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Linear Recurrences of Generalized Schreier Sets Revisited

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Abstract

For $p, q \in \mathbb{N}$, a finite nonempty set F is said to be (p, q)-*Schreier* (or *maximal* (p, q)-*Schreier*, respectively) if $q \min F \geq p|F|$ (or $q \min F = p|F|$, respectively). Using the inclusion-exclusion principle, Beanland et al. proved a linear recurrence for the counts of (p, q)-Schreier sets of the natural numbers. We show that the counts are taken periodically from Padovan-like sequences that satisfy simple recurrence relations. As an application, we obtain an alternative proof of Beanland et al.'s result. Furthermore, a similar result holds for the counts of maximal (p, q)-Schreier sets. We end with a discussion of the relation between (p, q)-Schreier and maximal (p, q)-Schreier sets.

1 Introduction

A finite nonempty subset F of natural numbers is called *Schreier* if $\min F \geq |F|$ and is called *maximal Schreier* if $\min F = |F|$. Bird [3] observed that for all $n \geq 1$, we have

$$|\{F \subset \mathbb{N} : F \text{ is Schreier and } \max F = n\}| \;=\; F_n, \tag{1}$$

where F_n is the n^{th} Fibonacci number defined as $F_{-1} = 1, F_0 = 0$ and $F_m = F_{m-1} + F_{m-2}$ for $m \geq 1$. Similarly, [4, Theorem 1] states that

$$|\{F \subset \mathbb{N} \,:\, F \text{ is maximal Schreier and } \max F = n\}| \;=\; F_{n-2}, \text{ for all } n \geq 1. \tag{2}$$

Previous work has generalized these results in various directions: Beanland et al. studied the more general condition $q \min F \geq p|F|$ and discovered linear recurrences of higher order; Chu et al. [5] connected Schreier multisets to higher-order Fibonacci sequences; more recently, Beanland et al. [2] counted unions of Schreier sets and provided a family of recursively defined sequences.

A set F of natural numbers is said to be (p, q)-*Schreier* (or *maximal* (p, q)-*Schreier*, respectively) if $q \min F \geq p|F|$ (or $q \min F = p|F|$, respectively). Let us recall [1, Theorem 1.1], which states that if

$$\mathcal{S}_n^{p/q} := \{F \subset \mathbb{N} \,:\, F \text{ is } (p,q)\text{-Schreier and } \max F = n\}, \text{ with } p, q, n \in \mathbb{N},$$

then

$$|\mathcal{S}_n^{p/q}| \;=\; \sum_{i=1}^{q} (-1)^{i+1} \binom{q}{i} |\mathcal{S}_{n-i}^{p/q}| + |\mathcal{S}_{n-(p+q)}^{p/q}|. \tag{3}$$

Continuing the work of Beanland et al. [1], we investigate the family of sequences $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$. While $(|\mathcal{S}_n^{1/3}|)_{n=1}^{\infty}$ and $(|\mathcal{S}_n^{1/4}|)_{n=1}^{\infty}$ were only recently added by the authors of the present paper to the On-Line Encyclopedia of Integer Sequences (OEIS) [6], these sequences are not far from the existing ones. Our main result shows that each $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ is a subsequence of a Padovan-like sequence whose linear recurrence is much simpler than (3). This relation reveals an alternative proof of (3), which employs characteristic polynomials instead of the inclusion-exclusion principle.

We are also interested in maximal (p, q)-Schreier sets: let

$$\mathcal{M}_n^{p/q} := \{F \subset \mathbb{N} \,:\, F \text{ is maximal } (p,q)\text{-Schreier and } \max F = n\}, \text{ with } p, q, n \in \mathbb{N}.$$

Tables 1 and 2 collect the initial values of $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ for different (p, q)'s.

Keen observation reveals that the terms of $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ are periodically taken from certain Padovan-like sequences. For $(p, q) \in \mathbb{N}^2$ with $p \geq q$, define the sequence $(a_{p,q,n})_{n=0}^{\infty}$ (Table 3) as follows:

$$\begin{aligned} a_{p,q,0} \;&=\; \cdots \;=\; a_{p,q,p-q-1} \;=\; 0, \\ a_{p,q,p-q} \;&=\; \cdots \;=\; a_{p,q,p+q-1} \;=\; 1, \text{ and} \\ a_{p,q,n} \;&=\; a_{p,q,n-q} + a_{p,q,n-p-q}, \text{ for } n \geq p+q. \end{aligned}$$

If $p < q$, let

$$\begin{aligned} a_{p,q,0} \;&=\; \cdots \;=\; a_{p,q,2p-1} \;=\; 1, \\ a_{p,q,2p} \;&=\; \cdots \;=\; a_{p,q,p+q-1} \;=\; 2, \text{ and} \\ a_{p,q,n} \;&=\; a_{p,q,n-q} + a_{p,q,n-p-q}, \text{ for } n \geq p+q. \end{aligned}$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert\mathcal{S}_n^{1/1}\rvert$	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$\lvert\mathcal{S}_n^{1/2}\rvert$	1	2	3	5	9	16	28	49	86	151	265	465	816	1432	2513
$\lvert\mathcal{S}_n^{1/3}\rvert$	1	2	4	7	12	21	38	70	129	236	429	778	1412	2567	4672
$\lvert\mathcal{S}_n^{1/4}\rvert$	1	2	4	8	15	27	48	86	157	292	549	1034	1939	3613	6697
$\lvert\mathcal{S}_n^{2/1}\rvert$	0	1	1	1	2	3	4	6	9	13	19	28	41	60	88
$\lvert\mathcal{S}_n^{2/3}\rvert$	1	1	2	4	7	12	20	33	55	93	159	273	468	799	1359

Table 1: The first 15 values of $(|\mathcal{S}_n^{p/q}|)_{n=1}^\infty$ with different values of (p,q). The sequences $(|\mathcal{S}_n^{1/1}|)_{n=1}^\infty$, $(|\mathcal{S}_n^{1/2}|)_{n=1}^\infty$, $(|\mathcal{S}_n^{2/1}|)_{n=1}^\infty$, and $(|\mathcal{S}_n^{2/3}|)_{n=1}^\infty$ are A000045, A005314, A078012, and A099558, respectively; the sequences $(|\mathcal{S}_n^{1/3}|)_{n=1}^\infty$ and $(|\mathcal{S}_n^{1/4}|)_{n=1}^\infty$ were newly added as A385106 and A385107, respectively.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert\mathcal{M}_n^{1/1}\rvert$	1	0	1	1	2	3	5	8	13	21	34	55	89	144	233
$\lvert\mathcal{M}_n^{1/2}\rvert$	0	1	1	1	2	4	7	12	21	37	65	114	200	351	616
$\lvert\mathcal{M}_n^{1/3}\rvert$	0	0	1	2	3	4	6	11	22	43	80	144	257	462	839
$\lvert\mathcal{M}_n^{1/4}\rvert$	0	0	0	1	3	6	10	15	22	35	64	129	265	529	1013
$\lvert\mathcal{M}_n^{2/1}\rvert$	0	1	0	0	1	1	1	2	3	4	6	9	13	19	28
$\lvert\mathcal{M}_n^{2/3}\rvert$	0	0	0	1	2	3	4	5	7	12	23	44	80	138	230

Table 2: The first 15 values of $(|\mathcal{M}_n^{p/q}|)_{n=1}^\infty$ with different values of (p,q). These sequences are A212804, A005251, A375169, A385142 (newly added), A078012, and A137357, respectively.

By changing the initial condition of the Padovan-like sequences $(a_{p,q,n})_{n=0}^\infty$, we arrive at a similar result for maximal (p,q)-Schreier sets. For $(p,q)\in\mathbb{N}^2$ and $p\ge q$, define the sequence $(a^{(m)}_{p,q,n})_{n=0}^\infty$ (Table 4) recursively as follows:

$$\begin{aligned} a^{(m)}_{p,q,0} &= \cdots = a^{(m)}_{p,q,p-q-1} = 0, \quad a^{(m)}_{p,q,p-q} = 1,\\ a^{(m)}_{p,q,p-q+1} &= \cdots = a^{(m)}_{p,q,p+q-1} = 0, \text{ and}\\ a^{(m)}_{p,q,n} &= a^{(m)}_{p,q,n-q} + a^{(m)}_{p,q,n-p-q}. \end{aligned}$$

If $p<q$, then

$$\begin{aligned} a^{(m)}_{p,q,0} &= \cdots = a^{(m)}_{p,q,2p-1} = 0, \quad a^{(m)}_{p,q,2p} = 1,\\ a^{(m)}_{p,q,2p+1} &= \cdots = a^{(m)}_{p,q,p+q-1} = 0, \text{ and}\\ a^{(m)}_{p,q,n} &= a^{(m)}_{p,q,n-q} + a^{(m)}_{p,q,n-p-q}, \end{aligned}$$

where the superscript (m) indicates that these sequences correspond to maximal (p,q)-Schreier sets.

For $(p,q) \in \mathbb{N}^2$ and $n \geq 0$, define

$$\Psi(p,q,n) \;:=\; \sum_{i=0}^{\left\lfloor \frac{n-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i}$$

and

$$\Phi(p,q,n) \;:=\; \sum_{\substack{i=0 \\ qi \equiv n \pmod{p}}}^{\left\lfloor \frac{n}{q} \right\rfloor} \binom{i}{\frac{n-iq}{p} - 2} + \begin{cases} 1, & \text{if } n = p-q; \\ 0, & \text{otherwise.} \end{cases}$$

We show later that

$$a_{p,q,n} \;=\; \Psi(p,q,n) \tag{4}$$

and

$$a^{(m)}_{p,q,n} \;=\; \Phi(p,q,n). \tag{5}$$

Tables 3 and 4 record the first few values of $(a_{p,q,n})_{n=0}^{\infty}$ and $(a^{(m)}_{p,q,n})_{n=0}^{\infty}$ for different (p,q)'s.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$a_{1,1,n}$	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597
$a_{1,2,n}$	1	1	2	2	3	4	5	7	9	12	16	21	28	37	49	65	86
$a_{1,3,n}$	1	1	2	2	2	3	4	4	5	7	8	9	12	15	17	21	27
$a_{1,4,n}$	1	1	2	2	2	2	3	4	4	4	5	7	8	8	9	12	15
$a_{2,1,n}$	0	1	1	1	2	3	4	6	9	13	19	28	41	60	88	129	189
$a_{2,3,n}$	1	1	1	1	2	2	2	3	3	4	5	5	7	8	9	12	13

Table 3: The first 17 values of $(a_{p,q,n})_{n=0}^{\infty}$ with different values of (p,q). Note that for $n \geq 0$, we have $a_{1,1,n}$ = A000045$(n+1)$, $a_{1,2,n}$ = A000931$(n+6)$, $a_{1,3,n}$ = A079398$(n+3)$, $a_{1,4,n}$ = A103372$(n+4)$, $a_{2,1,n}$ = A078012$(n+2)$, and $a_{2,3,n}$ = A226503$(n+2)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$a^{(m)}_{1,1,n}$	1	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987
$a^{(m)}_{1,2,n}$	0	0	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21	28
$a^{(m)}_{1,3,n}$	0	0	1	0	0	1	1	0	1	2	1	1	3	3	2	4	6	5
$a^{(m)}_{1,4,n}$	0	0	1	0	0	0	1	1	0	0	1	2	1	0	1	3	3	1
$a^{(m)}_{2,1,n}$	0	1	0	0	1	1	1	2	3	4	6	9	13	19	28	41	60	88
$a^{(m)}_{2,3,n}$	0	0	0	0	1	0	0	1	0	1	1	0	2	1	1	3	1	3

Table 4: The first 18 values of $(a^{(m)}_{p,q,n})_{n=0}^{\infty}$ with different values of (p,q). For $n \geq 0$, we have $a^{(m)}_{1,1,n}$ = A212804(n), $a^{(m)}_{1,2,n}$ = A000931$(n+1)$, $a^{(m)}_{1,3,n+2}$ = A017817(n), $a^{(m)}_{1,4,n+2}$ = A017827(n), $a^{(m)}_{2,1,n}$ = A135851$(n+1)$, and $a^{(m)}_{2,3,n+4}$ = A052920(n).

Theorem 1. *For $(p,q,n)\in\mathbb{N}^3$, we have*

$$|\mathcal{S}_n^{p/q}| \;=\; a_{p,q,(n-1)q} \tag{6}$$

and

$$|\mathcal{M}_n^{p/q}| \;=\; a^{(m)}_{p,q,(n-1)q}. \tag{7}$$

Corollary 2. *For $(p,q)\in\mathbb{N}^2$, the first $(p+q)$ terms of the sequence $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ are*

$$|\mathcal{S}_n^{p/q}| \;=\; \sum_{i=1}^{\left\lfloor \frac{(n+1)q}{p+q}\right\rfloor} \binom{n-\left\lceil \frac{ni}{q}\right\rceil}{i-1},\ \textit{for } 1\le n\le p+q.$$

Later terms satisfy the recurrence

$$|\mathcal{S}_n^{p/q}| \;=\; \sum_{i=1}^{q}(-1)^{i+1}\binom{q}{i}|\mathcal{S}_{n-i}^{p/q}| + |\mathcal{S}_{n-(p+q)}^{p/q}|,\ \textit{for } n\ge p+q+1.$$

Corollary 3. *For $(p,q)\in\mathbb{N}^2$, the first $(p+q)$ terms of the sequence $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ are*

$$|\mathcal{M}_n^{p/q}| \;=\; \sum_{k=1}^{\left\lfloor \frac{n+1}{p'+q'}\right\rfloor} \binom{n-p'k-1}{q'k-2},$$

where $p' = p/\gcd(p,q)$ and $q' = q/\gcd(p,q)$. Later terms satisfy the recurrence

$$|\mathcal{M}_n^{p/q}| \;=\; \sum_{i=1}^{q}(-1)^{i+1}\binom{q}{i}|\mathcal{M}_{n-i}^{p/q}| + |\mathcal{M}_{n-(p+q)}^{p/q}|,\ \textit{for } n\ge p+q+1.$$

Our paper is structured as follows: Section 2 establishes Formulas (4) and (5) for the sequences $(a_{p,q,n})_{n=0}^{\infty}$ and $(a^{(m)}_{p,q,n})_{n=0}^{\infty}$; Section 3 proves Theorem 1, which states that $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ are subsequences taken q-periodically out of $(a_{p,q,n})_{n=0}^{\infty}$ and $(a^{(m)}_{p,q,n})_{n=0}^{\infty}$, respectively. This enables us to obtain the recurrence of $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ using characteristic polynomials; Section 4 investigates the relation between $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ when either p or q is equal to 1 and suggests some problems for future studies.

2 Formulas for $(a_{p,q,n})_{n=0}^{\infty}$ and $(a^{(m)}_{p,q,n})_{n=0}^{\infty}$

In this section, we prove Formulas (4) and (5) by first confirming that these formulas produce the initial values of $(a_{p,q,n})_{n=0}^{\infty}$ and $(a^{(m)}_{p,q,n})_{n=0}^{\infty}$, respectively then showing that later values satisfy the recurrence given by the characteristic polynomial $1-x^q-x^{p+q}$.

Proof of (4). First, we verify that $\Psi(p,q,n)=a_{p,q,n}$ for $0\le n\le p+q-1$.

Case 1: $p\ge q$. When $n\le p-q-1$, the sum

$$\sum_{i=0}^{\lfloor \frac{n-p+q}{p+q}\rfloor}\binom{\lfloor \frac{n-p(i+1)}{q}\rfloor+1}{i}$$

is empty and thus, is equal to 0. When $p-q\le n\le p+q-1$, we obtain

$$\sum_{i=0}^{\lfloor \frac{n-p+q}{p+q}\rfloor}\binom{\lfloor \frac{n-p(i+1)}{q}\rfloor+1}{i} \;=\; \sum_{i=0}^{0}\binom{\lfloor \frac{n-p(i+1)}{q}\rfloor+1}{i} \;=\; \binom{\lfloor \frac{n-p}{q}\rfloor+1}{0} \;=\; 1.$$

Case 2: $p<q$. When $n\le 2p-1$, we have

$$\sum_{i=0}^{\lfloor \frac{n-p+q}{p+q}\rfloor}\binom{\lfloor \frac{n-p(i+1)}{q}\rfloor+1}{i} \;=\; \sum_{i=0}^{0}\binom{\lfloor \frac{n-p(i+1)}{q}\rfloor+1}{i} \;=\; \binom{\lfloor \frac{n-p}{q}\rfloor+1}{0} \;=\; 1;$$

when $2p\le n\le p+q-1$, we have

$$\begin{aligned}\sum_{i=0}^{\lfloor \frac{n-p+q}{p+q}\rfloor}\binom{\lfloor \frac{n-p(i+1)}{q}\rfloor+1}{i} &= \sum_{i=0}^{1}\binom{\lfloor \frac{n-p(i+1)}{q}\rfloor+1}{i}\\ &= \binom{\lfloor \frac{n-p}{q}\rfloor+1}{0}+\binom{\lfloor \frac{n-2p}{q}\rfloor+1}{1}\\ &= 1+1 \;=\; 2.\end{aligned}$$

It remains to prove that $(\Psi(p,q,n))_{n=1}^{\infty}$ and $(a_{p,q,n})_{n=1}^{\infty}$ satisfy the same recurrence: for all $n\ge p+q$, we verify the following identity

$$\sum_{i=0}^{\lfloor \frac{n-p+q}{p+q}\rfloor}\binom{\lfloor \frac{n-p(i+1)}{q}\rfloor+1}{i} \;=\; \sum_{i=0}^{\lfloor \frac{n-p}{p+q}\rfloor}\binom{\lfloor \frac{n-p(i+1)}{q}\rfloor}{i}+\sum_{i=0}^{\lfloor \frac{n-2p}{p+q}\rfloor}\binom{\lfloor \frac{n-p(i+2)}{q}\rfloor}{i}.$$

We have

$$\begin{aligned}&\sum_{i=0}^{\lfloor \frac{n-p+q}{p+q}\rfloor}\binom{\lfloor \frac{n-p(i+1)}{q}\rfloor+1}{i}-\sum_{i=0}^{\lfloor \frac{n-p}{p+q}\rfloor}\binom{\lfloor \frac{n-p(i+1)}{q}\rfloor}{i}\\ =\;&\sum_{i=1}^{\lfloor \frac{n-p+q}{p+q}\rfloor}\binom{\lfloor \frac{n-p(i+1)}{q}\rfloor+1}{i}-\sum_{i=1}^{\lfloor \frac{n-p}{p+q}\rfloor}\binom{\lfloor \frac{n-p(i+1)}{q}\rfloor}{i}\end{aligned}$$

$$= \sum_{i=\lfloor\frac{n-p}{p+q}\rfloor+1}^{\lfloor\frac{n-p+q}{p+q}\rfloor} \left(\binom{\lfloor\frac{n-p(i+1)}{q}\rfloor+1}{i}\right) + \sum_{i=1}^{\lfloor\frac{n-p}{p+q}\rfloor} \left(\left(\binom{\lfloor\frac{n-p(i+1)}{q}\rfloor+1}{i}\right) - \left(\binom{\lfloor\frac{n-p(i+1)}{q}\rfloor}{i}\right)\right)$$

$$= \sum_{i=\lfloor\frac{n+q}{p+q}\rfloor}^{\lfloor\frac{n-p+q}{p+q}\rfloor} \left(\binom{\lfloor\frac{n-p(i+1)}{q}\rfloor+1}{i}\right) + \sum_{i=1}^{\lfloor\frac{n-p}{p+q}\rfloor} \left(\binom{\lfloor\frac{n-p(i+1)}{q}\rfloor}{i-1}\right)$$

$$= \sum_{i=\lfloor\frac{n+q}{p+q}\rfloor}^{\lfloor\frac{n-p+q}{p+q}\rfloor} \left(\binom{\lfloor\frac{n-p(i+1)}{q}\rfloor+1}{i}\right) + \sum_{i=0}^{\lfloor\frac{n-2p-q}{p+q}\rfloor} \left(\binom{\lfloor\frac{n-p(i+2)}{q}\rfloor}{i}\right). \tag{8}$$

Write $n-p=(p+q)j+k$ for some $j\geq 0$ and $0\leq k\leq p+q-1$ and proceed by case analysis.

If $0\leq k\leq p-1$, then (8) is equal to

$$\begin{aligned}\sum_{i=j+1}^{j} \left(\binom{\lfloor\frac{n-p(i+1)}{q}\rfloor+1}{i}\right) + \sum_{i=0}^{j-1} \left(\binom{\lfloor\frac{n-p(i+2)}{q}\rfloor}{i}\right) &= \sum_{i=0}^{j-1} \left(\binom{\lfloor\frac{n-p(i+2)}{q}\rfloor}{i}\right) \\ &= \sum_{i=0}^{\lfloor\frac{n-2p}{p+q}\rfloor} \left(\binom{\lfloor\frac{n-p(i+2)}{q}\rfloor}{i}\right).\end{aligned}$$

If $p\leq k\leq p+q-1$, then (8) is equal to

$$\begin{aligned}&\sum_{i=j+1}^{j+1} \left(\binom{\lfloor\frac{n-p(i+1)}{q}\rfloor+1}{i}\right) + \sum_{i=0}^{j-1} \left(\binom{\lfloor\frac{n-p(i+2)}{q}\rfloor}{i}\right) \\ &= 1 + \sum_{i=0}^{j-1} \left(\binom{\lfloor\frac{n-p(i+2)}{q}\rfloor}{i}\right) \\ &= \sum_{i=0}^{j} \left(\binom{\lfloor\frac{n-p(i+2)}{q}\rfloor}{i}\right) = \sum_{i=0}^{\lfloor\frac{n-2p}{p+q}\rfloor} \left(\binom{\lfloor\frac{n-p(i+2)}{q}\rfloor}{i}\right).\end{aligned}$$

We have shown the identity

$$\sum_{i=0}^{\lfloor\frac{n-p+q}{p+q}\rfloor} \left(\binom{\lfloor\frac{n-p(i+1)}{q}\rfloor+1}{i}\right) - \sum_{i=0}^{\lfloor\frac{n-p}{p+q}\rfloor} \left(\binom{\lfloor\frac{n-p(i+1)}{q}\rfloor}{i}\right) = \sum_{i=0}^{\lfloor\frac{n-2p}{p+q}\rfloor} \left(\binom{\lfloor\frac{n-p(i+2)}{q}\rfloor}{i}\right)$$

holds for all $n\geq p+q$. $\square$

Proof of (5). First, we verify that $\Phi(p,q,n)=a_{p,q,n}$ for $0\leq n\leq p+q-1$.

Case 1: $p \geq q$. For $n \leq p+q-1$, we have

$$\frac{n-iq}{p} - 2 \;\leq\; \frac{p+q-1-iq}{p} - 2 \;\leq\; \frac{q-1}{p} - 1 \;<\; 0, \text{ for all } i \geq 0.$$

Hence, we have

$$\sum_{\substack{i=0 \\ qi \equiv n \pmod{p}}}^{\lfloor \frac{n}{q} \rfloor} \binom{i}{\frac{n-iq}{p} - 2} + \begin{cases} 1, & \text{if } n = p-q; \\ 0, & \text{otherwise} \end{cases} \;=\; \begin{cases} 1, & \text{if } n = p-q; \\ 0, & \text{if } 0 \leq n \leq p+q-1 \text{ and } n \neq p-q. \end{cases}$$

Case 2: $p < q$. For $n \leq 2p-1$, we have

$$\frac{n-iq}{p} - 2 \;\leq\; \frac{2p-1}{p} - 2 \;<\; 0, \text{ for all } i \geq 0,$$

which implies that

$$\sum_{\substack{i=0 \\ qi \equiv n \pmod{p}}}^{\lfloor \frac{n}{q} \rfloor} \binom{i}{\frac{n-iq}{p} - 2} \;=\; 0.$$

For $n = 2p$, we have

$$\sum_{\substack{i=0 \\ qi \equiv n \pmod{p}}}^{\lfloor \frac{n}{q} \rfloor} \binom{i}{\frac{n-iq}{p} - 2} \;=\; \sum_{\substack{i=0 \\ p|(qi)}}^{\lfloor \frac{2p}{q} \rfloor} \binom{i}{\frac{-iq}{p}} \;=\; \binom{0}{0} \;=\; 1$$

because every positive i makes $\binom{i}{-iq/p} = 0$. For $2p+1 \leq n \leq p+q-1$, we have

$$\frac{n-iq}{p} - 2 \;\leq\; \frac{p+q-1-iq}{p} - 2 \;=\; \frac{(1-i)q-1}{p} - 1 \;<\; 0, \text{ for all } i \geq 1,$$

which implies that

$$\sum_{\substack{i=0 \\ qi \equiv n \pmod{p}}}^{\lfloor \frac{n}{q} \rfloor} \binom{i}{\frac{n-iq}{p} - 2} \;=\; \begin{cases} \binom{0}{\frac{n}{p}-2}, & \text{if } p|n \\ 0, & \text{otherwise} \end{cases} \;=\; 0$$

because if $p|n$, then $n/p \geq 3$.

Next, we show that $(\Phi(p,q,n))_{n=0}^{\infty}$ follows the same recurrence as $(a_{p,q,n}^{(m)})_{n=0}^{\infty}$, namely

$$\Phi(p,q,n) \;=\; \Phi(p,q,n-q) + \Phi(p,q,n-p-q), \tag{9}$$

for $n \geq p+q$. To do so, we consider the two cases when $n = 2p$ and $n \neq 2p$.

Case A: $n = 2p$. We have

$$\Phi(p,q,n-p-q) \;=\; \Phi(p,q,p-q) \;=\; \sum_{\substack{i=0 \\ p|((i+1)q)}}^{\left\lfloor \frac{p}{q} \right\rfloor - 1} \binom{i}{\frac{-(i+1)q}{p} - 1} + 1 \;=\; 1$$

because $-(i+1)q/p - 1 < 0$ for all $i \geq 0$. Then

$$\Phi(p,q,2p) \;=\; \Phi(p,q,2p-q) + \Phi(p,q,p-q)$$

can be rewritten as

$$\sum_{\substack{i=0 \\ p|(qi)}}^{\left\lfloor \frac{2p}{q} \right\rfloor} \binom{i}{\frac{-iq}{p}} \;=\; \sum_{\substack{i=0 \\ p|((i+1)q)}}^{\left\lfloor \frac{2p}{q} \right\rfloor - 1} \binom{i}{\frac{-(i+1)q}{p}} + 1. \tag{10}$$

On the left side of (10), only $i = 0$ contributes a nonzero term to the sum, so

$$\sum_{\substack{i=0 \\ p|(qi)}}^{\left\lfloor \frac{2p}{q} \right\rfloor} \binom{i}{\frac{-iq}{p}} \;=\; 1.$$

On the right side of (10), each binomial in the sum is zero because $i \geq 0 > -(i+1)q/p$. This confirms (9) when $n = 2p$.

Case B: $n \neq 2p$. We write (9) as

$$\begin{aligned}
&\sum_{\substack{i=0 \\ qi \equiv n \pmod p}}^{\left\lfloor \frac{n}{q} \right\rfloor} \binom{i}{\frac{n-iq}{p} - 2} \\
&= \sum_{\substack{i=0 \\ q(i+1) \equiv n \pmod p}}^{\left\lfloor \frac{n}{q} \right\rfloor - 1} \binom{i}{\frac{n-(i+1)q}{p} - 2} + \sum_{\substack{i=0 \\ q(i+1) \equiv n \pmod p}}^{\left\lfloor \frac{n-p}{q} \right\rfloor - 1} \binom{i}{\frac{n-(i+1)q}{p} - 3}.
\end{aligned} \tag{11}$$

The right side of (11) is

$$\begin{aligned}
&\sum_{\substack{i=\left\lfloor \frac{n-p}{q} \right\rfloor \\ q(i+1) \equiv n \pmod p}}^{\left\lfloor \frac{n}{q} \right\rfloor - 1} \binom{i}{\frac{n-(i+1)q}{p} - 2} + \sum_{\substack{i=0 \\ q(i+1) \equiv n \pmod p}}^{\left\lfloor \frac{n-p}{q} \right\rfloor - 1} \left(\binom{i}{\frac{n-(i+1)q}{p} - 2} + \binom{i}{\frac{n-(i+1)q}{p} - 3} \right) \\
= &\sum_{\substack{i=\left\lfloor \frac{n-p}{q} \right\rfloor \\ q(i+1) \equiv n \pmod p}}^{\left\lfloor \frac{n}{q} \right\rfloor - 1} \binom{i}{\frac{n-(i+1)q}{p} - 2} + \sum_{\substack{i=0 \\ q(i+1) \equiv n \pmod p}}^{\left\lfloor \frac{n-p}{q} \right\rfloor - 1} \binom{i+1}{\frac{n-(i+1)q}{p} - 2}
\end{aligned}$$

$$= \sum_{\substack{i=\left\lfloor \frac{n-p}{q} \right\rfloor \\ q(i+1)\equiv n \pmod p}}^{\left\lfloor \frac{n}{q} \right\rfloor -1} \binom{i}{\frac{n-(i+1)q}{p}-2} + \sum_{\substack{i=1 \\ qi\equiv n \pmod p}}^{\left\lfloor \frac{n-p}{q} \right\rfloor} \binom{i}{\frac{n-iq}{p}-2}. \tag{12}$$

By (11) and (12), we need to prove

$$\sum_{\substack{i=\left\lfloor \frac{n-p}{q} \right\rfloor+1 \\ qi\equiv n \pmod p}}^{\left\lfloor \frac{n}{q} \right\rfloor} \binom{i}{\frac{n-iq}{p}-2} = \sum_{\substack{i=\left\lfloor \frac{n-p}{q} \right\rfloor \\ q(i+1)\equiv n \pmod p}}^{\left\lfloor \frac{n}{q} \right\rfloor -1} \binom{i}{\frac{n-(i+1)q}{p}-2}. \tag{13}$$

Note that we discard $\binom{0}{n/p-2}$ on the left side of (11) because the term is nonzero only when $n = 2p$. We claim that both sides of (13) vanish. On the left, we have

$$\frac{n-p}{q} < i \leq \frac{n}{q} \quad \text{and} \quad qi \equiv n \pmod p,$$

which give

$$-p < qi - n \leq 0 \quad \text{and} \quad qi \equiv n \pmod p.$$

Hence, we obtain $qi - n = 0$, and the corresponding binomial is $\binom{i}{-2} = 0$ because $i \geq 1$. Similarly, the right side of (13) is also 0. $\square$

3 Subsequences of Padovan-like sequences

3.1 Proof of Theorem 1

In this section, we show that the terms of $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ are taken q-periodically from the sequences $(a_{p,q,n})_{n=0}^{\infty}$ and $(a_{p,q,n}^{(m)})_{n=0}^{\infty}$. This allows us to deduce a linear recurrence for $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$.

First, we find a formula for $|\mathcal{S}_n^{p/q}|$. For $1 \leq i \leq n$, let F be an i-element set in $\mathcal{S}_n^{p/q}$; that is, we have $F \subset \{1, 2, \ldots, n\}$, $n \in F$, $|F| = i$, and $q \min F \geq pi$. Hence, to form F, we choose $i-1$ elements in $\left\{ \left\lceil \frac{pi}{q} \right\rceil, \ldots, n-1 \right\}$. Therefore, the number of i-element sets in $|\mathcal{S}_n^{p/q}|$ is

$$\binom{n - \left\lceil \frac{pi}{q} \right\rceil}{i-1}.$$

Here we require that $n - \lceil pi/q \rceil \geq i - 1$, or equivalently, we require that $i \leq \left\lfloor \frac{(n+1)q}{p+q} \right\rfloor$. Therefore, we have

$$|\mathcal{S}_n^{p/q}| = \sum_{i=1}^{\left\lfloor \frac{(n+1)q}{p+q} \right\rfloor} \binom{n - \left\lceil \frac{pi}{q} \right\rceil}{i-1}. \tag{14}$$

Next, we give a formula for $|\mathcal{M}_n^{p/q}|$. Let $p' = p/\gcd(p,q)$ and $q' = q/\gcd(p,q)$. Consider a (p,q)-maximal Schreier set $F \subset \{1,\ldots,n\}$ with $n \in F$ and $|F| = i$. Since $q \min F = p|F| = pi$, we know that i is a multiple of q'. Let $i = q'k$. Then $\min F = p'k$, and $F = \{p'k, n\} \cup F'$, where $F' \subset \{p'k+1, p'k+2, \ldots, n-1\}$ and $|F'| = q'k-2$. Hence, there are $\binom{n-p'k-1}{q'k-2}$ such sets F. Here we require $n - p'k - 1 \ge q'k - 2$, or equivalently, we require $k \le (n+1)/(p'+q')$. Therefore, we obtain

$$|\mathcal{M}_n^{p/q}| \;=\; \sum_{k=1}^{\left\lfloor \frac{n+1}{p'+q'} \right\rfloor} \binom{n-p'k-1}{q'k-2}. \tag{15}$$

Proof of Theorem 1. First, we prove (6). We have

$$\begin{aligned}
\Psi(p,q,(n-1)q) \;&=\; \sum_{i=0}^{\left\lfloor \frac{(n-1)q-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{(n-1)q-p(i+1)}{q} \right\rfloor + 1}{i} \\
&=\; \sum_{i=0}^{\left\lfloor \frac{nq-p}{p+q} \right\rfloor} \binom{n + \left\lfloor \frac{-p(i+1)}{q} \right\rfloor}{i} \\
&=\; \sum_{i=0}^{\left\lfloor \frac{nq-p}{p+q} \right\rfloor} \binom{n - \left\lceil \frac{p(i+1)}{q} \right\rceil}{i} \\
&=\; \sum_{i=1}^{\left\lfloor \frac{(n+1)q}{p+q} \right\rfloor} \binom{n - \left\lceil \frac{pi}{q} \right\rceil}{i-1} \;=\; |\mathcal{S}_n^{p/q}|.
\end{aligned}$$

Next, we prove (7). Let $p' = p/\gcd(p,q)$ and $q' = q/\gcd(p,q)$. Letting $n - i - 1 = p'k$ in the equations below, we have

$$\begin{aligned}
&\Phi(p,q,(n-1)q) \\
&= \sum_{\substack{i=0 \\ p' \mid (n-i-1)}}^{n-1} \binom{i}{\frac{q'(n-i-1)}{p'} - 2} + \begin{cases} 1, & \text{if } (n,q') = (p',1), \\ 0, & \text{otherwise,} \end{cases} \\
&= \sum_{k=0}^{\left\lfloor \frac{n-1}{p'} \right\rfloor} \binom{n-p'k-1}{q'k-2} + \begin{cases} 1, & \text{if } (n,q') = (p',1), \\ 0, & \text{otherwise,} \end{cases} \qquad (\text{with } n-i-1 = p'k) \\
&= \sum_{k=1}^{\left\lfloor \frac{n-1}{p'} \right\rfloor} \binom{n-p'k-1}{q'k-2} + \begin{cases} 1, & \text{if } (n,q') = (p',1), \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{16}$$

Thanks to (15) and (16), it suffices to verify that for $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{\left\lfloor \frac{n+1}{p'+q'} \right\rfloor} \binom{n-p'k-1}{q'k-2} \;=\; \sum_{k=1}^{\left\lfloor \frac{n-1}{p'} \right\rfloor} \binom{n-p'k-1}{q'k-2} + \begin{cases} 1, & \text{if } (n,q') = (p',1), \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

We proceed by case analysis.

Case 1: $(n-1)/p' \geq (n+1)/(p'+q')$ which is equivalent to $nq' \geq 2p'+q'$. Then $(n,q') \neq (p',1)$. We have

$$\sum_{k=1}^{\left\lfloor \frac{n-1}{p'} \right\rfloor} \binom{n-p'k-1}{q'k-2} - \sum_{k=1}^{\left\lfloor \frac{n+1}{p'+q'} \right\rfloor} \binom{n-p'k-1}{q'k-2} = \sum_{k=\left\lfloor \frac{n+1}{p'+q'} \right\rfloor+1}^{\left\lfloor \frac{n-1}{p'} \right\rfloor} \binom{n-p'k-1}{q'k-2}.$$

For each $k \geq \left\lfloor \frac{n+1}{p'+q'} \right\rfloor + 1$, we have $n-p'k-1 < q'k-2$, so the binomial $\binom{n-p'k-1}{q'k-2}$ is zero unless $q'k = 2$, which occurs when either $(q',k) = (2,1)$ or $(q',k) = (1,2)$.

a) *Case 1.1*: $(q',k) = (2,1)$. That $k = 1$ implies that $\left\lfloor \frac{n+1}{p'+q'} \right\rfloor = 0$, which in turn implies $n < p'+1$. However, the inequality $nq' \geq 2p'+q'$ gives $2n \geq 2p'+2$, which contradicts $n < p'+1$.

b) *Case 1.2*: $(q',k) = (1,2)$. Then $\left\lfloor \frac{n+1}{p'+q'} \right\rfloor = 1$, which implies

$$n \; < \; 2p'+2q'-1 \; = \; 2p'+1.$$

However, the inequality $nq' \geq 2p'+q'$ gives $n \geq 2p'+1$, which contradicts $n < 2p'+1$.

Case 2: $(n-1)/p' < (n+1)/(p'+q')$ which is equivalent to $n < 1+2p'/q'$. Then $(n+1)/(p'+q') < 2$.

If $\left\lfloor \frac{n+1}{p'+q'} \right\rfloor = 0$, then $(n,q') \neq (p',1)$, and $(n-1)/p' < (n+1)/(p'+q')$ implies that $\left\lfloor \frac{n-1}{p'} \right\rfloor = 0$. Hence, both sides of (17) are zero.

If $\left\lfloor \frac{n+1}{p'+q'} \right\rfloor = 1$, then $\left\lfloor \frac{n-1}{p'} \right\rfloor \in \{0,1\}$.

a) *Case 2.1*: If $\left\lfloor \frac{n-1}{p'} \right\rfloor = 1$, then $n \neq p'$, and (17) holds.

b) *Case 2.2*: If $\left\lfloor \frac{n-1}{p'} \right\rfloor = 0$, then $n < p'+1$. On the other hand, we deduce from $\left\lfloor \frac{n+1}{p'+q'} \right\rfloor = 1$ that $n \geq p'+q'-1$. Hence, we obtain

$$p'+1 \; > \; n \; \geq \; p'+q'-1 \implies q' = 1.$$

Then (17) becomes

$$\sum_{k=1}^{1} \binom{n-p'k-1}{k-2} = \sum_{k=1}^{0} \binom{n-p'k-1}{k-2} + \begin{cases} 1, & \text{if } n = p', \\ 0, & \text{otherwise,} \end{cases}$$

or equivalently, we have

$$\binom{n-p-1}{-1} \;=\; \begin{cases} 1, & \text{if } n = p, \\ 0, & \text{otherwise,} \end{cases}$$

which is true.

□

3.2 Recurrence of $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$

By Theorem 1, the recurrence of $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ is the same as the recurrence of subsequences taken q-periodically out of $(a_{p,q,n})_{n=0}^{\infty}$ and $(a_{p,q,n}^{(m)})_{n=0}^{\infty}$, respectively.

Definition 4. Let $p(x) = c_0 + c_1 x + \cdots + c_r x^r$ be a polynomial with real coefficients $(c_i)_{i=0}^{r}$. A sequence $(a_n)_{n=0}^{\infty}$ is said to *satisfy* $p(x)$ if

$$c_0 a_n + c_1 a_{n-1} + \cdots + c_r a_{n-r} \;=\; 0, \text{ for all } n \geq r.$$

Trivially, all sequences satisfy the zero polynomial.

We need the following lemma, whose proof is straightforward and therefore omitted.

Lemma 5. *If a sequence $(a_n)_{n=0}^{\infty}$ satisfies $p(x)$ and $p(x)$ divides $q(x)$, then $(a_n)_{n=0}^{\infty}$ satisfies $q(x)$.*

Proof of Corollaries 2 and 3. We verify initial terms and prove the linear recurrence for $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$. The formulas for initial terms are due to (14) and (15). We prove the recurrence relation. By definition, both $(a_{p,q,n})_{n=0}^{\infty}$ and $(a_{p,q,n}^{(m)})_{n=0}^{\infty}$ satisfy $g_{p,q}(x) := 1 - x^q - x^{p+q}$. Define

$$h_{p,q}(x) \;:=\; 1 - \sum_{i=1}^{q} (-1)^{i+1} \binom{q}{i} x^i - x^{p+q}.$$

We have

$$\begin{aligned} h_{p,q}(x^q) \;&=\; 1 + \sum_{i=1}^{q} (-1)^i \binom{q}{i} x^{qi} - x^{q(p+q)} \\ &=\; \sum_{i=0}^{q} (-1)^i \binom{q}{i} x^{qi} - x^{q(p+q)} \\ &=\; (1 - x^q)^q - (x^{p+q})^q \\ &=\; (1 - x^q - x^{p+q}) \sum_{i=0}^{q-1} (1 - x^q)^{q-1-i} x^{(p+q)i}, \end{aligned}$$

so $g_{p,q}(x)$ divides $h_{p,q}(x^q)$. By Lemma 5, both $(a_{p,q,n})_{n=0}^{\infty}$ and $(a_{p,q,n}^{(m)})_{n=0}^{\infty}$ satisfy $h_{p,q}(x^q)$. By Theorem 1, we have $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ satisfy $h_{p,q}(x)$. Here we use the obvious fact that for a fixed $k \in \mathbb{N}$, if $(b_n)_{n=1}^{\infty}$ satisfies $c_0 + c_k x^k + c_{2k}x^{2k} + \cdots + c_{nk}x^{nk}$, then the subsequence $(b_{(n-1)k})_{n=1}^{\infty}$ satisfies $c_0 + c_k x + c_{2k}x^2 + \cdots + c_{nk}x^n$. □

Remark 6. While one goal of the present paper is to establish the recurrence relation of $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ without using the inclusion-exclusion principle as in [1], we can use the same argument in the proof of Corollaries 2 and 3 to show that (6) is a corollary of [1, Theorem 1.1]. Indeed, Equality (3) states that $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ satisfies

$$h_{p,q}(x) \ = \ 1 + \sum_{i=1}^{q} (-1)^i \binom{q}{i} x^i - x^{p+q} \ = \ (1-x)^q - x^{p+q}.$$

The definition of $(a_{p,q,n})_{n=0}^{\infty}$ gives that $(a_{p,q,n})_{n=0}^{\infty}$ satisfies $g_{p,q}(x) = 1 - x^q - x^{p+q}$. Since $g_{p,q}(x)$ divides $h_{p,q}(x^q) = (1-x^q)^q - x^{q(p+q)}$, we know that $(a_{p,q,n})_{n=0}^{\infty}$ satisfies $h_{p,q}(x^q)$. Hence, the sequence $(a_{p,q,(n-1)q})_{n=1}^{\infty}$ satisfies $h_{p,q}(x)$. Since $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(a_{p,q,(n-1)q})_{n=1}^{\infty}$ satisfy the same polynomial and have the same initial terms, the two sequences are identical.

4 Relation between $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ and future investigations

It is natural to ask about the relation between the two sequences $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$: given one of the sequences, can we compute the other? This section presents the two special cases when $p = 1$ or $q = 1$ of the following problem.

Problem 7. For $(p, q) \in \mathbb{N}^2$, find the constants $(c_i)_{i=1}^{\infty}$ (if any) such that

$$|\mathcal{M}_n^{p/q}| = \sum_{i=1}^{\infty} c_i |\mathcal{S}_i^{p/q}|, \text{ for all } n \in \mathbb{N}.$$

Theorem 8. *For $p, n \in \mathbb{N}$, we have*

$$|\mathcal{S}_n^{p/1}| \ = \ |\mathcal{M}_{n+p+1}^{p/1}|.$$

Proof. By (14) and (15), we have

$$|\mathcal{S}_n^{p/1}| \ = \ \sum_{i=1}^{\left\lfloor \frac{n+1}{p+1} \right\rfloor} \binom{n-pi}{i-1} \text{ and } |\mathcal{M}_n^{p/1}| \ = \ \sum_{i=1}^{\left\lfloor \frac{n+1}{p+1} \right\rfloor} \binom{n-pi-1}{i-2}.$$

Hence, we obtain

$$
\begin{aligned}
|\mathcal{M}^{p/1}_{n+p+1}| \;&=\; \sum_{i=1}^{\left\lfloor \frac{n+p+2}{p+1} \right\rfloor} \binom{n+p+1-pi-1}{i-2} \;=\; \sum_{i=1}^{\left\lfloor \frac{n+1}{p+1} \right\rfloor+1} \binom{n-p(i-1)}{i-2} \\
&= \sum_{i=0}^{\left\lfloor \frac{n+1}{p+1} \right\rfloor} \binom{n-pi}{i-1} \\
&= \sum_{i=1}^{\left\lfloor \frac{n+1}{p+1} \right\rfloor} \binom{n-pi}{i-1} \;=\; |\mathcal{S}^{p/1}_{n}|.
\end{aligned}
$$

□

Theorem 9. *For $q, n \in \mathbb{N}$, we have*

$$|\mathcal{M}^{1/q}_{n}| \;=\; 2|\mathcal{S}^{1/q}_{n}| - |\mathcal{S}^{1/q}_{n+1}|. \tag{18}$$

Proof. Let $n \in \mathbb{N}$. When $q = 1$, we have $|\mathcal{S}^{1/1}_{n}| = F_n$, $|\mathcal{S}^{1/1}_{n+1}| = F_{n+1}$, and $|\mathcal{M}^{1/1}_{n}| = F_{n-2}$ (see (1) and (2)). Then (18) holds. For the rest of the proof, assume that $q \geq 2$.

We need to bijectively map two copies of $\mathcal{S}^{1/q}_{n}$ to $\mathcal{S}^{1/q}_{n+1} \cup \mathcal{M}^{1/q}_{n}$. Observe that

$$
\begin{aligned}
\phi :\; \mathcal{S}^{1/q}_{n} \;&\to\; \mathcal{S}^{1/q}_{n+1} \\
F \;&\mapsto\; F + q
\end{aligned}
$$

maps $\mathcal{S}^{1/q}_{n}$ injectively into $\mathcal{S}^{1/q}_{n+1}$. Furthermore, we know from the definition that $\mathcal{M}^{1/q}_{n} \subset \mathcal{S}^{1/q}_{n}$. Hence, it suffices to show that

$$|\mathcal{S}^{1/q}_{n} \backslash \mathcal{M}^{1/q}_{n}| \;=\; |\mathcal{S}^{1/q}_{n+1} \backslash \phi(\mathcal{S}^{1/q}_{n})|. \tag{19}$$

The left side of (19) is the cardinality of

$$L_{q,n} \;:=\; \{F \subset \{1, 2, \ldots, n\} \,:\, q \min F > |F| \text{ and } \max F = n\},$$

which can be partitioned into the collections

$$L_{q,n,i} \;:=\; \{F \subset \{1, 2, \ldots, n\} \,:\, q \min F > |F| = i \text{ and } \max F = n\}, \text{ with } i \geq 1.$$

For $i = 1$, the only set in $L_{q,n,1}$ is $\{n\}$, so we have $|L_{q,n,1}| = 1$. For $i \geq 2$, to form sets in $L_{q,n,i}$, we first choose the minimum j with $i/q < j \leq n-1$ then choose $i-2$ integers in $[j+1, n-1]$, which requires $i - 2 \leq n - j - 1$. Hence, for $i \geq 2$, we obtain

$$|L_{q,n,i}| \;=\; \sum_{\substack{i+1 \leq jq \leq (n-1)q \\ j \leq n-i+1}} \binom{n-j-1}{i-2} \;=\; \sum_{i+1 \leq jq \leq (n-i+1)q} \binom{n-j-1}{i-2}.$$

The right side of (19) is the cardinality of

$$R_{q,n} \ := \ \{F \subset \{1,2,\ldots,n+1\} \, : \, q\min F \geq |F| > q\min F - q, \max F = n+1\},$$

which can be partitioned into

$$R_{q,n,i} \ := \ \{F \subset \{1,2,\ldots,n+1\} \, : \, q\min F \geq |F| = i > q\min F - q, \\ \max F = n+1\}, \text{ with } i \geq 1.$$

We have $R_{q,n,1} = \emptyset$ because a nonempty $R_{q,n,1}$ requires $1 > (n+1)q - q = nq \geq 2$. For $i = 2$, an $F \in R_{q,n,2}$ must have the form $F = \{j, n+1\}$ with $1 \leq j \leq n$ and $jq \geq 2 > (j-1)q$. Since $q \geq 2$, the inequality $2 > (j-1)q$ implies that $j = 1$. Hence, the set F is $\{1, n+1\}$ and $|R_{q,n,2}| = 1$. To form sets in $R_{q,n,i}$ with $i \geq 3$, we first choose the minimum j with $j - 1 < i/q \leq j \leq n$ then choose $i-2$ integers in $[j+1, n]$, which requires $n - j \geq i - 2$. Hence, for $i \geq 3$, we obtain

$$|R_{q,n,i}| \ = \ \sum_{\substack{1+q(j-1)\leq i\leq jq\leq nq \\ j\leq n+2-i}} \binom{n-j}{i-2} \ = \ \sum_{\substack{i\leq jq\leq i+q-1 \\ j\leq n-i+2}} \binom{n-j}{i-2}.$$

To prove (19), we show that

$$|L_{q,n,i}| \ = \ |R_{q,n,i+1}|, \text{ for } i \geq 1. \tag{20}$$

The above analysis shows that (20) holds when $i = 1$. For $i \geq 2$, Equality (20) is the same as

$$\sum_{i+1\leq jq\leq (n-i+1)q} \binom{n-j-1}{i-2} \ = \ \sum_{\substack{i+1\leq jq\leq i+q \\ j\leq n-i+1}} \binom{n-j}{i-1}. \tag{21}$$

Case 1: $(n-i+1)q \leq i+q$. We have

$$\begin{aligned} &\sum_{i+1\leq jq\leq (n-i+1)q} \binom{n-j-1}{i-2} - \sum_{\substack{i+1\leq jq\leq i+q \\ j\leq n-i+1}} \binom{n-j}{i-1} \\ &= \sum_{i+1\leq jq\leq (n-i+1)q} \binom{n-j-1}{i-2} - \sum_{i+1\leq jq\leq (n-i+1)q} \binom{n-j}{i-1}. \end{aligned} \tag{22}$$

Since $(n-i+1)q \leq i+q$, we obtain

$$n \ \leq \ i + \frac{i}{q} \ = \ i + \frac{(i+1)-1}{q} \ \leq \ i + \frac{jq-1}{q} \ < \ i + j,$$

which implies $n \le i + j - 1$, but (22) sums over $j \le n - i + 1$. Therefore, $n = i + j - 1$ in (22), which gives

$$\begin{aligned} &\sum_{i+1 \le jq \le (n-i+1)q} \binom{n-j-1}{i-2} - \sum_{i+1 \le jq \le (n-i+1)q} \binom{n-j}{i-1} \\ &= \sum_{i+1 \le jq \le (n-i+1)q} 1 - \sum_{i+1 \le jq \le (n-i+1)q} 1 \ = \ 0. \end{aligned}$$

Case 2: $(n - i + 1)q > i + q$. Then (21) is equivalent to

$$\sum_{i+1 \le jq \le (n-i+1)q} \binom{n-j-1}{i-2} \ = \ \binom{n - \lceil \frac{i+1}{q} \rceil}{i-1},$$

which is precisely the hockey-stick identity (see [7, Theorem 1.2.3, item (5)]).

We have verified (21) and thus completed the proof. □

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