof. Well-definedness follows by induction on m. For $m = 4n + r$, the right-hand sides in (6) reduce to indices $< m$; for (7), the nested argument $2n + 1 - \mathcal{M}_1(n) \in \{2n, 2n + 1\} < 4n + 2 \leq m$. Balance follows from (5), since $\mathcal{M}_1(4n + 1) = 1 - \mathcal{M}_1(4n)$ and $\mathcal{M}_1(4n + 3) = 1 - \mathcal{M}_1(4n + 2)$. □

Remark 11. Since $\mathcal{M}_1$ is balanced, Lemma 5 gives

$$\mathcal{M}_1(2n + 1 - \mathcal{M}_1(n)) = \mathcal{M}_1(2n + 1) \oplus \mathcal{M}_1(n),$$

so (7) is equivalent to the non-nested relation $\mathcal{M}_1(4n + 2) = \mathcal{M}_1(2n + 1) \oplus \mathcal{M}_1(n)$.

3.1 A 4-state DFAO

Define

$$b(n) := \mathcal{M}_1(2n + 1 - \mathcal{M}_1(n)), \tag{8}$$

and consider the pair $s(n) := (\mathcal{M}_1(n), b(n))$. Informally, the following result gives us that the pair $s(4n + r)$ depends only on $s(n)$, for $r \in \{0, 1, 2, 3\}$ and for $n \geq 1$, disregarding the trivial $n = 0$ case whereby $\mathcal{M}_1(0) = 0$.

Lemma 12. *Let $x = \mathcal{M}_1(n)$ and $y = b(n)$, and let $n \geq 1$. Then*

$$s(4n) = (x, y), \tag{9}$$

$$s(4n+1) = (1 \oplus x,\ x \oplus y), \tag{10}$$

$$s(4n+2) = (y,\ 1 \oplus x), \text{ and} \tag{11}$$

$$s(4n+3) = (1 \oplus y,\ x \oplus y \oplus 1). \tag{12}$$

Proof. By Lemma 10 and Lemma 5, we have $y = b(n) = \mathcal{M}_1(2n+1) \oplus \mathcal{M}_1(n) = (1 - \mathcal{M}_1(2n)) \oplus x$, so that

$$\mathcal{M}_1(2n) = 1 \oplus x \oplus y, \qquad \mathcal{M}_1(2n+1) = x \oplus y.$$

Case $r = 0$. The first coordinate is $\mathcal{M}_1(4n) = \mathcal{M}_1(n) = x$. The latter coordinate satisfies $b(4n) = \mathcal{M}_1(8n + 1 - \mathcal{M}_1(4n)) = \mathcal{M}_1(8n + 1 - x)$. If $x = 0$, then $b(4n) = \mathcal{M}_1(8n+1) = \mathcal{M}_1(4(2n)+1) = 1 - \mathcal{M}_1(2n) = 1 - (1 \oplus x \oplus y) = 1 - (1 \oplus y) = y$. If $x = 1$, then $b(4n) = \mathcal{M}_1(8n) = \mathcal{M}_1(4(2n)) = \mathcal{M}_1(2n) = 1 \oplus x \oplus y = y$. In both cases, we find that (9) holds.

Case $r = 1$. The first coordinate is $\mathcal{M}_1(4n+1) = 1 - x = 1 \oplus x$. The latter coordinate satisfies $b(4n+1) = \mathcal{M}_1(8n+2+x)$. If $x = 0$, then $b(4n+1) = \mathcal{M}_1(8n+2) = \mathcal{M}_1(4(2n)+2) = b(2n)$. Using the property that $b(2n) = \mathcal{M}_1(4n + 1 - \mathcal{M}_1(2n))$, together with $\mathcal{M}_1(2n) = 1 \oplus y$, this yields $b(2n) = \mathcal{M}_1(4n + y)$. So, if $y = 0$, then $\mathcal{M}_1(4n) = x = 0$, and if $y = 1$, then $\mathcal{M}_1(4n+1) = 1 - x = 1$, so that $b(4n+1) = y$. Similarly, if $x = 1$, then $b(4n+1) = \mathcal{M}_1(8n+3) = \mathcal{M}_1(4(2n)+3) = 1 - b(2n)$. With $\mathcal{M}_1(2n) = y$, $b(2n) = \mathcal{M}_1(4n + 1 - y)$, we find that if $y = 0$, then $1 - \mathcal{M}_1(4n+1) = x = 1$, and if $y = 1$, then $1 - \mathcal{M}_1(4n) = 1 - x = 0$, so that $b(4n+1) = 1 \oplus y$. Consequently, we obtain that (10) holds.

Case $r = 2$. The first coordinate is $\mathcal{M}_1(4n+2) = b(n) = y$. The latter coordinate satisfies $b(4n+2) = \mathcal{M}_1(8n+5-y)$. If $y = 0$, then $\mathcal{M}_1(8n+5) = \mathcal{M}_1(4(2n+1)+1) = 1 - \mathcal{M}_1(2n+1) = 1 \oplus (x \oplus y) = 1 \oplus x$. If $y = 1$, then $\mathcal{M}_1(8n+4) = \mathcal{M}_1(4(2n+1)) = \mathcal{M}_1(2n+1) = x \oplus y = 1 \oplus x$. In both cases, we see that (11) holds.

Case $r = 3$. The first coordinate is $\mathcal{M}_1(4n+3) = 1 - y = 1 \oplus y$. The latter coordinate satisfies $b(4n+3) = \mathcal{M}_1(8n+6+y)$. If $y = 0$, then $\mathcal{M}_1(8n+6) = \mathcal{M}_1(4(2n+1)+2) = b(2n+1)$. Since $\mathcal{M}_1(2n+1) = x$, $b(2n+1) = \mathcal{M}_1(4n+3-x)$. So, if $x = 0$, then $\mathcal{M}_1(4n+3) = 1 \oplus y = 1$; if $x = 1$, then $\mathcal{M}_1(4n+2) = y = 0$; so $b(4n+3) = x \oplus y \oplus 1$. If $y = 1$, then $\mathcal{M}_1(8n+7) = \mathcal{M}_1(4(2n+1)+3) = 1 - b(2n+1)$. Since $\mathcal{M}_1(2n+1) = x \oplus 1$, we find that $b(2n+1) = \mathcal{M}_1(4n+2+x)$. So, if $x = 0$, then $1 - \mathcal{M}_1(4n+2) = 1 - y = 0$; if $x = 1$, then $1 - \mathcal{M}_1(4n+3) = y = 1$, so that $b(4n+3) = x \oplus y \oplus 1$. This gives us the desired relation in (12). □

Theorem 13. *The sequence $\mathcal{M}_1$ is 4-automatic and is computed (MSB-first, base 4) by the DFAO with states $\{0, 1, 2, 3\}$, initial state 0, output map $\tau(0) = \tau(2) = 0$, $\tau(1) = \tau(3) = 1$, and transition table*

δ	0	1	2	3
0	0	1	1	2
1	1	2	3	0
2	2	3	0	1
3	3	0	2	3

Proof. Lemma 12 gives four $\mathbb{F}_2$-affine transition rules:

$$(x, y) \xrightarrow{0} (x,\ y), \qquad (x, y) \xrightarrow{1} (1 \oplus x,\ x \oplus y),$$
$$(x, y) \xrightarrow{2} (y,\ 1 \oplus x), \quad (x, y) \xrightarrow{3} (1 \oplus y,\ x \oplus y \oplus 1).$$

Encoding the states as $\mathsf{0} = (0, 1)$, $\mathsf{1} = (1, 1)$, $\mathsf{2} = (0, 0)$, $\mathsf{3} = (1, 0)$, these yield the stated transition table. The initial state is $s(0) = (\mathcal{M}_1(0), b(0)) = (0, \mathcal{M}_1(1)) = (0, 1)$, corresponding to state $\mathsf{0}$. The output reads the first coordinate $x = \mathcal{M}_1(n)$. $\square$

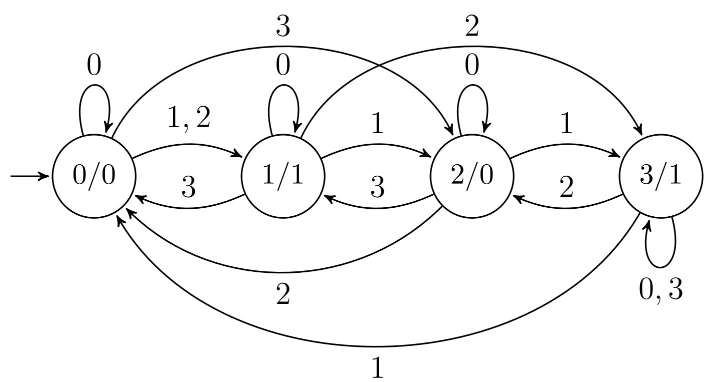

Figure 2: DFAO for $\mathcal{M}_1$ (4 states, base 4, MSB-first). State labels are *state*/*output*. The initial state is $\mathsf{0}$.

Remark 14. The four transition rules of the DFAO of $\mathcal{M}_1$ are $\mathbb{F}_2$-affine in (x, y); the same is true for $\mathcal{M}_2$ (Section 4). For each of the sequences $\mathcal{Q}$, $\mathcal{M}_1$, and $\mathcal{M}_2$, the affine form is a consequence of Lemma 5.

The defining recurrence of $\mathcal{M}_1$ is not directly an instance of Theorem 2, since the argument $2n + 1 - \mathcal{M}_1(n)$ in the nested recurrence in (7) depends on the value $\mathcal{M}_1(n)$. The b-sequence defined in (8) removes this dependence. Furthermore, Lemma 5 gives us that

$$\mathcal{M}_1(4n + 2) = \mathcal{M}_1(2n + 1) \oplus \mathcal{M}_1(n).$$

The XOR identity is a direct consequence of Lemma 5, since $\mathcal{M}_1$ is balanced.

Remark 15. By contrast, Allouche and Shallit [4] show that the frequency sequence $F(n)$ of the Hofstadter variant $V = Q_{1,4}$ satisfies a recurrence of the form $F(2a) = g(F(a-2), F(a-$

$1), F(a), F(a+1))$ with fixed shifts. This satisfies the required conditions in Theorem 2. In our setting, the sequences we consider *themselves*, as opposed to associated frequency sequences, are meta-automatic. In this direction, Lemma 5 provides a key tool, for our purposes.

Corollary 16. *The coding* $\tau\colon \mathsf{0}, \mathsf{2} \mapsto 0$ *and* $\mathsf{1}, \mathsf{3} \mapsto 1$ *applied to the fixed point of the* 4*-uniform morphism*

$$\mathsf{0} \to \mathsf{0112}, \quad \mathsf{1} \to \mathsf{1230}, \quad \mathsf{2} \to \mathsf{2301}, \quad \mathsf{3} \to \mathsf{3023}$$

starting from $\mathsf{0}$ *produces the sequence* $\mathcal{M}_1$*.*

Proof. For each state q, the morphism image is $\delta(q,0)\,\delta(q,1)\,\delta(q,2)\,\delta(q,3)$, read from the transition table of Theorem 13. ☐

4 The sequence $\mathcal{M}_2$

Definition 17. Define $\mathcal{M}_2(0) = 0$ and $\mathcal{M}_2(1) = 1$. For $n \geq 1$, set

$$\mathcal{M}_2(4n) = \mathcal{M}_2(2n + \mathcal{M}_2(n)), \qquad \mathcal{M}_2(4n+1) = 1 - \mathcal{M}_2(4n). \tag{13}$$

For $n \geq 0$, set

$$\mathcal{M}_2(4n+2) = \mathcal{M}_2(2n + 1 - \mathcal{M}_2(n)), \qquad \mathcal{M}_2(4n+3) = 1 - \mathcal{M}_2(4n+2). \tag{14}$$

Theorem 18. *The sequence* $\mathcal{M}_2$ *is well-defined and satisfies* (3)*.*

Proof. Well-definedness follows by induction as in Lemma 10. For the recurrence rules associated with the arguments $4n$ and $4n+1$, we have $n \geq 1$, so $2n + \mathcal{M}_2(n) \leq 2n+1 < 4n$. For the arguments $4n+2$ and $4n+3$, the nested argument satisfies $2n + 1 - \mathcal{M}_2(n) \leq 2n+1 < 4n+2$. Balance is immediate from the complementary definitions in (13)–(14). ☐

Define the pair

$$s(n) := \big(\mathcal{M}_2(n),\, \mathcal{M}_2(2n + \mathcal{M}_2(n))\big) \in \mathbb{F}_2^2. \tag{15}$$

Lemma 19. *If* $s(n) = (x, y)$*, then*

$$\mathcal{M}_2(2n) = x \oplus y, \qquad \mathcal{M}_2(2n+1) = x \oplus y \oplus 1.$$

Proof. If $x = 0$, then $y = \mathcal{M}_2(2n)$; if $x = 1$, then $y = \mathcal{M}_2(2n+1)$, and balance gives that $\mathcal{M}_2(2n) = 1 - y$. The identities follow. ☐

Theorem 20. *Writing* $s(n) = (x, y)$*, we have*

$$\begin{aligned} s(4n) &= (y,\ x \oplus y), \\ s(4n+1) &= (y \oplus 1,\ x \oplus y), \\ s(4n+2) &= (y \oplus 1,\ x \oplus y \oplus 1), \\ s(4n+3) &= (y,\ x \oplus y \oplus 1). \end{aligned}$$

Consequently $\mathcal{M}_2$ *is* 4*-automatic with at most* 4 *states.*

Proof. The case $n = 0$ is immediate from the initial values $\mathcal{M}_2(0) = 0$ and $\mathcal{M}_2(1) = 1$. We may therefore assume $n \geq 1$ whenever a rule for $4n$ or $4n+1$ is used. Write $s(n) = (x, y)$ so that $\mathcal{M}_2(2n) = x \oplus y$ and $\mathcal{M}_2(2n+1) = x \oplus y \oplus 1$.

Case $r = 0$. The first coordinate is given by $\mathcal{M}_2(4n) = \mathcal{M}_2(2n+x)$. By Lemma 19, we have that $\mathcal{M}_2(2n) = x \oplus y$ and $\mathcal{M}_2(2n+1) = x \oplus y \oplus 1$. In both cases, for $x = 0$ or $x = 1$, a direct check gives that $\mathcal{M}_2(2n+x) = y$. For the second coordinate $\mathcal{M}_2(8n+y)$, we see that if $y = 0$, then $\mathcal{M}_2(8n) = \mathcal{M}_2(4(2n)) = \mathcal{M}_2(2 \cdot 2n + \mathcal{M}_2(2n))$, and, since $\mathcal{M}_2(2n) = x \oplus y = x$, this equals $\mathcal{M}_2(4n+x)$. Simplifying both cases gives x. If $y = 1$, then $\mathcal{M}_2(8n+1) = 1 - \mathcal{M}_2(8n)$, and, by the same analysis as before, we get $x \oplus 1$. In both sub-cases the second coordinate equals $x \oplus y$. Therefore, we obtain

$$s(4n) = (y,\ x \oplus y).$$

Case $r = 1$. The first coordinate is given by $\mathcal{M}_2(4n+1) = 1 - \mathcal{M}_2(4n) = 1 - y = y \oplus 1$.

For the second coordinate of $s(4n+1)$, by definition, this is $\mathcal{M}_2(2(4n+1) + \mathcal{M}_2(4n+1)) = \mathcal{M}_2(8n+2+(y \oplus 1))$. When $y = 0$, we have $\mathcal{M}_2(8n+3) = \mathcal{M}_2(4(2n)+3) = 1 - \mathcal{M}_2(2(2n) + 1 - \mathcal{M}_2(2n))$. Since $\mathcal{M}_2(2n) = x$, this equals $1 - \mathcal{M}_2(4n+1-x) = x \oplus y$ for both values of x. When $y = 1$, we have $\mathcal{M}_2(8n+2) = \mathcal{M}_2(2(2n)+1-\mathcal{M}_2(2n))$. Since $\mathcal{M}_2(2n) = x \oplus 1$, this equals $\mathcal{M}_2(4n+x) = x \oplus y$ for both values of x. The second coordinate is therefore $x \oplus y$ in both cases. Therefore

$$s(4n+1) = (y \oplus 1,\ x \oplus y).$$

Case $r = 2$. For the first coordinate $\mathcal{M}_2(4n+2) = \mathcal{M}_2(2n+1-x)$, Lemma 5 gives $\mathcal{M}_2(2n+1) \oplus x = (x \oplus y \oplus 1) \oplus x = y \oplus 1$.

For the second coordinate of $s(4n+2)$, we find that $\mathcal{M}_2(2(4n+2) + \mathcal{M}_2(4n+2)) = \mathcal{M}_2(8n+4+(y \oplus 1))$. Since $\mathcal{M}_2(2n+1) = x \oplus 1$, we have $\mathcal{M}_2(8n+4) = \mathcal{M}_2(4n+3) = 1 - (y \oplus 1)$ for both values of x. When $y = 0$, this gives $\mathcal{M}_2(8n+4) = 0$, so $\mathcal{M}_2(8n+5) = 1 = x \oplus y \oplus 1$. When $y = 1$, since $\mathcal{M}_2(2n+1) = x$, we get $\mathcal{M}_2(8n+4) = \mathcal{M}_2(4n+2+x) = x \oplus y \oplus 1$ by a direct case check. In both cases the second coordinate is $x \oplus y \oplus 1$. It then follows that

$$s(4n+2) = (y \oplus 1,\ x \oplus y \oplus 1).$$

Case $r = 3$. The first coordinate is given by $\mathcal{M}_2(4n+3) = 1 - \mathcal{M}_2(4n+2) = 1 - (y \oplus 1) = y$.

For the second coordinate of $s(4n+3)$, we have that $\mathcal{M}_2(2(4n+3) + \mathcal{M}_2(4n+3)) = \mathcal{M}_2(8n+6+y)$. Since $\mathcal{M}_2(2n+1) = x \oplus 1$, the identity $\mathcal{M}_2(8n+6) = \mathcal{M}_2(2(2n+1) + 1 - \mathcal{M}_2(2n+1))$ gives $\mathcal{M}_2(8n+6) = x \oplus y \oplus 1$ by a direct two-case check on x; this handles $y = 0$. For $y = 1$, since $\mathcal{M}_2(2n+1) = x$, the same formula yields $1 - \mathcal{M}_2(4n+2+x) = x \oplus y \oplus 1$. The second coordinate is $x \oplus y \oplus 1$ in both cases. Consequently, we have that

$$s(4n+3) = (y,\ x \oplus y \oplus 1).$$

Each transition has the form $(x, y) \mapsto A\binom{x}{y} + \mathbf{v}(d)$ over $\mathbb{F}_2$, confirming the affine structure. □

Remark 21. The linear part $(x,y) \mapsto (y, x \oplus y)$ is given by multiplication by the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_2),$$

which has order 3 (indeed $A^3 = I$). The translation vectors are $\mathbf{v}(0) = \binom{0}{0}$, $\mathbf{v}(1) = \binom{1}{0}$, $\mathbf{v}(2) = \binom{1}{1}$, $\mathbf{v}(3) = \binom{0}{1}$.

Theorem 22. *The sequence $\mathcal{M}_2$ is computed (MSB-first, base 4) by the DFAO with states $\{0,1,2,3\}$, initial state 0, output $\tau(0) = \tau(2) = 0$, $\tau(1) = \tau(3) = 1$, and transition table*

δ	0	1	2	3
0	0	1	3	2
1	2	3	1	0
2	3	2	0	1
3	1	0	2	3

where the encoding is $\mathsf{0} = (0,0)$, $\mathsf{1} = (1,0)$, $\mathsf{2} = (0,1)$, $\mathsf{3} = (1,1)$.

Proof. This follows directly from the affine transitions in Theorem 20 applied to each encoded state. The initial state is $s(0) = (0,0)$, corresponding to state $\mathsf{0}$. □

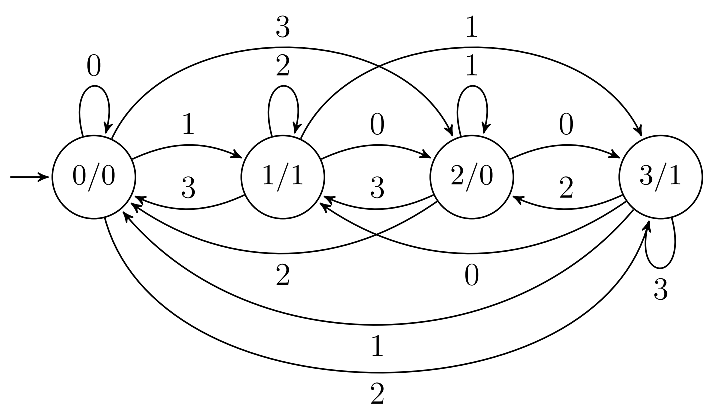

Figure 3: DFAO for $\mathcal{M}_2$ (4 states, base 4, MSB-first). State labels are *state*/*output*. The initial state is $\mathsf{0}$.

Corollary 23. *The coding $\tau\colon \mathsf{0}, \mathsf{2} \mapsto 0$ and $\mathsf{1}, \mathsf{3} \mapsto 1$ applied to the fixed point of the 4-uniform morphism*

$$\mathsf{0} \to \mathsf{0132}, \quad \mathsf{1} \to \mathsf{2310}, \quad \mathsf{2} \to \mathsf{3201}, \quad \mathsf{3} \to \mathsf{1023}$$

starting from $\mathsf{0}$ produces the sequence $\mathcal{M}_2$.

Remark 24. The four letters of the alphabet correspond to the states $(x, y) \in \mathbb{F}_2^2$ of the affine automaton (Theorem 20) via

$$\mathsf{0} = (0,0), \quad \mathsf{1} = (1,0), \quad \mathsf{2} = (0,1), \quad \mathsf{3} = (1,1).$$

With this identification, the image $\sigma_{\mathcal{M}_2}(\mathsf{r})$ lists the four successive states visited when the current state is r and the digits $d \in \{0,1,2,3\}$ are read in order, and the coding $\tau(\mathsf{r})$ is the first coordinate of the state. Under the relabeling $a = \mathsf{0}$, $b = \mathsf{2}$, $c = \mathsf{1}$, $d = \mathsf{3}$, the morphism reads $a \to acdb$, $b \to dbac$, $c \to bdca$, $d \to cabd$, with $\tau(a) = \tau(b) = 0$ and $\tau(c) = \tau(d) = 1$. This labeling is used in the right-special factor analysis below.

The defining recurrence of $\mathcal{M}_2$ is not directly an instance of Theorem 2, since the argument $2n + \mathcal{M}_2(n)$ in the nested recurrence in (13) and the argument $2n+1-\mathcal{M}_2(n)$ in (14) depend on the value $\mathcal{M}_2(n)$. This together with the sequence $\mathcal{M}_2(2n + \mathcal{M}_2(n))$ used in the pair (15) yields the system of recurrences in Theorem 20. Furthermore, Lemma 5 gives us that

$$\mathcal{M}_2(4n) = \mathcal{M}_2(2n) \oplus \mathcal{M}_2(n), \qquad \mathcal{M}_2(4n+2) = \mathcal{M}_2(2n+1) \oplus \mathcal{M}_2(n).$$

The XOR identities are direct consequences of Lemma 5, since $\mathcal{M}_2$ is balanced.

4.1 An explicit digit formula

Write the base-4 expansion $n = \sum_{k \geq 0} d_k 4^k$ with digits $d_k \in \{0,1,2,3\}$, and write $d_k = 2j_k + i_k$ with $i_k, j_k \in \{0,1\}$.

Theorem 25. *We have that*

$$\mathcal{M}_2(n) = \bigoplus_{k \equiv 0 \pmod 3} (i_k \oplus j_k) \;\oplus \bigoplus_{k \equiv 1 \pmod 3} j_k \;\oplus \bigoplus_{k \equiv 2 \pmod 3} i_k.$$

Proof. Let $d_{L-1} \cdots d_1 d_0$ be the base-4 expansion of n. Since the automaton reads digits MSB-first and the transition has the form $s \mapsto As + \mathbf{v}(d)$, starting from $s(0) = (0,0)$ gives

$$s(n) = \sum_{k=0}^{L-1} A^k \, \mathbf{v}(d_k) \pmod 2.$$

Since $A^3 = I$, the exponent depends only on k mod 3. Computing $A^k \mathbf{v}(d)$ for $k \in \{0,1,2\}$ and each digit d, we obtain

d	$A^0\mathbf{v}(d)$	$A^1\mathbf{v}(d)$	$A^2\mathbf{v}(d)$
0	$(0,0)$	$(0,0)$	$(0,0)$
1	$(1,0)$	$(0,1)$	$(1,1)$
2	$(1,1)$	$(1,0)$	$(0,1)$
3	$(0,1)$	$(1,1)$	$(1,0)$

Extracting the first coordinate and identifying $i_k = d_k \bmod 2$ and $j_k = \lfloor d_k/2 \rfloor$ yields the stated formula. $\square$

Theorem 26. *Write the binary expansion $n = \sum_{m\geq 0} b_m 2^m$ with $b_m \in \{0,1\}$, and define*

$$q(n) := \sum_{\substack{m\geq 0 \\ m\not\equiv 2 \pmod 3}} b_m 2^m, \tag{16}$$

so that $q(n)$ is obtained from n by zeroing the binary digits in positions $m \equiv 2 \pmod 3$. Then

$$\mathcal{M}_2(n) = \mathbf{t}(q(n)) \qquad (n \geq 0).$$

Equivalently, if $n = \sum_{k\geq 0} d_k 4^k$ with $d_k = 2j_k + i_k$, then $q(n) = \sum_{k\geq 0} d'_k 4^k$ where

$$d'_k = \begin{cases} d_k, & \text{if } k \equiv 0 \pmod 3; \\ 2j_k, & \text{if } k \equiv 1 \pmod 3; \\ i_k, & \text{otherwise.} \end{cases}$$

and $\mathcal{M}_2(n) = \mathbf{t}(q(n))$.

Proof. In Theorem 25, each base-4 digit d_k contributes to $\mathcal{M}_2(n)$ in a way that depends only on $k \bmod 3$: For $k \equiv 0 \bmod 3$, we take $i_k \oplus j_k$, for $k \equiv 1 \bmod 3$, we take only j_k, and for $k \equiv 2 \bmod 3$, we take i_k. Since $i_k = b_{2k}$ and $j_k = b_{2k+1}$, the binary positions $m = 2k$ and $m = 2k+1$ are kept or omitted, according to the residue class $k \bmod 3$, in the following manner. For $k \equiv 0 \pmod 3$, both positions are kept. For $k \equiv 1 \pmod 3$, the position $m = 2k$ is omitted. For $k \equiv 2 \pmod 3$, the position $m = 2k+1$ is omitted. In each of the cases where a position is omitted, the omitted position m satisfies $m \equiv 2 \pmod 3$. Conversely, every value m such that $m \equiv 2 \pmod 3$ falls into one of the two specified cases. Thus, the omitted positions are precisely those such that $m \equiv 2 \pmod 3$, and $\mathcal{M}_2(n)$ is the parity of the binary digit-sum of $q(n)$ defined in (16), and hence $\mathcal{M}_2(n) = \mathbf{t}(q(n))$. This gives us, in an equivalent way, the desired base-4 description of $q(n)$. □

5 Factor complexity

Let $p_a(n)$ denote the number of distinct length-n factors in the infinite word $a(0)a(1)a(2)\cdots$. A length-n factor u of a is called *right-special* if both $u0$ and $u1$ are factors of a. We write $\mathrm{RS}_a(n)$ in place of the set of right-special factors of a of length n.

Lemma 27. *For every infinite binary word a and every $n \geq 1$, we have that*

$$p_a(n+1) - p_a(n) = \#\,\mathrm{RS}_a(n).$$

Proof. For each length-n factor u, let $E_R(u) \subseteq \{0,1\}$ be the set of right extensions such that $u\alpha$ occurs in a. Then $p_a(n+1) = \sum_u |E_R(u)|$ and $p_a(n) = \sum_u 1$, so $p_a(n+1) - p_a(n) = \sum_u (|E_R(u)| - 1)$. In the binary case, $|E_R(u)| - 1$ equals 1 if u is right-special and 0 otherwise. □

Lemma 28. *If a is k-automatic, then the function $n \mapsto \#\mathrm{RS}_a(n)$ is k-regular in the sense of Allouche-Shallit [3], and is effectively computable from any DFAO for a. Moreover, if $\#\mathrm{RS}_a(n)$ is bounded, then it is k-automatic.*

Proof. The predicate "u is a length-n factor of a" and the predicate "u is right-special" are first-order definable over $(\mathbb{N}, +, V_k)$ relative to a DFAO for a, so the set of encodings of such pairs is regular. Counting right-special factors of a given length reduces to a finite-automaton computation, giving a k-regular sequence [3, Ch. 16]. A bounded k-regular sequence is k-automatic [3, Theorem 16.1.5]. □

Several statements in this paper can be certified by the `Walnut` theorem prover [29], which decides first-order properties of automatic sequences in Presburger arithmetic. This applies to the defining recurrences and the balance of $\mathcal{Q}$, $\mathcal{M}_1$, and $\mathcal{M}_2$ (Sections 3 and 4), and (using Lemma 28) to the right-special factor structure of $\mathcal{Q}$ and $\mathcal{M}_2$ in the present section. The right-special factor verifications for $\mathcal{M}_2$ use the CCL(S) determinization algorithm of Nicol and Frohme [25]. We give arguments that do not require Walnut wherever this is convenient.

5.1 Factor complexity of $\mathcal{Q}$

We first state a property of the factors of $\mathbf{t}$.

Lemma 29. *The set $\mathrm{Fac}(\mathbf{t})$ is closed under bitwise complementation: If $w \in \mathrm{Fac}(\mathbf{t})$, then $\bar{w} \in \mathrm{Fac}(\mathbf{t})$.*

Proof. The Thue-Morse morphism $\mu : 0 \mapsto 01,\ 1 \mapsto 10$ satisfies $\overline{\mu(w)} = \mu(\bar{w})$ for every binary word w. Since $\mathbf{t} = \mu^\omega(0)$ and both 0 and $1 = \bar{0}$ appear in $\mathbf{t}$, every factor w of $\mu^\omega(0)$ yields $\bar{w} = \overline{\mu^\omega(0)}|_{\text{same position}} = \mu^\omega(1)|_{\text{same position}}$, which is a factor of $\mu^\omega(1) = 1\,\mathbf{t}[1\,\infty)$, hence a factor of $\mathbf{t}$. □

Proposition 30. *Let J_n denote the set of length-n factors of $\mathcal{Q}$ that are not factors of $\mathbf{t}$. Then*

$$p_{\mathcal{Q}}(n) = p_{\mathbf{t}}(n) + |J_n|.$$

Proof. By Proposition 8, on each interval $[2^k, 2^{k+1})$, the sequence $\mathcal{Q}$ coincides with $\mathbf{t}$ or $1-\mathbf{t}$, according to the parity of k. The cases give us that a factor of $\mathcal{Q}$ of length n is either a factor of $\mathbf{t}$ or the complement of a factor of $\mathbf{t}$, which, by Lemma 29, is again a factor of $\mathbf{t}$. In any case, such factors are in $\mathrm{Fac}(\mathbf{t})$.

For an interval of the form $[2^k, 2^{k+1})$, suppose that the indices associated with a factor of length n properly contains this interval. This can only occur when its left endpoint is at a position $< 2^k$ and its right endpoint is at a position $\geq 2^{k+1}$ for some k, which forces $n \geq 2^k$. In every case, such a factor has the form $u_0\bar{u}_1u_2\bar{u}_3\cdots$ obtained by concatenating consecutive pieces of $\mathbf{t}$ with alternating bitwise complementation, where the concatenation $u_0u_1u_2u_3\cdots$ is a factor of $\mathbf{t}$. By Lemma 29, the complementation pattern preserves membership in $\mathrm{Fac}(\mathbf{t})$ for the simplest one-boundary case $u_0\bar{u}_1$ if and only if $\bar{u}_0u_1 \in \mathrm{Fac}(\mathbf{t})$, and this need not hold in general. According to the above definition of J_n, we have that J_n as the set of length-n factors that have at least one index strictly less than 2^k and at least one index greater than or equal to 2^k and that are not in $\mathrm{Fac}(\mathbf{t})$.

Since every factor of $\mathbf{t}$ appears in $\mathcal{Q}$, the set of factors of $\mathcal{Q}$ of length n is exactly $\mathrm{Fac}(\mathbf{t}) \cap \{0,1\}^n \cup J_n$, giving us that $p_{\mathcal{Q}}(n) = p_{\mathbf{t}}(n) + |J_n|$. □

Theorem 31. *For all $k \geq 4$, we have that*

$$p_{\mathcal{Q}}(2^k) = 13 \cdot 2^{k-2} - 2.$$

Equivalently, $p_{\mathcal{Q}}(2^k) = p_{\mathbf{t}}(2^k) + 2^{k-2}$, so $|J_{2^k}| = 2^{k-2}$.

Proof. The value $p_{\mathbf{t}}(2^k) = 3 \cdot 2^k - 2$ is known [6, 24]. By Proposition 30, we have that $p_{\mathcal{Q}}(2^k) = p_{\mathbf{t}}(2^k) + |J_{2^k}|$. Also, Lemma 27 gives us that

$$\#\mathrm{RS}_{\mathcal{Q}}(n) = p_{\mathcal{Q}}(n+1) - p_{\mathcal{Q}}(n). \tag{17}$$

The right-special count in (17) takes values in $\{2,3,4\}$, and this is certified for all $n \geq 2$ by Walnut. The desired result follows from the recurrence

$$p_{\mathcal{Q}}(2^{k+2}) = 4\,p_{\mathcal{Q}}(2^k) + 6 \qquad (k \geq 4),$$

with $p_{\mathcal{Q}}(16) = 50$, which has the unique solution $p_{\mathcal{Q}}(2^k) = 13 \cdot 2^{k-2} - 2$, also verified computationally for $4 \leq k \leq 7$. □

Remark 32. The set J_{2^k} has size 2^{k-2}, so factors in J_{2^k} account for one quarter of the length-2^k factors of $\mathcal{Q}$ beyond those of $\mathbf{t}$.

Since $\#\mathrm{RS}_{\mathcal{Q}}(n) \in \{2,3,4\}$ for all $n \geq 1$, the values of $p_{\mathcal{Q}}$ are determined by the support of each slope.

Lemma 27 is implicit in the following Proposition, according to (18).

Proposition 33. *For $n \geq 16$, set $k = \lfloor \log_2 n \rfloor$ and write $n = 2^k + i$ with $0 \leq i < 2^k$. Then*

$$p_{\mathcal{Q}}(n) = \begin{cases} 13 \cdot 2^{k-2} - 2, & \text{if } i = 0; \\ 4n - 3 \cdot 2^{k-2} - 4, & \text{if } 1 \leq i \leq 2^{k-1}; \\ 3n + 3 \cdot 2^{k-2} - 3, & \text{if } 2^{k-1} < i \leq 3 \cdot 2^{k-2}; \\ 2n + 5 \cdot 2^{k-1} - 2, & \text{if } 3 \cdot 2^{k-2} < i < 2^k. \end{cases}$$

Moreover, we have that

$$\#\mathrm{RS}_{\mathcal{Q}}(n) = p_{\mathcal{Q}}(n+1) - p_{\mathcal{Q}}(n). \tag{18}$$

The left-hand side of (18) equals 4 for $n \in (2^k,\, 3 \cdot 2^{k-1}]$, and 3 for $n \in (3 \cdot 2^{k-1},\, 7 \cdot 2^{k-2}]$, and 2 for $n \in (7 \cdot 2^{k-2},\, 2^{k+1})$.

Proof. This is certified by Walnut. □

By Proposition 33, $n + 1 \leq p_{\mathcal{Q}}(n) = O(n)$ for all $n \geq 1$.

5.2 Factor complexity of $\mathcal{M}_1$ and $\mathcal{M}_2$

Proposition 34. *We have* $p_{\mathcal{M}_1}(n) = \Theta(n)$ *and* $p_{\mathcal{M}_2}(n) = \Theta(n)$.

Proof. The incidence matrices of the morphisms in Corollaries 16 and 23 are

$$M_{\mathcal{M}_1} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}, \qquad M_{\mathcal{M}_2} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{19}$$

The incidence matrices in (19) give us that both of the associated morphisms are primitive. A result of Pansiot [26] gives us that the fixed point of a primitive k-uniform morphism has factor complexity $O(n)$, and the same upper bound holds after applying a coding. Neither $\mathcal{M}_1$ nor $\mathcal{M}_2$ is ultimately periodic, as can be checked from the DFAOs of Theorems 13 and 22, or certified by Walnut. By the Morse–Hedlund theorem [3, Theorem 10.2.6], the factor complexity of a non-ultimately-periodic word is at least $n+1$ for every $n \geq 0$. Hence both complexities are $\Theta(n)$. □

An explicit evaluation of the factor complexity of $\mathcal{M}_2$ is provided below.

Example 35. We obtain the initial values

$$p_{\mathcal{M}_1}(1 \ldots 15) = (2, 4, 6, 9, 12, 17, 22, 28, 34, 40, 46, 52, 58, 64, 70),$$

and

$$p_{\mathcal{M}_2}(1 \ldots 15) = (2, 4, 6, 10, 12, 14, 16, 18, 20, 24, 28, 32, 36, 40, 44).$$

Proposition 36. *For all* $k \geq 3$, *we have that*

$$p_{\mathcal{M}_2}(2^k) = \begin{cases} \frac{5}{2} \cdot 2^k - 2, & \text{if } 3 \mid k; \\ 3 \cdot 2^k - 2, & \text{otherwise.} \end{cases}$$

Moreover, the recurrence

$$p_{\mathcal{M}_2}(2^{k+3}) = 8\, p_{\mathcal{M}_2}(2^k) + 14 \qquad (k \geq 3)$$

holds true.

Proof. This is certified by Walnut. □

Since $\#\mathrm{RS}_{\mathcal{M}_2}(n) \in \{2, 4\}$, the values of $p_{\mathcal{M}_2}$ are determined by the positions at which $\#\mathrm{RS}_{\mathcal{M}_2}(n) = 4$. The following proposition extends Proposition 36 to a complete evaluation.

Proposition 37. *We have $p_{\mathcal{M}_2}(1) = 2$. For $n \geq 2$, set $k = \lfloor \log_2 n \rfloor$ and $r = k \bmod 3$.*

Case $r = 0$: *On the interval $[2^k, 2^{k+1}]$, we have that*

$$p_{\mathcal{M}_2}(n) = \begin{cases} 2n + (2^{k-1} - 2), & \text{if } 2^k \leq n \leq 2^k + 1; \\ 4n - (3 \cdot 2^{k-1} + 4), & \text{if } 2^k + 1 \leq n \leq 7 \cdot 2^{k-2} + 1; \\ 2n + (2^{k+1} - 2), & \text{if } 7 \cdot 2^{k-2} + 1 \leq n \leq 2^{k+1}. \end{cases}$$

Case $r = 1$: *On the interval $[2^k, 2^{k+1}]$, we have that*

$$p_{\mathcal{M}_2}(n) = \begin{cases} 2n + (2^{k} - 2), & \text{if } 2^k \leq n \leq 2^k + 1; \\ 4n - (2^{k} + 4), & \text{if } 2^k + 1 \leq n \leq 3 \cdot 2^{k-1} + 1; \\ 2n + (2^{k+1} - 2), & \text{if } 3 \cdot 2^{k-1} + 1 \leq n \leq 2^{k+1}. \end{cases}$$

Case $r = 2$: *On the interval $[2^k, 2^{k+1}]$, we have that*

$$p_{\mathcal{M}_2}(n) = 2n + (2^k - 2).$$

Proof. This is certified by Walnut. □

6 Conclusion

Consider replacing $a(2n)+a(2n+1) = 1$ by $a(3n)+a(3n+1)+a(3n+2) = 2$, with a taking values in $\{0, 1\}$. This gives rise to expressions of the form $a(3n+r-a(n))$, and the analogue of Lemma 5 converts each such expression into a fixed-argument expression. A small number of initial configurations produce 3-automatic binary sequences with small DFAOs; we plan to treat this family separately.

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8 Minimal DFAO sizes

Minimal DFAO sizes for the meta-automatic sequences considered in this paper are listed in Table 1.

$F(n)$	$G(n)$	DFAO	Nested	Notes
$a(n)$	$a(n)$	2	0	Thue-Morse
$a(n)$	$a(2n)$	**3**	**0**	$\mathcal{Q}$ (Thue-Morse Quarto)
$a(n)$	$a(2n+1)$	2	0	Thue-Morse
$a(n)$	$a(2n+a(n))$	4	1	
$a(n)$	$a(2n+1-a(n))$	**4**	**1**	$\mathcal{M}_1$
$a(2n)$	$a(n)$	4	0	
$a(2n)$	$a(2n)$	—	0	ultimately periodic
$a(2n)$	$a(2n+1)$	2	0	Thue-Morse
$a(2n)$	$a(2n+a(n))$	3	1	
$a(2n)$	$a(2n+1-a(n))$	3	1	
$a(2n+1)$	$a(n)$	5	0	
$a(2n+1)$	$a(2n)$	3	0	complement of $\mathcal{Q}$
$a(2n+1)$	$a(2n+1)$	5	0	
$a(2n+1)$	$a(2n+a(n))$	5	1	
$a(2n+1)$	$a(2n+1-a(n))$	5	1	
$a(2n+a(n))$	$a(n)$	4	1	
$a(2n+a(n))$	$a(2n)$	4	1	
$a(2n+a(n))$	$a(2n+1)$	4	1	
$a(2n+a(n))$	$a(2n+a(n))$	4	2	
$a(2n+a(n))$	$a(2n+1-a(n))$	**4**	**2**	$\mathcal{M}_2$
$a(2n+1-a(n))$	$a(n)$	5	1	
$a(2n+1-a(n))$	$a(2n)$	4	1	
$a(2n+1-a(n))$	$a(2n+1)$	5	1	
$a(2n+1-a(n))$	$a(2n+a(n))$	5	2	
$a(2n+1-a(n))$	$a(2n+1-a(n))$	5	2	

Table 1: Minimal DFAO size for each specialization. The column "Nested" counts the number of branch rules (F and G) whose right-hand side contains an expression of the form $a(\cdot \pm a(n))$. Boldface marks the three sequences studied in this paper.

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