Journal of Integer Sequences, Vol. 29 (2026),
Article 26.2.2

Linear Recurrences of Generalized Schreier Sets Revisited

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Abstract

For $p, q \in \mathbb{N}$, a finite nonempty set F is said to be (p, q)-*Schreier* (or *maximal* (p, q)-*Schreier*, respectively) if $q \min F \geq p|F|$ (or $q \min F = p|F|$, respectively). Using the inclusion-exclusion principle, Beanland et al. proved a linear recurrence for the counts of (p, q)-Schreier sets of the natural numbers. We show that the counts are taken periodically from Padovan-like sequences that satisfy simple recurrence relations. As an application, we obtain an alternative proof of Beanland et al.'s result. Furthermore, a similar result holds for the counts of maximal (p, q)-Schreier sets. We end with a discussion of the relation between (p, q)-Schreier and maximal (p, q)-Schreier sets.

1 Introduction

A finite nonempty subset F of natural numbers is called *Schreier* if $\min F \geq |F|$ and is called *maximal Schreier* if $\min F = |F|$. Bird [3] observed that for all $n \geq 1$, we have

$$|\{F \subset \mathbb{N} : F \text{ is Schreier and } \max F = n\}| = F_n, \tag{1}$$

where F_n is the n^{th} Fibonacci number defined as $F_{-1} = 1, F_0 = 0$ and $F_m = F_{m-1} + F_{m-2}$ for $m \geq 1$. Similarly, [4, Theorem 1] states that

$$|\{F \subset \mathbb{N} : F \text{ is maximal Schreier and } \max F = n\}| = F_{n-2}, \text{ for all } n \geq 1. \tag{2}$$

Previous work has generalized these results in various directions: Beanland et al. studied the more general condition $q \min F \geq p|F|$ and discovered linear recurrences of higher order; Chu et al. [5] connected Schreier multisets to higher-order Fibonacci sequences; more recently, Beanland et al. [2] counted unions of Schreier sets and provided a family of recursively defined sequences.

A set F of natural numbers is said to be (p, q)-*Schreier* (or *maximal* (p, q)-*Schreier*, respectively) if $q \min F \geq p|F|$ (or $q \min F = p|F|$, respectively). Let us recall [1, Theorem 1.1], which states that if

$$\mathcal{S}_n^{p/q} := \{F \subset \mathbb{N} : F \text{ is } (p,q)\text{-Schreier and } \max F = n\}, \text{ with } p, q, n \in \mathbb{N},$$

then

$$|\mathcal{S}_n^{p/q}| = \sum_{i=1}^{q} (-1)^{i+1} \binom{q}{i} |\mathcal{S}_{n-i}^{p/q}| + |\mathcal{S}_{n-(p+q)}^{p/q}|. \tag{3}$$

Continuing the work of Beanland et al. [1], we investigate the family of sequences $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$. While $(|\mathcal{S}_n^{1/3}|)_{n=1}^{\infty}$ and $(|\mathcal{S}_n^{1/4}|)_{n=1}^{\infty}$ were only recently added by the authors of the present paper to the On-Line Encyclopedia of Integer Sequences (OEIS) [6], these sequences are not far from the existing ones. Our main result shows that each $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ is a subsequence of a Padovan-like sequence whose linear recurrence is much simpler than (3). This relation reveals an alternative proof of (3), which employs characteristic polynomials instead of the inclusion-exclusion principle.

We are also interested in maximal (p, q)-Schreier sets: let

$$\mathcal{M}_n^{p/q} := \{F \subset \mathbb{N} : F \text{ is maximal } (p,q)\text{-Schreier and } \max F = n\}, \text{ with } p, q, n \in \mathbb{N}.$$

Tables 1 and 2 collect the initial values of $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ for different (p, q)'s.

Keen observation reveals that the terms of $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ are periodically taken from certain Padovan-like sequences. For $(p, q) \in \mathbb{N}^2$ with $p \geq q$, define the sequence $(a_{p,q,n})_{n=0}^{\infty}$ (Table 3) as follows:

$$\begin{aligned}
a_{p,q,0} &= \cdots = a_{p,q,p-q-1} = 0, \\
a_{p,q,p-q} &= \cdots = a_{p,q,p+q-1} = 1, \text{ and} \\
a_{p,q,n} &= a_{p,q,n-q} + a_{p,q,n-p-q}, \text{ for } n \geq p+q.
\end{aligned}$$

If $p < q$, let

$$\begin{aligned}
a_{p,q,0} &= \cdots = a_{p,q,2p-1} = 1, \\
a_{p,q,2p} &= \cdots = a_{p,q,p+q-1} = 2, \text{ and} \\
a_{p,q,n} &= a_{p,q,n-q} + a_{p,q,n-p-q}, \text{ for } n \geq p+q.
\end{aligned}$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert\mathcal{S}_n^{1/1}\rvert$	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$\lvert\mathcal{S}_n^{1/2}\rvert$	1	2	3	5	9	16	28	49	86	151	265	465	816	1432	2513
$\lvert\mathcal{S}_n^{1/3}\rvert$	1	2	4	7	12	21	38	70	129	236	429	778	1412	2567	4672
$\lvert\mathcal{S}_n^{1/4}\rvert$	1	2	4	8	15	27	48	86	157	292	549	1034	1939	3613	6697
$\lvert\mathcal{S}_n^{2/1}\rvert$	0	1	1	1	2	3	4	6	9	13	19	28	41	60	88
$\lvert\mathcal{S}_n^{2/3}\rvert$	1	1	2	4	7	12	20	33	55	93	159	273	468	799	1359

Table 1: The first 15 values of $(|\mathcal{S}_n^{p/q}|)_{n=1}^\infty$ with different values of (p,q). The sequences $(|\mathcal{S}_n^{1/1}|)_{n=1}^\infty$, $(|\mathcal{S}_n^{1/2}|)_{n=1}^\infty$, $(|\mathcal{S}_n^{2/1}|)_{n=1}^\infty$, and $(|\mathcal{S}_n^{2/3}|)_{n=1}^\infty$ are A000045, A005314, A078012, and A099558, respectively; the sequences $(|\mathcal{S}_n^{1/3}|)_{n=1}^\infty$ and $(|\mathcal{S}_n^{1/4}|)_{n=1}^\infty$ were newly added as A385106 and A385107, respectively.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert\mathcal{M}_n^{1/1}\rvert$	1	0	1	1	2	3	5	8	13	21	34	55	89	144	233
$\lvert\mathcal{M}_n^{1/2}\rvert$	0	1	1	1	2	4	7	12	21	37	65	114	200	351	616
$\lvert\mathcal{M}_n^{1/3}\rvert$	0	0	1	2	3	4	6	11	22	43	80	144	257	462	839
$\lvert\mathcal{M}_n^{1/4}\rvert$	0	0	0	1	3	6	10	15	22	35	64	129	265	529	1013
$\lvert\mathcal{M}_n^{2/1}\rvert$	0	1	0	0	1	1	1	2	3	4	6	9	13	19	28
$\lvert\mathcal{M}_n^{2/3}\rvert$	0	0	0	1	2	3	4	5	7	12	23	44	80	138	230

Table 2: The first 15 values of $(|\mathcal{M}_n^{p/q}|)_{n=1}^\infty$ with different values of (p,q). These sequences are A212804, A005251, A375169, A385142 (newly added), A078012, and A137357, respectively.

By changing the initial condition of the Padovan-like sequences $(a_{p,q,n})_{n=0}^\infty$, we arrive at a similar result for maximal (p,q)-Schreier sets. For $(p,q)\in\mathbb{N}^2$ and $p\ge q$, define the sequence $(a^{(m)}_{p,q,n})_{n=0}^\infty$ (Table 4) recursively as follows:

$$\begin{aligned} a^{(m)}_{p,q,0} &= \cdots = a^{(m)}_{p,q,p-q-1} = 0, \quad a^{(m)}_{p,q,p-q} = 1, \\ a^{(m)}_{p,q,p-q+1} &= \cdots = a^{(m)}_{p,q,p+q-1} = 0, \text{ and} \\ a^{(m)}_{p,q,n} &= a^{(m)}_{p,q,n-q} + a^{(m)}_{p,q,n-p-q}. \end{aligned}$$

If $p<q$, then

$$\begin{aligned} a^{(m)}_{p,q,0} &= \cdots = a^{(m)}_{p,q,2p-1} = 0, \quad a^{(m)}_{p,q,2p} = 1, \\ a^{(m)}_{p,q,2p+1} &= \cdots = a^{(m)}_{p,q,p+q-1} = 0, \text{ and} \\ a^{(m)}_{p,q,n} &= a^{(m)}_{p,q,n-q} + a^{(m)}_{p,q,n-p-q}, \end{aligned}$$

where the superscript (m) indicates that these sequences correspond to maximal (p,q)-Schreier sets.

For $(p,q) \in \mathbb{N}^2$ and $n \geq 0$, define

$$\Psi(p,q,n) \ := \ \sum_{i=0}^{\left\lfloor \frac{n-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i}$$

and

$$\Phi(p,q,n) \ := \ \sum_{\substack{i=0 \\ qi \equiv n \pmod{p}}}^{\left\lfloor \frac{n}{q} \right\rfloor} \binom{i}{\frac{n-iq}{p} - 2} + \begin{cases} 1, & \text{if } n = p-q; \\ 0, & \text{otherwise.} \end{cases}$$

We show later that

$$a_{p,q,n} \ = \ \Psi(p,q,n) \tag{4}$$

and

$$a^{(m)}_{p,q,n} \ = \ \Phi(p,q,n). \tag{5}$$

Tables 3 and 4 record the first few values of $(a_{p,q,n})_{n=0}^{\infty}$ and $(a^{(m)}_{p,q,n})_{n=0}^{\infty}$ for different (p,q)'s.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$a_{1,1,n}$	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597
$a_{1,2,n}$	1	1	2	2	3	4	5	7	9	12	16	21	28	37	49	65	86
$a_{1,3,n}$	1	1	2	2	2	3	4	4	5	7	8	9	12	15	17	21	27
$a_{1,4,n}$	1	1	2	2	2	2	3	4	4	4	5	7	8	8	9	12	15
$a_{2,1,n}$	0	1	1	1	2	3	4	6	9	13	19	28	41	60	88	129	189
$a_{2,3,n}$	1	1	1	1	2	2	2	3	3	4	5	5	7	8	9	12	13

Table 3: The first 17 values of $(a_{p,q,n})_{n=0}^{\infty}$ with different values of (p,q). Note that for $n \geq 0$, we have $a_{1,1,n} = $ A000045$(n+1)$, $a_{1,2,n} = $ A000931$(n+6)$, $a_{1,3,n} = $ A079398$(n+3)$, $a_{1,4,n} = $ A103372$(n+4)$, $a_{2,1,n} = $ A078012$(n+2)$, and $a_{2,3,n} = $ A226503$(n+2)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$a^{(m)}_{1,1,n}$	1	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987
$a^{(m)}_{1,2,n}$	0	0	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21	28
$a^{(m)}_{1,3,n}$	0	0	1	0	0	1	1	0	1	2	1	1	3	3	2	4	6	5
$a^{(m)}_{1,4,n}$	0	0	1	0	0	0	1	1	0	0	1	2	1	0	1	3	3	1
$a^{(m)}_{2,1,n}$	0	1	0	0	1	1	1	2	3	4	6	9	13	19	28	41	60	88
$a^{(m)}_{2,3,n}$	0	0	0	0	1	0	0	1	0	1	1	0	2	1	1	3	1	3

Table 4: The first 18 values of $(a^{(m)}_{p,q,n})_{n=0}^{\infty}$ with different values of (p,q). For $n \geq 0$, we have $a^{(m)}_{1,1,n} = $ A212804(n), $a^{(m)}_{1,2,n} = $ A000931$(n+1)$, $a^{(m)}_{1,3,n+2} = $ A017817(n), $a^{(m)}_{1,4,n+2} = $ A017827(n), $a^{(m)}_{2,1,n} = $ A135851$(n+1)$, and $a^{(m)}_{2,3,n+4} = $ A052920(n).

Theorem 1. *For $(p,q,n)\in\mathbb{N}^3$, we have*

$$|\mathcal{S}_n^{p/q}| \;=\; a_{p,q,(n-1)q} \tag{6}$$

and

$$|\mathcal{M}_n^{p/q}| \;=\; a^{(m)}_{p,q,(n-1)q}. \tag{7}$$

Corollary 2. *For $(p,q)\in\mathbb{N}^2$, the first $(p+q)$ terms of the sequence $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ are*

$$|\mathcal{S}_n^{p/q}| \;=\; \sum_{i=1}^{\left\lfloor \frac{(n+1)q}{p+q}\right\rfloor} \binom{n-\left\lceil \frac{ni}{q}\right\rceil}{i-1},\ \textit{for } 1\le n\le p+q.$$

Later terms satisfy the recurrence

$$|\mathcal{S}_n^{p/q}| \;=\; \sum_{i=1}^{q}(-1)^{i+1}\binom{q}{i}|\mathcal{S}_{n-i}^{p/q}| + |\mathcal{S}_{n-(p+q)}^{p/q}|,\ \textit{for } n\ge p+q+1.$$

Corollary 3. *For $(p,q)\in\mathbb{N}^2$, the first $(p+q)$ terms of the sequence $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ are*

$$|\mathcal{M}_n^{p/q}| \;=\; \sum_{k=1}^{\left\lfloor \frac{n+1}{p'+q'}\right\rfloor} \binom{n-p'k-1}{q'k-2},$$

where $p'=p/\gcd(p,q)$ and $q'=q/\gcd(p,q)$. Later terms satisfy the recurrence

$$|\mathcal{M}_n^{p/q}| \;=\; \sum_{i=1}^{q}(-1)^{i+1}\binom{q}{i}|\mathcal{M}_{n-i}^{p/q}| + |\mathcal{M}_{n-(p+q)}^{p/q}|,\ \textit{for } n\ge p+q+1.$$

Our paper is structured as follows: Section 2 establishes Formulas (4) and (5) for the sequences $(a_{p,q,n})_{n=0}^{\infty}$ and $(a^{(m)}_{p,q,n})_{n=0}^{\infty}$; Section 3 proves Theorem 1, which states that $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ are subsequences taken q-periodically out of $(a_{p,q,n})_{n=0}^{\infty}$ and $(a^{(m)}_{p,q,n})_{n=0}^{\infty}$, respectively. This enables us to obtain the recurrence of $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ using characteristic polynomials; Section 4 investigates the relation between $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ when either p or q is equal to 1 and suggests some problems for future studies.

2 Formulas for $(a_{p,q,n})_{n=0}^{\infty}$ and $(a^{(m)}_{p,q,n})_{n=0}^{\infty}$

In this section, we prove Formulas (4) and (5) by first confirming that these formulas produce the initial values of $(a_{p,q,n})_{n=0}^{\infty}$ and $(a^{(m)}_{p,q,n})_{n=0}^{\infty}$, respectively then showing that later values satisfy the recurrence given by the characteristic polynomial $1-x^q-x^{p+q}$.

Proof of (4). First, we verify that $\Psi(p,q,n) = a_{p,q,n}$ for $0 \le n \le p+q-1$.

Case 1: $p \ge q$. When $n \le p-q-1$, the sum

$$\sum_{i=0}^{\left\lfloor \frac{n-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i}$$

is empty and thus, is equal to 0. When $p-q \le n \le p+q-1$, we obtain

$$\sum_{i=0}^{\left\lfloor \frac{n-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} \;=\; \sum_{i=0}^{0} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} \;=\; \binom{\left\lfloor \frac{n-p}{q} \right\rfloor + 1}{0} \;=\; 1.$$

Case 2: $p < q$. When $n \le 2p-1$, we have

$$\sum_{i=0}^{\left\lfloor \frac{n-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} \;=\; \sum_{i=0}^{0} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} \;=\; \binom{\left\lfloor \frac{n-p}{q} \right\rfloor + 1}{0} \;=\; 1;$$

when $2p \le n \le p+q-1$, we have

$$\begin{aligned}\sum_{i=0}^{\left\lfloor \frac{n-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} \;&=\; \sum_{i=0}^{1} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} \\ &=\; \binom{\left\lfloor \frac{n-p}{q} \right\rfloor + 1}{0} + \binom{\left\lfloor \frac{n-2p}{q} \right\rfloor + 1}{1} \\ &=\; 1+1 \;=\; 2.\end{aligned}$$

It remains to prove that $(\Psi(p,q,n))_{n=1}^{\infty}$ and $(a_{p,q,n})_{n=1}^{\infty}$ satisfy the same recurrence: for all $n \ge p+q$, we verify the following identity

$$\sum_{i=0}^{\left\lfloor \frac{n-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} \;=\; \sum_{i=0}^{\left\lfloor \frac{n-p}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor}{i} + \sum_{i=0}^{\left\lfloor \frac{n-2p}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+2)}{q} \right\rfloor}{i}.$$

We have

$$\begin{aligned}&\sum_{i=0}^{\left\lfloor \frac{n-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} - \sum_{i=0}^{\left\lfloor \frac{n-p}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor}{i} \\ =\; &\sum_{i=1}^{\left\lfloor \frac{n-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} - \sum_{i=1}^{\left\lfloor \frac{n-p}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor}{i}\end{aligned}$$

$$= \sum_{i=\left\lfloor \frac{n-p}{p+q} \right\rfloor+1}^{\left\lfloor \frac{n-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} + \sum_{i=1}^{\left\lfloor \frac{n-p}{p+q} \right\rfloor} \left(\binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} - \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor}{i} \right)$$
$$= \sum_{i=\left\lfloor \frac{n+q}{p+q} \right\rfloor}^{\left\lfloor \frac{n-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} + \sum_{i=1}^{\left\lfloor \frac{n-p}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor}{i-1}$$
$$= \sum_{i=\left\lfloor \frac{n+q}{p+q} \right\rfloor}^{\left\lfloor \frac{n-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} + \sum_{i=0}^{\left\lfloor \frac{n-2p-q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+2)}{q} \right\rfloor}{i}. \tag{8}$$

Write $n - p = (p+q)j + k$ for some $j \geq 0$ and $0 \leq k \leq p + q - 1$ and proceed by case analysis.

If $0 \leq k \leq p - 1$, then (8) is equal to

$$\sum_{i=j+1}^{j} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} + \sum_{i=0}^{j-1} \binom{\left\lfloor \frac{n-p(i+2)}{q} \right\rfloor}{i} = \sum_{i=0}^{j-1} \binom{\left\lfloor \frac{n-p(i+2)}{q} \right\rfloor}{i}$$
$$= \sum_{i=0}^{\left\lfloor \frac{n-2p}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+2)}{q} \right\rfloor}{i}.$$

If $p \leq k \leq p + q - 1$, then (8) is equal to

$$\sum_{i=j+1}^{j+1} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} + \sum_{i=0}^{j-1} \binom{\left\lfloor \frac{n-p(i+2)}{q} \right\rfloor}{i}$$
$$= 1 + \sum_{i=0}^{j-1} \binom{\left\lfloor \frac{n-p(i+2)}{q} \right\rfloor}{i}$$
$$= \sum_{i=0}^{j} \binom{\left\lfloor \frac{n-p(i+2)}{q} \right\rfloor}{i} = \sum_{i=0}^{\left\lfloor \frac{n-2p}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+2)}{q} \right\rfloor}{i}.$$

We have shown the identity

$$\sum_{i=0}^{\left\lfloor \frac{n-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor + 1}{i} - \sum_{i=0}^{\left\lfloor \frac{n-p}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+1)}{q} \right\rfloor}{i} = \sum_{i=0}^{\left\lfloor \frac{n-2p}{p+q} \right\rfloor} \binom{\left\lfloor \frac{n-p(i+2)}{q} \right\rfloor}{i}$$

holds for all $n \geq p + q$. □

Proof of (5). First, we verify that $\Phi(p, q, n) = a_{p,q,n}$ for $0 \leq n \leq p + q - 1$.

Case 1: $p \geq q$. For $n \leq p+q-1$, we have

$$\frac{n-iq}{p} - 2 \;\leq\; \frac{p+q-1-iq}{p} - 2 \;\leq\; \frac{q-1}{p} - 1 \;<\; 0, \text{ for all } i \geq 0.$$

Hence, we have

$$\sum_{\substack{i=0 \\ qi \equiv n \pmod p}}^{\lfloor \frac{n}{q} \rfloor} \binom{i}{\frac{n-iq}{p} - 2} + \begin{cases} 1, & \text{if } n = p-q; \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & \text{if } n = p-q; \\ 0, & \text{if } 0 \leq n \leq p+q-1 \text{ and } n \neq p-q. \end{cases}$$

Case 2: $p < q$. For $n \leq 2p-1$, we have

$$\frac{n-iq}{p} - 2 \;\leq\; \frac{2p-1}{p} - 2 \;<\; 0, \text{ for all } i \geq 0,$$

which implies that

$$\sum_{\substack{i=0 \\ qi \equiv n \pmod p}}^{\lfloor \frac{n}{q} \rfloor} \binom{i}{\frac{n-iq}{p} - 2} \;=\; 0.$$

For $n = 2p$, we have

$$\sum_{\substack{i=0 \\ qi \equiv n \pmod p}}^{\lfloor \frac{n}{q} \rfloor} \binom{i}{\frac{n-iq}{p} - 2} \;=\; \sum_{\substack{i=0 \\ p|(qi)}}^{\lfloor \frac{2p}{q} \rfloor} \binom{i}{\frac{-iq}{p}} \;=\; \binom{0}{0} \;=\; 1$$

because every positive i makes $\binom{i}{-iq/p} = 0$. For $2p+1 \leq n \leq p+q-1$, we have

$$\frac{n-iq}{p} - 2 \;\leq\; \frac{p+q-1-iq}{p} - 2 \;=\; \frac{(1-i)q-1}{p} - 1 \;<\; 0, \text{ for all } i \geq 1,$$

which implies that

$$\sum_{\substack{i=0 \\ qi \equiv n \pmod p}}^{\lfloor \frac{n}{q} \rfloor} \binom{i}{\frac{n-iq}{p} - 2} \;=\; \begin{cases} \binom{0}{\frac{n}{p}-2}, & \text{if } p|n \\ 0, & \text{otherwise} \end{cases} \;=\; 0$$

because if $p|n$, then $n/p \geq 3$.

Next, we show that $(\Phi(p,q,n))_{n=0}^{\infty}$ follows the same recurrence as $(a_{p,q,n}^{(m)})_{n=0}^{\infty}$, namely

$$\Phi(p,q,n) \;=\; \Phi(p,q,n-q) + \Phi(p,q,n-p-q), \tag{9}$$

for $n \geq p+q$. To do so, we consider the two cases when $n = 2p$ and $n \neq 2p$.

Case A: $n = 2p$. We have

$$\Phi(p,q,n-p-q) \;=\; \Phi(p,q,p-q) \;=\; \sum_{\substack{i=0 \\ p|((i+1)q)}}^{\lfloor \frac{p}{q} \rfloor - 1} \binom{i}{\frac{-(i+1)q}{p} - 1} + 1 \;=\; 1$$

because $-(i+1)q/p - 1 < 0$ for all $i \geq 0$. Then

$$\Phi(p,q,2p) \;=\; \Phi(p,q,2p-q) + \Phi(p,q,p-q)$$

can be rewritten as

$$\sum_{\substack{i=0 \\ p|(qi)}}^{\lfloor \frac{2p}{q} \rfloor} \binom{i}{\frac{-iq}{p}} \;=\; \sum_{\substack{i=0 \\ p|((i+1)q)}}^{\lfloor \frac{2p}{q} \rfloor - 1} \binom{i}{\frac{-(i+1)q}{p}} + 1. \tag{10}$$

On the left side of (10), only $i = 0$ contributes a nonzero term to the sum, so

$$\sum_{\substack{i=0 \\ p|(qi)}}^{\lfloor \frac{2p}{q} \rfloor} \binom{i}{\frac{-iq}{p}} \;=\; 1.$$

On the right side of (10), each binomial in the sum is zero because $i \geq 0 > -(i+1)q/p$. This confirms (9) when $n = 2p$.

Case B: $n \neq 2p$. We write (9) as

$$\begin{aligned}
&\sum_{\substack{i=0 \\ qi \equiv n \pmod{p}}}^{\lfloor \frac{n}{q} \rfloor} \binom{i}{\frac{n-iq}{p} - 2} \\
&= \sum_{\substack{i=0 \\ q(i+1) \equiv n \pmod{p}}}^{\lfloor \frac{n}{q} \rfloor - 1} \binom{i}{\frac{n-(i+1)q}{p} - 2} + \sum_{\substack{i=0 \\ q(i+1) \equiv n \pmod{p}}}^{\lfloor \frac{n-p}{q} \rfloor - 1} \binom{i}{\frac{n-(i+1)q}{p} - 3}.
\end{aligned} \tag{11}$$

The right side of (11) is

$$\begin{aligned}
&\sum_{\substack{i=\lfloor \frac{n-p}{q} \rfloor \\ q(i+1) \equiv n \pmod{p}}}^{\lfloor \frac{n}{q} \rfloor - 1} \binom{i}{\frac{n-(i+1)q}{p} - 2} + \sum_{\substack{i=0 \\ q(i+1) \equiv n \pmod{p}}}^{\lfloor \frac{n-p}{q} \rfloor - 1} \left(\binom{i}{\frac{n-(i+1)q}{p} - 2} + \binom{i}{\frac{n-(i+1)q}{p} - 3} \right) \\
&= \sum_{\substack{i=\lfloor \frac{n-p}{q} \rfloor \\ q(i+1) \equiv n \pmod{p}}}^{\lfloor \frac{n}{q} \rfloor - 1} \binom{i}{\frac{n-(i+1)q}{p} - 2} + \sum_{\substack{i=0 \\ q(i+1) \equiv n \pmod{p}}}^{\lfloor \frac{n-p}{q} \rfloor - 1} \binom{i+1}{\frac{n-(i+1)q}{p} - 2}
\end{aligned}$$

$$= \sum_{\substack{i=\lfloor \frac{n-p}{q} \rfloor \\ q(i+1)\equiv n \pmod p}}^{\lfloor \frac{n}{q} \rfloor -1} \binom{i}{\frac{n-(i+1)q}{p}-2} + \sum_{\substack{i=1 \\ qi\equiv n \pmod p}}^{\lfloor \frac{n-p}{q} \rfloor} \binom{i}{\frac{n-iq}{p}-2}. \tag{12}$$

By (11) and (12), we need to prove

$$\sum_{\substack{i=\lfloor \frac{n-p}{q} \rfloor+1 \\ qi\equiv n \pmod p}}^{\lfloor \frac{n}{q} \rfloor} \binom{i}{\frac{n-iq}{p}-2} = \sum_{\substack{i=\lfloor \frac{n-p}{q} \rfloor \\ q(i+1)\equiv n \pmod p}}^{\lfloor \frac{n}{q} \rfloor -1} \binom{i}{\frac{n-(i+1)q}{p}-2}. \tag{13}$$

Note that we discard $\binom{0}{n/p-2}$ on the left side of (11) because the term is nonzero only when $n = 2p$. We claim that both sides of (13) vanish. On the left, we have

$$\frac{n-p}{q} < i \leq \frac{n}{q} \quad \text{and} \quad qi \equiv n \pmod p,$$

which give

$$-p < qi - n \leq 0 \quad \text{and} \quad qi \equiv n \pmod p.$$

Hence, we obtain $qi - n = 0$, and the corresponding binomial is $\binom{i}{-2} = 0$ because $i \geq 1$. Similarly, the right side of (13) is also 0. □

3 Subsequences of Padovan-like sequences

3.1 Proof of Theorem 1

In this section, we show that the terms of $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ are taken q-periodically from the sequences $(a_{p,q,n})_{n=0}^{\infty}$ and $(a_{p,q,n}^{(m)})_{n=0}^{\infty}$. This allows us to deduce a linear recurrence for $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$.

First, we find a formula for $|\mathcal{S}_n^{p/q}|$. For $1 \leq i \leq n$, let F be an i-element set in $\mathcal{S}_n^{p/q}$; that is, we have $F \subset \{1, 2, \dots, n\}$, $n \in F$, $|F| = i$, and $q \min F \geq pi$. Hence, to form F, we choose $i-1$ elements in $\left\{\left\lceil \frac{pi}{q} \right\rceil, \dots, n-1\right\}$. Therefore, the number of i-element sets in $|\mathcal{S}_n^{p/q}|$ is

$$\binom{n - \left\lceil \frac{pi}{q} \right\rceil}{i-1}.$$

Here we require that $n - \lceil pi/q \rceil \geq i - 1$, or equivalently, we require that $i \leq \left\lfloor \frac{(n+1)q}{p+q} \right\rfloor$. Therefore, we have

$$|\mathcal{S}_n^{p/q}| = \sum_{i=1}^{\left\lfloor \frac{(n+1)q}{p+q} \right\rfloor} \binom{n - \left\lceil \frac{pi}{q} \right\rceil}{i-1}. \tag{14}$$

Next, we give a formula for $|\mathcal{M}_n^{p/q}|$. Let $p' = p/\gcd(p,q)$ and $q' = q/\gcd(p,q)$. Consider a (p,q)-maximal Schreier set $F \subset \{1,\ldots,n\}$ with $n \in F$ and $|F| = i$. Since $q \min F = p|F| = pi$, we know that i is a multiple of q'. Let $i = q'k$. Then $\min F = p'k$, and $F = \{p'k, n\} \cup F'$, where $F' \subset \{p'k+1, p'k+2, \ldots, n-1\}$ and $|F'| = q'k - 2$. Hence, there are $\binom{n-p'k-1}{q'k-2}$ such sets F. Here we require $n - p'k - 1 \ge q'k - 2$, or equivalently, we require $k \le (n+1)/(p'+q')$. Therefore, we obtain

$$|\mathcal{M}_n^{p/q}| \;=\; \sum_{k=1}^{\left\lfloor \frac{n+1}{p'+q'} \right\rfloor} \binom{n-p'k-1}{q'k-2}. \tag{15}$$

Proof of Theorem 1. First, we prove (6). We have

$$\begin{aligned}
\Psi(p,q,(n-1)q) \;&=\; \sum_{i=0}^{\left\lfloor \frac{(n-1)q-p+q}{p+q} \right\rfloor} \binom{\left\lfloor \frac{(n-1)q-p(i+1)}{q} \right\rfloor + 1}{i} \\
&=\; \sum_{i=0}^{\left\lfloor \frac{nq-p}{p+q} \right\rfloor} \binom{n + \left\lfloor \frac{-p(i+1)}{q} \right\rfloor}{i} \\
&=\; \sum_{i=0}^{\left\lfloor \frac{nq-p}{p+q} \right\rfloor} \binom{n - \left\lceil \frac{p(i+1)}{q} \right\rceil}{i} \\
&=\; \sum_{i=1}^{\left\lfloor \frac{(n+1)q}{p+q} \right\rfloor} \binom{n - \left\lceil \frac{pi}{q} \right\rceil}{i-1} \;=\; |\mathcal{S}_n^{p/q}|.
\end{aligned}$$

Next, we prove (7). Let $p' = p/\gcd(p,q)$ and $q' = q/\gcd(p,q)$. Letting $n - i - 1 = p'k$ in the equations below, we have

$$\begin{aligned}
&\Phi(p,q,(n-1)q) \\
&= \sum_{\substack{i=0 \\ p' \mid (n-i-1)}}^{n-1} \binom{i}{\frac{q'(n-i-1)}{p'} - 2} + \begin{cases} 1, & \text{if } (n,q') = (p',1), \\ 0, & \text{otherwise}, \end{cases} \\
&= \sum_{k=0}^{\left\lfloor \frac{n-1}{p'} \right\rfloor} \binom{n-p'k-1}{q'k-2} + \begin{cases} 1, & \text{if } (n,q') = (p',1), \\ 0, & \text{otherwise}, \end{cases} \qquad (\text{with } n-i-1 = p'k) \\
&= \sum_{k=1}^{\left\lfloor \frac{n-1}{p'} \right\rfloor} \binom{n-p'k-1}{q'k-2} + \begin{cases} 1, & \text{if } (n,q') = (p',1), \\ 0, & \text{otherwise}. \end{cases}
\end{aligned} \tag{16}$$

Thanks to (15) and (16), it suffices to verify that for $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{\left\lfloor \frac{n+1}{p'+q'} \right\rfloor} \binom{n-p'k-1}{q'k-2} \;=\; \sum_{k=1}^{\left\lfloor \frac{n-1}{p'} \right\rfloor} \binom{n-p'k-1}{q'k-2} + \begin{cases} 1, & \text{if } (n,q') = (p',1), \\ 0, & \text{otherwise}. \end{cases} \tag{17}$$

We proceed by case analysis.

Case 1: $(n-1)/p' \geq (n+1)/(p'+q')$ which is equivalent to $nq' \geq 2p'+q'$. Then $(n, q') \neq (p', 1)$. We have

$$\sum_{k=1}^{\left\lfloor \frac{n-1}{p'} \right\rfloor} \binom{n-p'k-1}{q'k-2} - \sum_{k=1}^{\left\lfloor \frac{n+1}{p'+q'} \right\rfloor} \binom{n-p'k-1}{q'k-2} = \sum_{k=\left\lfloor \frac{n+1}{p'+q'} \right\rfloor+1}^{\left\lfloor \frac{n-1}{p'} \right\rfloor} \binom{n-p'k-1}{q'k-2}.$$

For each $k \geq \left\lfloor \frac{n+1}{p'+q'} \right\rfloor + 1$, we have $n - p'k - 1 < q'k - 2$, so the binomial $\binom{n-p'k-1}{q'k-2}$ is zero unless $q'k = 2$, which occurs when either $(q', k) = (2, 1)$ or $(q', k) = (1, 2)$.

a) *Case 1.1*: $(q', k) = (2, 1)$. That $k = 1$ implies that $\left\lfloor \frac{n+1}{p'+q'} \right\rfloor = 0$, which in turn implies $n < p' + 1$. However, the inequality $nq' \geq 2p' + q'$ gives $2n \geq 2p' + 2$, which contradicts $n < p' + 1$.

b) *Case 1.2*: $(q', k) = (1, 2)$. Then $\left\lfloor \frac{n+1}{p'+q'} \right\rfloor = 1$, which implies

$$n < 2p' + 2q' - 1 = 2p' + 1.$$

However, the inequality $nq' \geq 2p' + q'$ gives $n \geq 2p' + 1$, which contradicts $n < 2p' + 1$.

Case 2: $(n-1)/p' < (n+1)/(p'+q')$ which is equivalent to $n < 1 + 2p'/q'$. Then $(n+1)/(p'+q') < 2$.

If $\left\lfloor \frac{n+1}{p'+q'} \right\rfloor = 0$, then $(n, q') \neq (p', 1)$, and $(n-1)/p' < (n+1)/(p'+q')$ implies that $\left\lfloor \frac{n-1}{p'} \right\rfloor = 0$. Hence, both sides of (17) are zero.

If $\left\lfloor \frac{n+1}{p'+q'} \right\rfloor = 1$, then $\left\lfloor \frac{n-1}{p'} \right\rfloor \in \{0, 1\}$.

a) *Case 2.1*: If $\left\lfloor \frac{n-1}{p'} \right\rfloor = 1$, then $n \neq p'$, and (17) holds.

b) *Case 2.2*: If $\left\lfloor \frac{n-1}{p'} \right\rfloor = 0$, then $n < p'+1$. On the other hand, we deduce from $\left\lfloor \frac{n+1}{p'+q'} \right\rfloor = 1$ that $n \geq p' + q' - 1$. Hence, we obtain

$$p' + 1 > n \geq p' + q' - 1 \implies q' = 1.$$

Then (17) becomes

$$\sum_{k=1}^{1} \binom{n-p'k-1}{k-2} = \sum_{k=1}^{0} \binom{n-p'k-1}{k-2} + \begin{cases} 1, & \text{if } n = p', \\ 0, & \text{otherwise,} \end{cases}$$

or equivalently, we have

$$\binom{n-p-1}{-1} \;=\; \begin{cases} 1, & \text{if } n = p, \\ 0, & \text{otherwise,} \end{cases}$$

which is true.

□

3.2 Recurrence of $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$

By Theorem 1, the recurrence of $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$ is the same as the recurrence of subsequences taken q-periodically out of $(a_{p,q,n})_{n=0}^{\infty}$ and $(a_{p,q,n}^{(m)})_{n=0}^{\infty}$, respectively.

Definition 4. Let $p(x) = c_0 + c_1x + \cdots + c_rx^r$ be a polynomial with real coefficients $(c_i)_{i=0}^{r}$. A sequence $(a_n)_{n=0}^{\infty}$ is said to *satisfy* $p(x)$ if

$$c_0a_n + c_1a_{n-1} + \cdots + c_ra_{n-r} \;=\; 0, \text{ for all } n \geq r.$$

Trivially, all sequences satisfy the zero polynomial.

We need the following lemma, whose proof is straightforward and therefore omitted.

Lemma 5. *If a sequence $(a_n)_{n=0}^{\infty}$ satisfies $p(x)$ and $p(x)$ divides $q(x)$, then $(a_n)_{n=0}^{\infty}$ satisfies $q(x)$.*

Proof of Corollaries 2 and 3. We verify initial terms and prove the linear recurrence for $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^{\infty}$. The formulas for initial terms are due to (14) and (15). We prove the recurrence relation. By definition, both $(a_{p,q,n})_{n=0}^{\infty}$ and $(a_{p,q,n}^{(m)})_{n=0}^{\infty}$ satisfy $g_{p,q}(x) := 1 - x^q - x^{p+q}$. Define

$$h_{p,q}(x) \;:=\; 1 - \sum_{i=1}^{q}(-1)^{i+1}\binom{q}{i}x^i - x^{p+q}.$$

We have

$$\begin{aligned} h_{p,q}(x^q) \;&=\; 1 + \sum_{i=1}^{q}(-1)^i\binom{q}{i}x^{qi} - x^{q(p+q)} \\ &=\; \sum_{i=0}^{q}(-1)^i\binom{q}{i}x^{qi} - x^{q(p+q)} \\ &=\; (1-x^q)^q - (x^{p+q})^q \\ &=\; (1 - x^q - x^{p+q})\sum_{i=0}^{q-1}(1-x^q)^{q-1-i}x^{(p+q)i}, \end{aligned}$$

so $g_{p,q}(x)$ divides $h_{p,q}(x^q)$. By Lemma 5, both $(a_{p,q,n})_{n=0}^\infty$ and $(a_{p,q,n}^{(m)})_{n=0}^\infty$ satisfy $h_{p,q}(x^q)$. By Theorem 1, we have $(|\mathcal{S}_n^{p/q}|)_{n=1}^\infty$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^\infty$ satisfy $h_{p,q}(x)$. Here we use the obvious fact that for a fixed $k \in \mathbb{N}$, if $(b_n)_{n=1}^\infty$ satisfies $c_0 + c_k x^k + c_{2k}x^{2k} + \cdots + c_{nk}x^{nk}$, then the subsequence $(b_{(n-1)k})_{n=1}^\infty$ satisfies $c_0 + c_k x + c_{2k}x^2 + \cdots + c_{nk}x^n$. □

Remark 6. While one goal of the present paper is to establish the recurrence relation of $(|\mathcal{S}_n^{p/q}|)_{n=1}^\infty$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^\infty$ without using the inclusion-exclusion principle as in [1], we can use the same argument in the proof of Corollaries 2 and 3 to show that (6) is a corollary of [1, Theorem 1.1]. Indeed, Equality (3) states that $(|\mathcal{S}_n^{p/q}|)_{n=1}^\infty$ satisfies

$$h_{p,q}(x) \ = \ 1 + \sum_{i=1}^{q}(-1)^i\binom{q}{i}x^i - x^{p+q} \ = \ (1-x)^q - x^{p+q}.$$

The definition of $(a_{p,q,n})_{n=0}^\infty$ gives that $(a_{p,q,n})_{n=0}^\infty$ satisfies $g_{p,q}(x) = 1 - x^q - x^{p+q}$. Since $g_{p,q}(x)$ divides $h_{p,q}(x^q) = (1-x^q)^q - x^{q(p+q)}$, we know that $(a_{p,q,n})_{n=0}^\infty$ satisfies $h_{p,q}(x^q)$. Hence, the sequence $(a_{p,q,(n-1)q})_{n=1}^\infty$ satisfies $h_{p,q}(x)$. Since $(|\mathcal{S}_n^{p/q}|)_{n=1}^\infty$ and $(a_{p,q,(n-1)q})_{n=1}^\infty$ satisfy the same polynomial and have the same initial terms, the two sequences are identical.

4 Relation between $(|\mathcal{S}_n^{p/q}|)_{n=1}^\infty$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^\infty$ and future investigations

It is natural to ask about the relation between the two sequences $(|\mathcal{S}_n^{p/q}|)_{n=1}^\infty$ and $(|\mathcal{M}_n^{p/q}|)_{n=1}^\infty$: given one of the sequences, can we compute the other? This section presents the two special cases when $p = 1$ or $q = 1$ of the following problem.

Problem 7. For $(p,q) \in \mathbb{N}^2$, find the constants $(c_i)_{i=1}^\infty$ (if any) such that

$$|\mathcal{M}_n^{p/q}| = \sum_{i=1}^{\infty} c_i|\mathcal{S}_i^{p/q}|, \text{ for all } n \in \mathbb{N}.$$

Theorem 8. *For $p, n \in \mathbb{N}$, we have*

$$|\mathcal{S}_n^{p/1}| \ = \ |\mathcal{M}_{n+p+1}^{p/1}|.$$

Proof. By (14) and (15), we have

$$|\mathcal{S}_n^{p/1}| \ = \ \sum_{i=1}^{\lfloor\frac{n+1}{p+1}\rfloor}\binom{n-pi}{i-1} \text{ and } |\mathcal{M}_n^{p/1}| \ = \ \sum_{i=1}^{\lfloor\frac{n+1}{p+1}\rfloor}\binom{n-pi-1}{i-2}.$$

Hence, we obtain

$$\begin{aligned}
|\mathcal{M}^{p/1}_{n+p+1}| &= \sum_{i=1}^{\lfloor\frac{n+p+2}{p+1}\rfloor}\binom{n+p+1-pi-1}{i-2} = \sum_{i=1}^{\lfloor\frac{n+1}{p+1}\rfloor+1}\binom{n-p(i-1)}{i-2}\\
&= \sum_{i=0}^{\lfloor\frac{n+1}{p+1}\rfloor}\binom{n-pi}{i-1}\\
&= \sum_{i=1}^{\lfloor\frac{n+1}{p+1}\rfloor}\binom{n-pi}{i-1} = |\mathcal{S}^{p/1}_{n}|.
\end{aligned}$$

□

Theorem 9. *For $q, n \in \mathbb{N}$, we have*

$$|\mathcal{M}^{1/q}_n| = 2|\mathcal{S}^{1/q}_n| - |\mathcal{S}^{1/q}_{n+1}|. \tag{18}$$

Proof. Let $n \in \mathbb{N}$. When $q = 1$, we have $|\mathcal{S}^{1/1}_n| = F_n$, $|\mathcal{S}^{1/1}_{n+1}| = F_{n+1}$, and $|\mathcal{M}^{1/1}_n| = F_{n-2}$ (see (1) and (2)). Then (18) holds. For the rest of the proof, assume that $q \geq 2$.

We need to bijectively map two copies of $\mathcal{S}^{1/q}_n$ to $\mathcal{S}^{1/q}_{n+1} \cup \mathcal{M}^{1/q}_n$. Observe that

$$\begin{aligned}
\phi:\ \mathcal{S}^{1/q}_n &\to \mathcal{S}^{1/q}_{n+1}\\
F &\mapsto F+q
\end{aligned}$$

maps $\mathcal{S}^{1/q}_n$ injectively into $\mathcal{S}^{1/q}_{n+1}$. Furthermore, we know from the definition that $\mathcal{M}^{1/q}_n \subset \mathcal{S}^{1/q}_n$. Hence, it suffices to show that

$$|\mathcal{S}^{1/q}_n \backslash \mathcal{M}^{1/q}_n| = |\mathcal{S}^{1/q}_{n+1} \backslash \phi(\mathcal{S}^{1/q}_n)|. \tag{19}$$

The left side of (19) is the cardinality of

$$L_{q,n} := \{F \subset \{1,2,\ldots,n\} : q\min F > |F| \text{ and } \max F = n\},$$

which can be partitioned into the collections

$$L_{q,n,i} := \{F \subset \{1,2,\ldots,n\} : q\min F > |F| = i \text{ and } \max F = n\}, \text{ with } i \geq 1.$$

For $i = 1$, the only set in $L_{q,n,1}$ is $\{n\}$, so we have $|L_{q,n,1}| = 1$. For $i \geq 2$, to form sets in $L_{q,n,i}$, we first choose the minimum j with $i/q < j \leq n-1$ then choose $i-2$ integers in $[j+1, n-1]$, which requires $i-2 \leq n-j-1$. Hence, for $i \geq 2$, we obtain

$$|L_{q,n,i}| = \sum_{\substack{i+1\leq jq\leq (n-1)q\\ j\leq n-i+1}}\binom{n-j-1}{i-2} = \sum_{i+1\leq jq\leq (n-i+1)q}\binom{n-j-1}{i-2}.$$

The right side of (19) is the cardinality of

$$R_{q,n} \ := \ \{F \subset \{1,2,\ldots,n+1\} \,:\, q\min F \geq |F| > q\min F - q, \max F = n+1\},$$

which can be partitioned into

$$R_{q,n,i} \ := \ \{F \subset \{1,2,\ldots,n+1\} \,:\, q\min F \geq |F| = i > q\min F - q, \\ \max F = n+1\}, \text{ with } i \geq 1.$$

We have $R_{q,n,1} = \emptyset$ because a nonempty $R_{q,n,1}$ requires $1 > (n+1)q - q = nq \geq 2$. For $i = 2$, an $F \in R_{q,n,2}$ must have the form $F = \{j, n+1\}$ with $1 \leq j \leq n$ and $jq \geq 2 > (j-1)q$. Since $q \geq 2$, the inequality $2 > (j-1)q$ implies that $j = 1$. Hence, the set F is $\{1, n+1\}$ and $|R_{q,n,2}| = 1$. To form sets in $R_{q,n,i}$ with $i \geq 3$, we first choose the minimum j with $j - 1 < i/q \leq j \leq n$ then choose $i - 2$ integers in $[j+1, n]$, which requires $n - j \geq i - 2$. Hence, for $i \geq 3$, we obtain

$$|R_{q,n,i}| \ = \sum_{\substack{1+q(j-1)\leq i\leq jq\leq nq\\ j\leq n+2-i}} \binom{n-j}{i-2} \ = \sum_{\substack{i\leq jq\leq i+q-1\\ j\leq n-i+2}} \binom{n-j}{i-2}.$$

To prove (19), we show that

$$|L_{q,n,i}| \ = \ |R_{q,n,i+1}|, \text{ for } i \geq 1. \tag{20}$$

The above analysis shows that (20) holds when $i = 1$. For $i \geq 2$, Equality (20) is the same as

$$\sum_{i+1\leq jq\leq (n-i+1)q} \binom{n-j-1}{i-2} \ = \sum_{\substack{i+1\leq jq\leq i+q\\ j\leq n-i+1}} \binom{n-j}{i-1}. \tag{21}$$

Case 1: $(n-i+1)q \leq i + q$. We have

$$\begin{aligned} &\sum_{i+1\leq jq\leq (n-i+1)q} \binom{n-j-1}{i-2} - \sum_{\substack{i+1\leq jq\leq i+q\\ j\leq n-i+1}} \binom{n-j}{i-1} \\ &= \sum_{i+1\leq jq\leq (n-i+1)q} \binom{n-j-1}{i-2} - \sum_{i+1\leq jq\leq (n-i+1)q} \binom{n-j}{i-1}. \end{aligned} \tag{22}$$

Since $(n-i+1)q \leq i+q$, we obtain

$$n \ \leq \ i + \frac{i}{q} \ = \ i + \frac{(i+1)-1}{q} \ \leq \ i + \frac{jq-1}{q} \ < \ i + j,$$

which implies $n \leq i + j - 1$, but (22) sums over $j \leq n - i + 1$. Therefore, $n = i + j - 1$ in (22), which gives

$$\begin{aligned} &\sum_{i+1 \leq jq \leq (n-i+1)q} \binom{n-j-1}{i-2} - \sum_{i+1 \leq jq \leq (n-i+1)q} \binom{n-j}{i-1} \\ &= \sum_{i+1 \leq jq \leq (n-i+1)q} 1 - \sum_{i+1 \leq jq \leq (n-i+1)q} 1 \;=\; 0. \end{aligned}$$

Case 2: $(n - i + 1)q > i + q$. Then (21) is equivalent to

$$\sum_{i+1 \leq jq \leq (n-i+1)q} \binom{n-j-1}{i-2} \;=\; \binom{n - \lceil \frac{i+1}{q} \rceil}{i-1},$$

which is precisely the hockey-stick identity (see [7, Theorem 1.2.3, item (5)]).

We have verified (21) and thus completed the proof. □

5 Acknowledgments

The authors would like to thank the anonymous referee for a careful reading of the paper and for the helpful suggestions.

This work is partially supported by the College of Arts & Sciences at Texas A&M University. The second named author is an undergraduate at Texas A&M University, working under the guidance of the first named author.

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2020 *Mathematics Subject Classification*: Primary 05A19; Secondary 11B37, 11Y55, 05A15.
Keywords: Schreier set, recurrence, Pascal triangle.

(Concerned with sequences A000045, A000931, A005251, A005314, A017817, A017827, A052920, A078012, A079398, A099558, A103372, A135851, A137357, A212804, A226503, A375169, A385106, A385107, and A385142.)

Received June 23 2025; revised versions received December 23 2025; March 3 2026. Published in *Journal of Integer Sequences*, March 23 2026.

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