Journal of Integer Sequences, Vol. 29 (2026),
Article 26.3.5

On the Expansion of $(XV)^n$ and Bessel Numbers

Abdelhay Benmoussa
Secteur Scolaire Ansis, Unité Alanfoukt
Tiqqi 80324, Agadir Ida Outanane
Morocco
abdelhay.benmoussa@taalim.ma

Abstract

We expand the operator $(XV)^n$, where X is multiplication by x and V is the Volterra operator. The resulting coefficients are shown to be the Bessel numbers (OEIS A001498). We also present two applications and a conjectural generalization.

1 Introduction

Let

$$(Xf)(x) = xf(x), \qquad D = \frac{d}{dx}.$$

A well-known result in combinatorics states that [3]

$$(XD)^n = \sum_{k=0}^{n} S(n,k) X^k D^k,$$

where $S(n,k)$ denote the Stirling numbers of the second kind (OEIS A008277).

In this note, we investigate an alternative setting obtained by replacing the derivative operator with the Volterra operator

$$V(f)(x) = \int_0^x f(t)\,dt.$$

We show that the expansion of the operator $(XV)^n$, defined by

$$(XV)^n(f)(x) := \begin{cases} f(x), & \text{if } n = 0; \\ x \displaystyle\int_0^x (XV)^{n-1}(f)(t)\, dt, & \text{if } \ n \geq 1, \end{cases}$$

leads naturally to an explicit formula whose coefficients are the Bessel numbers (OEIS A001498), given by

$$a(n,k) = [x^k]\, y_n(x) = \frac{(n+k)!}{2^k\, k!\, (n-k)!},$$

where $y_n(x)$ is the n-th Bessel polynomial [2]. Then we present two applications, and conclude with a conjecture that generalizes the obtained formula.

2 Main result

Theorem 1. *Let $n \geq 1$. Then*

$$(XV)^n = \sum_{k=0}^{n-1} (-1)^k\, a(n-1,k)\, X^{n-k} V^{n+k}. \tag{1}$$

Proof. Applying $(XV)^n$ to a function f gives

$$\begin{aligned} (XV)^n(f)(x) &= x \int_0^x x_{n-1} \int_0^{x_{n-1}} \cdots x_1 \int_0^{x_1} f(t)\, dt\, dx_1 \cdots dx_{n-1} \\ &= x \int_0^x f(t) \left(\iiint_{t \leq x_1 \leq \cdots \leq x_{n-1} \leq x} x_1 x_2 \cdots x_{n-1}\, dx_1 \cdots dx_{n-1} \right) dt. \end{aligned}$$

Now, by induction we have

$$\iiint_{t \leq x_1 \leq \cdots \leq x_{n-1} \leq x} x_1 x_2 \cdots x_{n-1}\, dx_1 \cdots dx_{n-1} = \frac{(x^2 - t^2)^{n-1}}{2^{n-1}(n-1)!}.$$

Thus

$$\begin{aligned} (XV)^n(f)(x) &= \frac{x}{2^{n-1}(n-1)!} \int_0^x f(t)(x^2 - t^2)^{n-1}\, dt \\ &= \frac{x}{2^{n-1}(n-1)!} \int_0^x f(t)(x-t)^{n-1}(x+t)^{n-1}\, dt \\ &= \frac{x}{2^{n-1}(n-1)!} \int_0^x f(t)(x-t)^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{k} (t-x)^k (2x)^{n-1-k}\, dt \end{aligned}$$

$$= \frac{x}{2^{\,n-1}(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(2x)^{\,n-1-k}\int_0^x f(t)(x-t)^{n+k-1}\,dt$$
$$= \sum_{k=0}^{n-1}(-1)^k\frac{1}{2^k(n-1-k)!k!}x^{\,n-k}\int_0^x f(t)(x-t)^{n+k-1}\,dt.$$

Applying Cauchy's formula for repeated integration, we obtain

$$\begin{aligned}(XV)^n(f)(x) &= \sum_{k=0}^{n-1}(-1)^k\frac{(n+k-1)!}{2^k(n-1-k)!k!}x^{\,n-k}\,V^{n+k}(f)(x)\\ &= \sum_{k=0}^{n-1}(-1)^k a(n-1,k)\,x^{\,n-k}V^{n+k}(f)(x)\\ &= \left(\sum_{k=0}^{n-1}(-1)^k a(n-1,k)\,X^{\,n-k}V^{n+k}\right)(f)(x).\end{aligned}$$

$\square$

3 Two applications

3.1 Power functions

Let $\alpha > 0$. We first compute

$$\begin{aligned}(XV)^{n+1}(t^{\alpha-1})(x) &= x\int_0^x \frac{(x^2-t^2)^n}{2^n n!}t^{\alpha-1}\,dt\\ &= \frac{x}{2^n n!}\int_0^1 \left(x^2(1-u)\right)^n\left(xu^{1/2}\right)^{\alpha-1}\frac{x}{2}u^{-1/2}\,du \qquad (t = x\sqrt{u})\\ &= \frac{x}{2^n n!}\cdot\frac{x}{2}\,x^{2n}x^{\alpha-1}\int_0^1 (1-u)^n u^{(\alpha/2)-1}\,du\\ &= \frac{x^{\alpha+2n+1}}{2^{n+1}n!}B(\alpha/2, n+1)\\ &= \frac{x^{\alpha+2n+1}\Gamma(\alpha/2)}{2^{n+1}\,\Gamma(\alpha/2+n+1)}\\ &= \frac{x^{\alpha+2n+1}}{\alpha(\alpha\ +2)\cdots(\alpha+2n)},\end{aligned}$$

where $B(x,y)$ is the Beta function [1].

On the other hand,

$$
\begin{aligned}
(XV)^{n+1}(t^{\alpha-1})(x) &= \sum_{k=0}^{n}(-1)^k a(n,k)\, x^{n+1-k}\, V^{n+1+k}(t^{\alpha-1})(x) \\
&= \sum_{k=0}^{n}(-1)^k a(n,k)\, x^{n+1-k} \int_0^x \frac{(x-t)^{n+k}}{(n+k)!} t^{\alpha-1}\, dt \\
&= \sum_{k=0}^{n} \frac{(-1)^k a(n,k)}{(n+k)!}\, x^{n+1-k} \int_0^1 x^{n+k}(1-u)^{n+k} x^{\alpha-1} u^{\alpha-1}\, x\, du \qquad (t = xu) \\
&= \sum_{k=0}^{n} \frac{(-1)^k a(n,k)}{(n+k)!}\, x^{2n+\alpha+1} \int_0^1 (1-u)^{n+k} u^{\alpha-1}\, du \\
&= x^{2n+\alpha+1} \sum_{k=0}^{n} \frac{(-1)^k a(n,k)}{(n+k)!} B(\alpha, n+k+1) \\
&= x^{2n+\alpha+1} \sum_{k=0}^{n} \frac{(-1)^k a(n,k)\Gamma(\alpha)}{\Gamma(\alpha+n+k+1)} \\
&= x^{\alpha+2n+1} \sum_{k=0}^{n} \frac{(-1)^k\, a(n,k)}{\alpha(\alpha+1)\cdots(\alpha+n+k)}.
\end{aligned}
$$

By comparing the two expressions, we obtain the generating function

$$
\frac{1}{\alpha(\alpha\ +2)\cdots(\alpha+2n)} = \sum_{k=0}^{n} \frac{(-1)^k\, a(n,k)}{\alpha(\alpha+1)\cdots(\alpha+n+k)}. \tag{2}
$$

3.2 Exponential function

Next, we consider the exponential function. Using Formula , we obtain

$$
\begin{aligned}
(XV)^{n+1}(e^t)(x) &= \sum_{k=0}^{n}(-1)^k a(n,k)\, x^{n+1-k}\, V^{n+1+k}(e^t)(x) \\
&= \sum_{k=0}^{n}(-1)^k a(n,k)\, x^{n+1-k} \int_0^x \frac{(x-t)^{n+k}}{(n+k)!} e^t\, dt \\
&= e^x \sum_{k=0}^{n}(-1)^k a(n,k)\, x^{n+1-k}\, \frac{\gamma(n+k+1,x)}{(n+k)!} \\
&= e^x \sum_{k=0}^{n}(-1)^k a(n,k)\, x^{n+1-k} \left(1 - e^{-x}\sum_{j=0}^{n+k}\frac{x^j}{j!}\right) \\
&= e^x \sum_{k=0}^{n}(-1)^k a(n,k)\, x^{\,n+1-k} - \sum_{k=0}^{n}\sum_{j=0}^{n+k}(-1)^k a(n,k)\, \frac{x^{\,n+j+1-k}}{j!},
\end{aligned}
$$

Here, $\gamma(x,y)$ is the lower incomplete gamma function [1]. Rosengren proved [4] that

$$\sum_{k=0}^{n}\sum_{j=0}^{n+k}(-1)^k a(n,k)\,\frac{x^{\,n+j-k}}{j!} = \sum_{k=0}^{n}(-1)^{\,n-k}\,\frac{(2(n-k)-1)!!}{(2k)!!}\,x^{\,2k}.$$

Hence

$$(XV)^{n+1}(e^t)(x) = x^{\,n+1}e^x\,y_n\left(-\frac{1}{x}\right) - \sum_{k=0}^{n}(-1)^{\,n-k}\,\frac{(2(n-k)-1)!!}{(2k)!!}\,x^{\,2k+1}. \tag{3}$$

Evaluating at $x=1$ yields the following Dobinski-like identity for the sequence $a(n) = y_n(-1)$ (OEIS A000806):

$$a(n) = \frac{1}{e}\left((XV)^{n+1}(e^t)(1) + \sum_{k=0}^{n}(-1)^{\,n-k}\frac{(2(n-k)-1)!!}{(2k)!!}\right). \tag{4}$$

4 Generalization

Using Cauchy's formula for repeated integration, we obtain

$$XV - VX = V^2.$$

Motivated by this identity, consider the general commutation relation [3]

$$uv - vu = hv^s, \qquad h \in \mathbb{C}\setminus\{0\},\ s \in \mathbb{R}. \tag{5}$$

We conclude with the following conjecture.

Conjecture 2. Let u and v be two variables satisfying (5). Then, for any integer $n \geq 1$,

$$(uv)^n = \sum_{k=1}^{n}\mathfrak{S}_{s;h}(n,k)\,u^{\,k}\,v^{\,s(n-k)+k}, \tag{6}$$

where $\mathfrak{S}_{s;h}(n,k)$ are the generalized Stirling numbers introduced by Mansour and Schork [3].

5 Acknowledgments

The author thanks the referee and the editor for their valuable comments, which improved the manuscript.

References

[1] M. Abramowitz and I. A. Stegun, editors. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing*. New York: Dover.

[2] E. Grosswald. *Bessel Polynomials*, Springer, 1978.

[3] T. Mansour, M. Schork and M. Shattuck. The generalized Stirling and Bell numbers revisited. *J. Integer Sequences* **15** (2012), Article 12.8.3.

[4] H. Rosengren. Proving that a certain factorial double sum collapses to a double–factorial series. MathOverflow posting (2025). Available at `https://mathoverflow.net/questions/499957`.

2020 *Mathematics Subject Classification*: Primary 11B83; Secondary 11C08, 26A36.
Keywords: Bessel number, Volterra operator.

(Concerned with sequences A000806, A001497, A001498, A008277, A122848, and A122850.)

Received July 26 2025; revised versions received July 30 2025; July 31 2025; December 6 2025; December 7 2025; May 19 2026. Published in *Journal of Integer Sequences*, May 20 2026.

Return to Journal of Integer Sequences home page.