Journal of Integer Sequences, Vol. 29 (2026),
Article 26.3.4

A Counterexample to a Purported Construction of Normal Numbers

John M. Campbell
Department of Mathematics and Statistics
Dalhousie University
Halifax, NS B3H 4R2
Canada
jh241966@dal.ca

Abstract

Pollack and Vandehey noted that a purported construction of normal numbers given by Szüsz and Volkmann requires an additional condition. As we demonstrate, not only is this condition required in part of the proof of Szüsz and Volkmann's main result, but their original claim is false. We give an explicit counterexample via a decimal expansion that satisfies the conditions of Szüsz and Volkmann but is not normal. This counterexample provides a non-normal analogue of the normal number $0.(1)(4)(9)(16)(25)\cdots$ considered by Besicovitch.

1 Introduction

In the field of probabilistic number theory, the study of distributions of sequences of integers and digits plays a fundamental role. Such distributions give rise to questions as in the following. For a given real number r, does a given sequence of digits appear, in some specified way, in the decimal expansion of r as often as every other sequence of the same length? Informally, a real number satisfying this property is said to be *normal* (in base 10). There is a long and rich history concerning normal numbers, and a notable construction of normal numbers is due to Szüsz and Volkmann [5]. A corrected version of this construction given by Pollack and Vandehey [4] implies the normality of one of the most famous constants in the study of normal numbers.

By taking the sequence $(1, 2, 3, 4, \ldots)$ of consecutive positive integers, and forming the real number

$$0.123456789101112\ldots \tag{1}$$

by concatenating their decimal expansions, we get *Champernowne's constant*, indexed in the On-Line Encyclopedia of Integer Sequences [6] as sequence A033307. This constant was proved to be normal by Champernowne in 1933 [2] and provides the first known and explicit construction of a normal number. Similarly, by taking the sequence $(2, 3, 5, 7, \ldots)$ of increasing prime numbers (given in the OEIS as A000040), we obtain the constant

$$0.235711131719232\ldots \tag{2}$$

known as the *Copeland–Erdős constant* (see A033308). This constant was proved to be normal by Copeland and Erdős in 1946 [3]. The research interest surrounding the constants in (1) and (2) raises questions as to whether or not a given sequence of nonnegative integers gives rise to a normal number using the analogous construction.

Informally, a real number α is said to be *normal in base* $g \geq 2$ if, for a sequence of n base-g digits, this sequence appears with frequency g^{-n} in the base-g expansion of α. For a nonnegative, real-valued function f defined on $\mathbb{N}$, we form the base-g expansion

$$0.\lfloor f(1)\rfloor \lfloor f(2)\rfloor \ldots \tag{3}$$

by analogy with the base-10 concatenations in (1) and (2). A construction due to Szüsz and Volkmann [5] purports to ensure the base-g normality of a concatenation of the form indicated in (3) for a differentiable function f satisfying certain growth properties. Pollack and Vandehey [4] noted (as discussed in more detail below) that Szüsz and Volkmann's method requires that

$$\lim_{x\to\infty} \frac{\log f(x)}{\log x} \leq 1, \tag{4}$$

but the condition in (4) was not included in Szüsz and Volkmann's work. The corrected version of Szüsz and Volkmann's construction provides a way of proving the normality of Champernowne's constant, for the $f(n) = n$ case of (3).

Since the required condition (4) was not included in Szüsz and Volkmann's main result [5], this raises the problem of constructing an explicit counterexample. We give such a construction proving the non-normality of our constructed constant, which may be seen as a non-normal analogue of the normal number

$$0.149162536496481\ldots \tag{5}$$

related to the work of Besicovitch [1] (see also A001191), where (5) is given by the $f(n) = n^2$ case of (3). Observe that by setting $f(x) = x^2$, the required condition in (4) is not satisfied, so that Szüsz and Volkmann's method cannot be used to prove the normality of (5) [4]. Explicitly, Pollack and Vandehey noted the following, where $\eta(f) = \beta$ in the notation of Szüsz and Volkmann:

> "Szüsz and Volkmann mistakenly claim that their result is strong enough to prove [the normality of (5)]. In Theorem 2 of their paper, they need an additional condition that $\beta \leq 1$, because if $\beta > 1$, then the bound in line (3.11) would be $M_k = O(1)$, which would cause their condition (v) to fail." [4, p. 759]

1.1 Szüsz and Volkmann's purported construction

Our terminology and notation concerning normal numbers are mainly based upon Szüsz and Volkmann's work [5]. In this regard, we let $g \geq 2$ denote a fixed base, we let α be a real value, we let E denote a block of digits, and we let $A_E(\alpha, n)$ denote the number of copies of E within the first n digits of α. The value α is *normal of order* k in base g if

$$\lim_{n\to\infty} \frac{A_E(\alpha, n)}{n} = \frac{1}{g^{\ell(E)}} \tag{6}$$

for all blocks E satisfying $\ell(E) = k$, letting $\ell(E)$ denote the number of digits in E, counting multiplicities. The value α is *normal* in base g if it is normal (in base g) for all possible orders $k \geq 1$.

Let f denote a real-valued function on a domain containing $\mathbb{N}$, and let f satisfy $f(n) \geq 0$ and $f(n) > 0$ for sufficiently large n. We then let $\alpha(f) = \alpha_g(f)$ denote the real number such that the base-g expansion of $\alpha(f)$ equals $0.b_1b_2...$, writing b_n in place of the base-g expansion of $\lfloor f(n) \rfloor$. In this regard, a *Champernowne function* is a function f satisfying $\lim_{x\to\infty} f(x) = \infty$ and such that $\alpha_g(f)$ is normal in base g for all possible integers $g \geq 2$. To construct Champernowne functions, according to the purported construction given by Szüsz and Volkmann [5], this requires constraints on the growth of f, formulated in terms of the η-function given as follows. For a function h that is real-valued and positive, we set $\eta(h) = \lim_{x\to\infty} \frac{\log h(x)}{\log x}$, assuming this limit exists.

The main result in Szüsz and Volkmann's paper is given as Theorem 2 [5] and asserts that if f is differentiable and if f is monotonically increasing and positive for all sufficiently large arguments and if $\eta(f)$ and $\eta(f')$ exist and $\eta(f) > 0$, then f is a Champernowne function. Since the condition such that $\eta(f) \leq 1$ would also be required in the proof of this result [4], this leads us to the problem of constructing a function f that satisfies all of Szüsz and Volkmann's conditions and such that (3) is not normal, i.e., with $\eta(f) > 1$.

2 Main construction

For positive integers n, we let

$$h_1(n) := \begin{cases} 0, & \text{if } n < 10; \\ 10^{\lfloor \log_{10}(n) \rfloor}, & \text{otherwise.} \end{cases}$$

Also, for positive integers n, we let

$$h_2(n) := 10^{\lfloor \log_{10}(n) \rfloor} \left\lfloor \frac{\sum_{i=1}^{10^{\lfloor \log_{10}(n) \rfloor}} h_1(i)}{10^{\lfloor \log_{10}(n) \rfloor}} \right\rfloor + \sum_{i=10^{\lfloor \log_{10}(n) \rfloor}+1}^{n} h_1(i). \tag{7}$$

The integer sequence $(h_2(n))_{n \geq 1}$ is given in the OEIS as A380904.

Example 1. Initial entries in the integer sequence given by (7) are such that

$$(h_2(n))_{n \geq 1} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, \ldots),$$

and the behavior of the integer sequence $(h_2(n))_{n \geq 1}$ is further illustrated in Section 2.1 below.

2.1 Outline

To provide an explicit counterexample to Szüsz and Volkmann's claim, we construct a function $f(x)$ (satisfying the required conditions in Szüsz and Volkmann's claim) on $\mathbb{R}_{\geq 0}$, as below, such that the integer part of $f(n)$ is $h_2(n)$ for positive integers n, and we prove the desired equality $\lfloor f(n) \rfloor = h_2(n)$ in Lemma 3 below. The motivation behind the use of the integer sequence $(h_2(n))_{n \geq 1}$ for the purposes of this paper is outlined below.

Informally, the sequence $(h_2(n))_{n \geq 1}$ is defined in such a way so that it grows quadratically, with $\eta(f) = 2$, but in such a way so that the number of trailing zeroes in the entries of $(h_2(n))_{n \geq 1}$ is made to be sufficiently large so as to prevent normality. Moreover, this needs to be in such a way so that f is sufficiently well-behaved, i.e., so that the limit associated with $\eta(f')$ exists. This is illustrated below.

For $n \geq 9$, differences between consecutive terms in $(h_2(n))_{n \geq 1}$ are all positive powers of 10. Informally, these powers of 10 are determined in such a way so that the number of trailing zeroes among the terms in $(h_2(n))_{n \geq 1}$ continues to grow in a sufficiently well-behaved way. With regard to the initial values of h_2 shown in Example 1, starting from $h_2(10) = 10$, we add the value 10 iteratively to form new terms until we reach an index of the form $10^m - 1$, with $(h_2(n))_{10 \leq n \leq 99} = (10, 20, 30, \ldots, 890, 900)$. The next term $h_2(100)$ is the next value after $h_2(99) = 900$ with at least 2 trailing zeroes, with $h_2(100) = 1000$. We then add the value 100 iteratively to form new terms until we again reach an index of the form $10^m - 1$, with $(h_2(n))_{100 \leq n \leq 999} = (1000, 1100, 1200, \ldots, 90800, 90900)$, and, mimicking a previous step, the next term $h_2(1000)$ is the next value after $h_2(999) = 90900$ with at least 3 trailing zeroes, with $h_2(1000) = 91000$. We then continue in the suggested manner. The explicit formula in (7) formalizes the construction illustrated above.

2.2 Lemmata

Let $y_1(n) := h_2\left(10^{n-1}\right)$ and $y_2(n) := h_2\big(10^n - 1\big)$. Our construction requires the following properties concerning the functions h_1, y_1, and y_2.

Lemma 2. *For positive integers ℓ, the relation*

$$\sum_{i=1}^{10^{\ell+1}-1} h_1(i) = \underbrace{9090\cdots 90}_{\ell}0 \tag{8}$$

holds. For integers $\ell \geq 2$, the relations

$$\frac{y_2(\ell) - y_1(\ell)}{9 \cdot 10^{\ell-1} - 1} = 10^{\ell-1} \text{ and} \tag{9}$$

$$y_1(\ell+1) - y_2(\ell) = 10^{2\left\lfloor \frac{\ell}{2} \right\rfloor} \tag{10}$$

hold.

Proof. For positive integers ℓ, for $j \in \{1, 2, \dots, \ell\}$, and for $i \in \{10^j, 10^{j+1} - 1\}$, since $h_1(i) = 10^j$ is constant, we rewrite the left-hand side of (8) as

$$\sum_{i=10^1}^{10^2-1} h_1(i) + \sum_{i=10^2}^{10^3-1} h_1(i) + \cdots + \sum_{i=10^\ell}^{10^{\ell+1}-1} h_1(i) = \sum_{i=10^1}^{10^2-1} 10^1 + \sum_{i=10^2}^{10^3-1} 10^2 + \cdots + \sum_{i=10^\ell}^{10^{\ell+1}-1} 10^\ell$$

to obtain the decimal expansion on the right of (8).

We find that the relation

$$h_2\left(10^\ell\right) = 10^\ell \left\lfloor \frac{\sum_{i=1}^{10^\ell} h_1(i)}{10^\ell} \right\rfloor \tag{11}$$

follows in a direct way by setting $n = 10^\ell$ in (7). Setting $n = 10^\ell - 1$ in (7), we find that

$$h_2\left(10^\ell - 1\right) = 10^{\ell-1} \left\lfloor \frac{\sum_{i=1}^{10^{\ell-1}} h_1(i)}{10^{\ell-1}} \right\rfloor + \sum_{i=10^{\ell-1}+1}^{10^\ell-1} h_1(i),$$

and, since $h_1(i) = 10^{\ell-1}$ is constant for $i \in \{10^{\ell-1}+1, 10^{\ell-1}+2, \dots, 10^\ell - 1\}$, we obtain the right-hand expression in the relation

$$h_2\left(10^\ell - 1\right) = 10^{\ell-1} \left\lfloor \frac{\sum_{i=1}^{10^{\ell-1}} h_1(i)}{10^{\ell-1}} \right\rfloor + 10^{\ell-1}\left(10^\ell - 10^{\ell-1} - 1\right) \tag{12}$$

for $\ell \geq 2$. Since

$$\frac{y_2(\ell) - y_1(\ell)}{9 \cdot 10^{\ell-1} - 1} = \frac{h_2(10^\ell - 1) - h_2(10^{\ell-1})}{9 \cdot 10^{\ell-1} - 1}, \tag{13}$$

we may evaluate the right-hand side of (13) in closed form using (11) and (12) together for $\ell \geq 2$.

The evaluation in (8) gives us that

$$\left\lfloor \frac{\sum_{i=1}^{10^\ell} h_1(i)}{10^\ell} \right\rfloor = \begin{cases} 10^{\ell-1}, & \text{if } \ell \leq 2; \\ \underbrace{9090\cdots90}_{\lfloor\frac{\ell-1}{2}\rfloor-1}91, & \text{if } \ell \neq 1 \text{ is odd;} \\ \underbrace{9090\cdots90}_{\lfloor\frac{\ell-1}{2}\rfloor-1}910, & \text{if } \ell \neq 2 \text{ is even.} \end{cases} \tag{14}$$

holds. Consequently, from (11), we find that

$$y_1(\ell) = \underbrace{9090\cdots90}_{\lfloor\frac{\ell}{2}\rfloor-2}91\,\underbrace{00\cdots0}_{2\lfloor\frac{\ell-1}{2}\rfloor+1} \tag{15}$$

holds for $\ell \geq 4$. Similarly, from (12) and (14) together, we find that

$$y_2(\ell) = \underbrace{9090\cdots90}_{\lfloor\frac{\ell+1}{2}\rfloor}\underbrace{00\cdots0}_{2\lfloor\frac{\ell}{2}\rfloor-1} \tag{16}$$

holds for $\ell \geq 2$. From (15) and (16) together with the base cases for $\ell \leq 3$, we obtain that (10) holds for $\ell \geq 2$. $\square$

For $0 \leq x \leq \frac{100}{11}$, we set

$$f(x) = \frac{89}{110}e^{\frac{100}{89}(11x-100)} + \frac{1}{10}. \tag{17}$$

For $\frac{100}{11} \leq x \leq 10$, we set $f(x) = 10x - 90$. Now, let $m \geq 2$ be an integer, and set

$$\mathcal{H}(m) := y_1(m+1) - y_2(m), \tag{18}$$

$$a(m) := \frac{20\mathcal{H}(m) - 11\cdot 10^m}{9} + 9\cdot 10^{m-1}, \tag{19}$$

$$b(m) := \frac{(10^m-1)\,(-20\,\mathcal{H}(m) + 11\cdot 10^m)}{9} + 10^m - 9\cdot 10^{2m-1}, \tag{20}$$

$$c(m) := -20\mathcal{H}(m) + 9\cdot 10^{m-1} + 11\cdot 10^m, \tag{21}$$

$$d(m) := 20\cdot 10^m\,\mathcal{H}(m) - 11\cdot 10^{2m} - 9\cdot 10^{2m-1} + 10^m. \tag{22}$$

Again for $m \in \mathbb{N}_{\geq 2}$, we define the function $h_3^{(m)}$ on the interval $[10^m - 1, 10^m]$ so that

$$h_3^{(m)}(x) := \begin{cases} a(m)x + b(m), & \text{if } 10^m - 1 \leq x \leq 10^m - \frac{1}{10}; \\ c(m)x + d(m), & \text{if } 10^m - \frac{1}{10} \leq x \leq 10^m. \end{cases}$$

We then set

$$f(x) := \frac{x(y_2(m) - y_1(m))}{9\cdot 10^{m-1} - 1} - \frac{y_1(m) + 10^{m-1}y_2(m) - 10^m y_1(m)}{9\cdot 10^{m-1} - 1} \tag{23}$$

if $10^{m-1} \leq x \leq 10^m - 1$ and we set

$$f(x) := h_2(10^m - 1) + \int_{10^m-1}^{x} h_3^{(m)}(t)\, dt$$

if $10^m - 1 \leq x \leq 10^m$, again with the understanding that $m \geq 2$.

Lemma 3. *The equality $\lfloor f(n) \rfloor = h_2(n)$ holds for positive integers n. Moreover, the equality $f(n) = h_2(n)$ holds for integers $n \geq 10$.*

Proof. The sequence $(h_2(n))_{n \geq 1}$ may be constructed in an equivalent way as follows. Let $\mathcal{S}(0) = 0$. Let n be a positive integer. If n is not of the form 10^j for $j > 1$, we set

$$\mathcal{S}(n) = \mathcal{S}(n-1) + h_1(n). \tag{24}$$

Otherwise, we set $\mathcal{S}(n) = n \left\lfloor \frac{\mathcal{S}(n-1)+h_1(n)}{n} \right\rfloor$. We claim that $\mathcal{S}(n) = h_2(n)$ for all natural numbers n. This leads us to show that $h_2(n)$ satisfies the same recurrence as $\mathcal{S}(n)$, and we find that the base case holds. If n is not of the form 10^j for $j > 1$, then the desired recurrence $h_2(n) = h_2(n-1) + h_1(n)$ follows in a direct way from the definition in (7) by using the property that $\lfloor \log_{10}(n) \rfloor = \lfloor \log_{10}(n-1) \rfloor$. Now, suppose that n is of the form 10^j for $j > 1$. Then $n-1$ is not of this form, so, inductively, we have that

$$n \left\lfloor \frac{\mathcal{S}(n-1) + h_1(n)}{n} \right\rfloor = 10^j \left\lfloor \frac{10^{j-1} \left\lfloor \frac{\sum_{i=1}^{10^{j-1}} h_1(i)}{10^{j-1}} \right\rfloor + \sum_{i=10^{j-1}+1}^{10^j} h_1(i)}{10^j} \right\rfloor,$$

so it remans to prove that

$$\left\lfloor \frac{\sum_{i=1}^{10^j} h_1(i)}{10^j} \right\rfloor = \left\lfloor \frac{10^{j-1} \left\lfloor \frac{\sum_{i=1}^{10^{j-1}} h_1(i)}{10^{j-1}} \right\rfloor + \sum_{i=10^{j-1}+1}^{10^j} h_1(i)}{10^j} \right\rfloor. \tag{25}$$

We then find that (25) is equivalent to

$$\left\lfloor \frac{\sum_{i=1}^{10^j} h_1(i)}{10^j} \right\rfloor = \left\lfloor \frac{1}{10} \left\lfloor \frac{\sum_{i=1}^{10^{j-1}} h_1(i)}{10^{j-1}} \right\rfloor + \frac{9}{10} \left(1 + 10^{j-1}\right) \right\rfloor. \tag{26}$$

For $j \geq 3$, the desired recurrence in (26) then follows in a direct way from the evaluation in (14).

From the definition for $f(x)$ for $x \leq 10$, it is immediate that $\lfloor f(n) \rfloor = 0 = h_2(n)$ for $n \leq 10$, and we may similarly routinely verify that $f(n) = h_2(n)$ holds for $10 \leq n \leq 100$.

Suppose that n satisfies $10^{m-1} < n \leq 10^m - 1$, again for some $m \in \mathbb{N}_{\geq 2}$. From the first case of the definition for f, we find that

$$f(n) - f(n-1) = \frac{y_2(m) - y_1(m)}{9 \cdot 10^{m-1} - 1}.$$

The relation (9) in Lemma 2 then gives us that $f(n) - f(n-1) = 10^{m-1} = h_1(n)$, i.e., so that the same recurrence in (24) holds if n is not of the form 10^j for $j > 1$. Similarly, by setting $x = 10^{m-1}$ in (23), we obtain that $f(n) = h_2(n)$. □

Lemma 4. *The mapping $f \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is differentiable on $\mathbb{R}_{>0}$.*

Proof. The elementary functions used to define f for $x \in \left[0, \frac{100}{11}\right]$ and for $x \in \left[\frac{100}{11}, 10\right]$ are such that f is differentiable on $\left(0, \frac{100}{11}\right)$ and on $\left(\frac{100}{11}, 10\right)$, and, moreover, we have that

$$\lim_{x \to \left(\frac{100}{11}\right)^-} f(x) = \lim_{x \to \left(\frac{100}{11}\right)^+} f(x) = f\left(\frac{100}{11}\right) = \frac{10}{11}$$

and that

$$\lim_{x \to \left(\frac{100}{11}\right)^-} f'(x) = \lim_{x \to \left(\frac{100}{11}\right)^+} f'(x) = 10.$$

We also find that the evaluation $f(x) = 10x - 90$ holds for $x \in \left[\frac{100}{11}, 99\right]$.

Let $m \in \mathbb{N}_{\geq 2}$. For $x \in (10^{m-1}, 10^m - 1)$, we have that $f(x)$ agrees with a polynomial of degree 1, with

$$f'(x) = \frac{y_2(m) - y_1(m)}{9 \cdot 10^{m-1} - 1} = 10^{m-1}$$

as the leading coefficient, by the relation (9) in Lemma 2. Similarly, for $x \in (10^m - 1, 10^m)$, we have that $f'(x) = h_3^{(m)}(x)$, and the expansions among (18)–(22) give us that $\lim_{x \to (10^m - 1)^+} h_3^{(m)}(x) = 10^{m-1}$.

Finally, the expansions among (18)–(22) give us that $\lim_{x \to (10^m)^-} h_3^{(m)}(x) = 10^m$, and this value agrees with $f'(x)$ for $x \in (10^m,\ 10^{m+1}\ -1)$. □

Lemma 5. *The mapping $f \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is monotonically increasing on its domain.*

Proof. The elementary function in (17) gives us that f is monotonically increasing for $x \leq \frac{100}{11}$, and, as above, we have that $f(x) = 10x - 90$ for $x \in \left[\frac{100}{11}, 99\right]$. For $m \in \mathbb{N}_{\geq 2}$, and for $10^{m-1} \leq x \leq 10^m - 1$, we have, as above, that $f(x)$ agrees with a degree-1 polynomial with 10^{m-1} as its leading coefficient. For $10^m - 1 \leq x \leq 10^m$, we consider two cases. First suppose that $10^m - 1 \leq x \leq 10^m - \frac{1}{10}$. In this case, we have that $h_3^{(m)}(x)$ agrees with a line segment, and its value at $10^m - 1$ is 10^{m-1} and its value at $10^m - \frac{1}{10}$ is

$$2y_1(m+1) - 2y_2(m) - 19 \cdot 10^{m-2}. \tag{27}$$

From (27), together with (10) in Lemma 2 and the inequality $\frac{9}{10} > \frac{1}{9}\left(10 - 2 \cdot 10^{1-m+2\lfloor \frac{m}{2} \rfloor}\right)$, we find that the expression displayed in (27) is positive. For $10^m - \frac{1}{10} \leq x \leq 10^m$, we have that $h_3^{(m)}(x)$ agrees with a line segment, and its value at $10^m - \frac{1}{10}$ again equals (27), and its value at 10^m is 10^m. $\square$

Lemma 6. *The limit associated with $\eta(f)$ exists, with $\eta(f) = 2$.*

Proof. The monotonicity of f gives that for every $x \in [10^{m-1}, 10^m)$, it holds that $y_1(m) \leq f(x) \leq y_1(m+1)$, and thus

$$\frac{9}{100} 10^{2\lfloor \log_{10}(x) \rfloor} \leq f(x) \leq 10 \cdot 10^{2\lfloor \log_{10}(x) \rfloor}. \tag{28}$$

From (28), we write

$$\frac{\log\left(\frac{9}{100} 10^{2\lfloor \log_{10}(x) \rfloor}\right)}{\log(x)} \leq \frac{\log(f(x))}{\log(x)} \leq \frac{\log\left(10 \cdot 10^{2\lfloor \log_{10}(x) \rfloor}\right)}{\log(x)}. \tag{29}$$

By taking the limits as $x \to \infty$ of the upper and lower bounds in (29), we find that $\eta(f)$ exists, with $\eta(f) = 2$. $\square$

Lemma 7. *The limit associated with $\eta(f')$ exists, with $\eta(f') = 1$.*

Proof. For $x \in (10^{m-1}, 10^m - 1)$, we have that $f'(x) = 10^{m-1}$. For $x \in (10^m - 1, 10^m)$, we have that $f'(x) = h_3^{(m)}(x)$. As indicated above, on the interval $[10^m - 1, 10^m]$, the function $h_3^{(m)}(x)$ agrees with a line segment from 10^{m-1} to

$$2 \cdot 10^{2\lfloor \frac{m}{2} \rfloor} - 19 \cdot 10^{m-2} \tag{30}$$

and a line segment from (30) to 10^m. By considering the parity of m in (30), we have, regardless of the parity of m, that

$$10^{m-2} \leq f'(x) \leq 2 \cdot 10^m, \tag{31}$$

letting $m \in \mathbb{N}_{\geq 2}$. The upper and lower bounds in (31) hold for $10^{m-1} < x < 10^m$, giving us that

$$10^{\lfloor \log_{10}(x) \rfloor - 1} \leq f'(x) \leq 2 \cdot 10^{\lfloor \log_{10}(x) \rfloor + 1}, \tag{32}$$

for x not equal to a power of 10, and a similar argument gives us that (32) also holds for x equal to a power of 10. From (32), we write

$$\frac{\log\left(10^{\lfloor \log_{10}(x) \rfloor - 1}\right)}{\log x} \leq \frac{\log\left(f'(x)\right)}{\log x} \leq \frac{\log\left(2 \cdot 10^{\lfloor \log_{10}(x) \rfloor + 1}\right)}{\log x}, \tag{33}$$

Taking the limit as $x \to \infty$, the limits of the upper and lower bounds in (33) both reduce to 1. $\square$

2.3 Non-normality

As below, for a block B of digits, we let $\ell^0(B)$ denote the total number of zeroes, counting repetitions, in B.

Theorem 8. *The concatenation* $0.h_2(1)h_2(2)\ldots$ *is not normal in base 10.*

Proof. From the lower and upper bounds in (28), we obtain that

$$2\lfloor\log_{10}(n)\rfloor - 2 < \log_{10}(f(n)) \leq 2\lfloor\log_{10}(n)\rfloor + 1. \tag{34}$$

The inequalities in (34) then give us that

$$2\lfloor\log_{10}(n)\rfloor - 2 \leq \#\text{ of digits in } h_2(n) \leq 2\lfloor\log_{10}(n)\rfloor + 1 \tag{35}$$

for all positive integers n. So, by taking the concatenation $0.h_2(1)h_2(2)\ldots$, and by then taking the sequence of digits to the right of the decimal point given by the concatenation of the digits in the first $m \in \mathbb{N}$ entries of the integer sequence $(h_2(n) : n \in \mathbb{N})$, the total number of digits in this sequence is $\sum_{n=1}^{m}(\#\text{ of digits in } h_2(n))$, and this satisfies, from (35), the inequalities such that

$$\sum_{n=1}^{m}(2\lfloor\log_{10}(n)\rfloor - 2) \leq \sum_{n=1}^{m}(\#\text{ of digits in } h_2(n)) \leq \sum_{n=1}^{m}(2\lfloor\log_{10}(n)\rfloor + 1). \tag{36}$$

For the sequence $h_2(1)h_2(2)\ldots$ (denoted as an infinite word), we take an initial and finite segment S of this sequence. Then there is a well-defined value $m = m(S)$ such that

$$\sum_{n=1}^{m}(\#\text{ of digits in } h_2(n)) \leq \ell(S) < \sum_{n=1}^{m+1}(\#\text{ of digits in } h_2(n)). \tag{37}$$

We then write

$$\ell(S) = \sum_{n=1}^{m}(\#\text{ of digits in } h_2(n)) + \ell\left(\overline{S}\right) \tag{38}$$

for a possibly empty string $\overline{S}$ given by taking the $\ell(S) - \sum_{n=1}^{m}(\#\text{ of digits in } h_2(n))$ digits to the right of S. From the well-definedness of $m(S)$ given by (37), we find that

$$0 \leq \ell\left(\overline{S}\right) < \#\text{ of digits in } h_2(m+1). \tag{39}$$

From (36), we obtain that

$$\sum_{n=1}^{m}(2\lfloor\log_{10}(n)\rfloor - 2) + \ell\left(\overline{S}\right) \leq \ell(S) \leq \sum_{n=1}^{m}(2\lfloor\log_{10}(n)\rfloor + 1) + \ell\left(\overline{S}\right). \tag{40}$$

From (39) and (40), we find that

$$\sum_{n=1}^{m}(2\lfloor\log_{10}(n)\rfloor-2)\leq\ell(S)<\sum_{n=1}^{m}(2\lfloor\log_{10}(n)\rfloor+1)+\#\text{ of digits in }h_2(m+1).$$

This and (35) give us that

$$\sum_{n=1}^{m}(2\lfloor\log_{10}(n)\rfloor-2)\leq\ell(S)<\sum_{n=1}^{m+1}(2\lfloor\log_{10}(n)\rfloor+1).$$

Using the recursion for h_2 given in the proof of Lemma 3, we have that the following holds. For an integer n such that $10^k\leq n\leq 10^{k+1}-1$, the base-10 digit expansion of $h_2(n)$ is such that its last k digits are all equal to 0, letting k be a positive integer. So, the number of zeroes appearing in the digital expansion of $h_2(n)$ is at least $\lfloor\log_{10}(n)\rfloor$, with

$$\lfloor\log_{10}(n)\rfloor\leq\#\text{ of zeroes in }h_2(n)\leq 2\lfloor\log_{10}(n)\rfloor+1,$$

and this holds for positive integers n. By analogy with (38), and for the strings S and $\overline{S}$ as before, we find that $\ell^0(S)=\sum_{n=1}^{m}(\#\text{ of zeroes in }h_2(n))+\ell^0\left(\overline{S}\right)$. Since

$$\sum_{n=1}^{m}\lfloor\log_{10}(n)\rfloor+\ell^0\left(\overline{S}\right)\leq\sum_{n=1}^{m}(\#\text{ of zeroes in }h_2(n))+\ell^0\left(\overline{S}\right)\leq \sum_{n=1}^{m}(2\lfloor\log_{10}(n)\rfloor+1)+\ell^0\left(\overline{S}\right),$$

we find that $\sum_{n=1}^{m}\lfloor\log_{10}(n)\rfloor\leq\ell^0(S)<\sum_{n=1}^{m+1}(2\lfloor\log_{10}(n)\rfloor+1)$, so that

$$\frac{\sum_{n=1}^{m}\lfloor\log_{10}(n)\rfloor}{\sum_{n=1}^{m+1}\left(2\lfloor\log_{10}(n)\rfloor+1\right)}\leq\frac{\ell^0(S)}{\ell(S)}\leq 1. \tag{41}$$

Using an Abel summation formula, we have that $\sum_{n=1}^{m}\lfloor\log_{10}(n)\rfloor\sim\frac{m\log m}{\log 10}$ and that the partial sum in the denominator of the lower bound in (41) is asymptotic to $\frac{2m\log(m)}{\log(10)}$. So, the limit as $m\to\infty$ of the lower bound in (41) is equal to $\frac{1}{2}$. So, with respect to the notation in (6), by setting α as the real value $0.\lfloor h_2(1)\rfloor\lfloor h_2(2)\rfloor\ldots$, and by setting E as the block of digits given by the single digit 0, the limit $\lim_{n\to\infty}\frac{A_E(\alpha,n)}{n}$ would not be equal to the desired target value of $\frac{1}{10}$. So, the value α is not normal in base 10. ∎

We have shown in Section 2.2 that f satisfies all of the required conditions given in Theorem 2 from the work of Szüsz and Volkmann [5]. However, the concatenation $0.\lfloor f(1)\rfloor\lfloor f(2)\rfloor\ldots$ we have constructed is not normal in base 10, according to Lemma 3 and Theorem 8, yielding the desired counterexample. The non-normality of our new constant is illustrated in Table 1.

p	Ratio of zero digits among the digits of $(h_2(n))_{1\leq n\leq 10^p}$
1	0.909091
2	0.405109
3	0.446318
4	0.476029
5	0.492968
6	0.503644
7	0.510957

Table 1: An illustration of the non-normality of $0.\lfloor f(1)\rfloor\lfloor f(2)\rfloor\ldots$.

3 Acknowledgments

The author was supported by a Killam Postdoctoral Fellowship from the Killam Trusts and thanks Karl Dilcher for many useful and engaging discussions related to this paper. The author is thankful for the exceptionally helpful reviewer feedback that has been provided and that has led to many substantial improvements to this research paper.

References

[1] A. S. Besicovitch, The asymptotic distribution of the numerals in the decimal representation of the squares of the natural numbers, *Math. Z.* **39** (1935), 146–156.

[2] D. G. Champernowne, The construction of decimals normal in the scale of ten, *J. London Math. Soc.* **8** (1933), 254–260.

[3] A. H. Copeland and P. Erdős, Note on normal numbers, *Bull. Amer. Math. Soc.* **52** (1946), 857–860.

[4] P. Pollack and J. Vandehey, Besicovitch, bisection, and the normality of 0.(1)(4)(9) (16)(25) $\cdots$, *Amer. Math. Monthly* **122** (2015), 757–765.

[5] P. Szüsz and B. Volkmann, A combinatorial method for constructing normal numbers, *Forum Math.* **6** (1994), 399–414.

[6] N. J. A. Sloane et al., The On-Line Encyclopedia of Integer Sequences, 2025. Available at https://oeis.org.

2020 *Mathematics Subject Classification*: Primary 11K16.
Keywords: normal number, decimal expansion, integer sequence.

(Concerned with sequences A001191, A033307, A033308, and A380904.)

Received January 29 2025; revised versions received February 25 2025; January 12 2026. Published in *Journal of Integer Sequences*, May 20 2026.

Return to Journal of Integer Sequences home page.