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Third-Order Recurrence Sequences with Infinitely Many Terms not Equal to the Sum of Two Prime Powers

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Abstract

We show in each of the Tribonacci, Padovan, Van der Laan, Perrin, Leonardo, and Narayana's cows sequences that there are infinitely many terms that cannot be represented as the sum of two prime powers.

1 Introduction

One of the oldest and best-known unsolved problems in mathematics is Goldbach's conjecture originating in 1742. It states that every even integer at least 4 is the sum of two prime numbers. When considering positive integers that cannot be represented as the sum of two primes (or not the sum of two prime powers) one is drawn toward the odd integers. Cohen and Selfridge [4] proved there exist infinitely many odd integers that are not the sum and not the difference of a power of two and a prime. Z.-W. Sun [10] found a smaller such integer when proving there exist infinitely many integers not of the form $\pm p^a \pm q^b$ with primes p and q and nonnegative integers a and b. Moreover, it has been proven that there exist infinitely many polygonal numbers that are not the sum and not the difference of two prime powers [1].

That is, there exist infinitely many polygonal numbers that cannot be represented as $p^a \pm q^b$ with primes p and q and nonnegative integers a and b.

Luca and Stănică [9] showed there exist infinitely many Fibonacci numbers that cannot be represented as a sum of two prime powers. Similar results were proven for the Lucas numbers [2]. In this work, we prove analogous results for some third-order recurrence sequences. Moreover, we provide a general procedure for producing such results like those mentioned in the works above. Covering systems have been the main tool utilized in these works above and are the main tool used in this work.

A *covering system* of congruences (or more often simply called a *covering*) for the integers is a finite collection of congruences with the property that every integer satisfies at least one congruence in the system. More precisely, a *covering* for the integers is a finite set $\{(a_1, b_1), (a_2, b_2), \ldots, (a_t, b_t)\}$ for which each integer n satisfies $n \equiv a_i \pmod{b_i}$ for some $1 \leq i \leq t$. The notion of coverings originated in the seminal paper of Erdős [6] who proved there are infinitely many odd numbers that are not of the form $2^k + p$, where p is a prime.

Let $\mathbf{r} = (r_k)_{k\geq 0}$ denote an integer valued linear homogeneous recurrence sequence. It is well known that such sequences are periodic modulo a positive integer m (e.g., [3, 5]). For a positive integer m, let $h(m) = h_{\mathbf{r}}(m)$ be the period of the sequence modulo m, and for an integer x let

$$\mathcal{A}(x, m) = \mathcal{A}_{\mathbf{r}}(x, m) := \{y \pmod{h(m)} : r_y \equiv x \pmod{m} \text{ with } 0 \leq y \leq h(m) - 1\}.$$

That is, $\mathcal{A}(x, m)$ is the set of all congruence classes y modulo $h(m)$ such that $r_y \equiv x \pmod{m}$. Throughout for some degree three linear homogeneous recurrence sequences, we compute such $\mathcal{A}$ sets. We simply use $\mathcal{A}(x, m)$ for each recurrence sequence. Since we consider each recurrence sequence in separate sections, there should be no confusion to the underlying sequence.

2 General procedure

In what follows, we present sufficient conditions and a general procedure for proving that a linear homogeneous recurrence sequence of integers has infinitely many terms (a positive proportion) that cannot be represented as the sum of two prime powers (i.e., cannot be represented as $p^a + q^b$ for some primes p, q and nonnegative integers a, b).

Theorem 1. *Let $(r_k)_{k\geq 0}$ be a linear homogeneous recurrence sequence of integers. For each $1 \leq i \leq t$, let $a_i \geq 0$ and $b_i > 0$ be integers and p_i be prime. Consider the following system of congruences:*

$$\begin{aligned}
n &\equiv a_1 \pmod{b_1} \quad \& \quad r_k \equiv 2^{a_1} \pmod{p_1}\\
n &\equiv a_2 \pmod{b_2} \quad \& \quad r_k \equiv 2^{a_2} \pmod{p_2}\\
&\vdots\\
n &\equiv a_t \pmod{b_t} \quad \& \quad r_k \equiv 2^{a_t} \pmod{p_t}
\end{aligned} \tag{1}$$

Let $a_0 = 0$ and $p_0 = 2$. Furthermore, assume

(i) The congruences involving n form a covering of the integers.

(ii) We have $2^{b_i} \equiv 1 \pmod{p_i}$ for each $1 \le i \le t$.

(iii) There exists an integer $k_0 \in \bigcap_{0\le i\le t} \mathcal{A}(2^{a_i}, p_i)$.

Then for every $k \equiv k_0 \pmod{\operatorname{lcm}(h(p_0), h(p_1), \ldots, h(p_t))}$ that is sufficiently large we have r_k cannot be represented as $p^a + q^b$ for some primes p, q and nonnegative integers a, b.

Proof. Suppose the congruences involving n from Eqs. (1) form a covering of the integers and $2^{b_i} \equiv 1 \pmod{p_i}$ for each $1 \le i \le t$. Let $a_0 = 0$ and $p_0 = 2$. Also, suppose that there exists an integer

$$k_0 \in \bigcap_{0\le i\le t} \mathcal{A}(2^{a_i}, p_i). \tag{2}$$

Since the intersection in (2) is nonempty and contains k_0, note that every $k \equiv k_0 \pmod L$ with $L = \operatorname{lcm}(h(p_0), h(p_1), \ldots, h(p_t))$ will also be in the intersection. Lastly, note that $k \in \mathcal{A}(1, 2)$ ensures that r_k is odd.

Now, suppose some r_k can be expressed as a sum of two prime powers; that is, suppose $r_k = p^a + q^b$ for some primes p, q and nonnegative integers a, b. Since r_k is odd, one of the primes must be 2. Without loss of generality, we can write $r_k = 2^a + q^b$. Since the congruences involving n in Eqs. (1) form a covering of the integers, a must fit into a residue class expressed in one of the rows of the table. Suppose we have $a \equiv a_i \pmod{b_i}$, where $n \equiv a_i \pmod{b_i}$ & $r_k \equiv 2^{a_i} \pmod{p_i}$ is row i in Eqs. (1). Since $2^{b_i} \equiv 1 \pmod{p_i}$, we deduce that $r_k \equiv 2^a \pmod{p_i}$. In particular, $p_i \mid (r_k - 2^a)$. However, $r_k - 2^a = q^b$, which implies that $q = p_i$. Thus, $q \in \{p_1, p_2, \ldots, p_t\}$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ denote the characteristic roots of the characteristic equation of $\mathbf{r} = (r_j)_{j\ge 0}$. Since $\mathbf{r}$ is a linear homogeneous recurrence sequence of integers, there exists complex numbers $A_1, A_2, \ldots, A_s$ such that

$$r_j = A_1\alpha_1^j + A_2\alpha_2^j + \cdots + A_s\alpha_s^j$$

for each $j \ge 0$. Consider the equation

$$A_1\alpha_1^j + A_2\alpha_2^j + \cdots + A_s\alpha_s^j = 2^a + q^b.$$

This equation with $q \in \{p_1, p_2, \ldots, p_t\}$ can be viewed as an $\mathcal{S}$-unit equation known to have only finitely many solutions (j, a, b) [7]. Thus, for sufficiently large k, we deduce that there are no solutions to $r_k = 2^a + q^b$ with $q \in \{p_1, p_2, \ldots, p_t\}$. It follows that if k is sufficiently large and $k \equiv k_0 \pmod L$, then r_k is not a sum of two prime powers. Hence, the conclusion of the theorem holds. □

We now use Theorem 1 to prove results about some third-order recurrence sequences.

3 Tribonacci

Let $(T_k)_{k\geq 0}$ be the sequence of Tribonacci numbers defined by $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, and $T_k = T_{k-1} + T_{k-2} + T_{k-3}$ for all $k \geq 3$. As our first application of Theorem 1, we prove a theorem about the Tribonacci numbers.

Theorem 2. *There are infinitely many Tribonacci numbers T_k that cannot be represented as $p^a + q^b$ for some primes p, q and nonnegative integers a, b. In fact, this occurs for every sufficiently large $k \equiv 3817 \pmod{163843680}$.*

Proof. First consider the following table.

$$\begin{array}{lcl}
n \equiv 1 \pmod 2 & \& & T_k \equiv 2^1 \pmod 3 \\
n \equiv 2 \pmod 4 & \& & T_k \equiv 2^2 \pmod 5 \\
n \equiv 0 \pmod 8 & \& & T_k \equiv 2^0 \pmod{17} \\
n \equiv 1 \pmod 3 & \& & T_k \equiv 2^1 \pmod 7 \\
n \equiv 0 \pmod{12} & \& & T_k \equiv 2^0 \pmod{13} \\
n \equiv 20 \pmod{24} & \& & T_k \equiv 2^{20} \pmod{241}
\end{array}$$

Table 1: Tribonacci congruences.

Note assumptions (i) and (ii) in Theorem 1 hold. The solutions for k such that T_k is odd and for each of the congruences involving T_k in Table 1 are

$$\begin{array}{lcl}
\mathcal{A}(1,2) & = & \{1,2\} \pmod 4 \\
\mathcal{A}(2^1,3) & = & \{3,8,10\} \pmod{13} \\
\mathcal{A}(2^2,5) & = & \{4,7,8,10,11,12,17,23,28\} \pmod{31} \\
\mathcal{A}(2^0,17) & = & \{1, 2, 17, 20, 21, 34, 52, 60, 62, 67, 69, 72, 73, 83, 89, 94\} \pmod{96} \\
\mathcal{A}(2^1,7) & = & \{3,8,10,20,25,30,33,34,43\} \pmod{48} \\
\mathcal{A}(2^0,13) & = & \{1, 2, 11, 17, 36, 42, 72, 79, 82, 83, 93, 121, 127, 134, 141, 150, 161, \\
& & 166\} \pmod{168} \\
\mathcal{A}(2^{20},241) & = & \{458, 569, 745, 1117, 1686, 2137, 2443, 3146, 3449, 3817, 3915, 4195, \\
& & 4410, 4524, 5176, 5669, 5701, 5738, 6115, 6685, 8014, 8339, 8345, 8709, \\
& & 8938, 9066, 9194, 9196, 9526, 10166, 10216, 10239, 10510, 10719, 10793, \\
& & 11213, 11286, 11636, 12067, 12081, 12869, 13126, 13133, 13633, 14402, \\
& & 14666, 14861, 15553, 15667, 15974, 16172, 16291, 16618, 16687, 16698, \\
& & 16825, 17251, 17459, 17512, 17612, 17628, 17772, 17950, 18000, 18426, \\
& & 18484, 18851, 18905, 19041, 19070, 19357, 19483, 19509, 20028, 20234, \\
& & 20316, 20581, 20716, 21015, 21034, 21101, 21112, 21520, 22166, 22560, \\
& & 22605, 23044, 23175, 23322, 24371, 24654, 25004, 25405, 25567, 25692, \\
& & 25755, 26204, 26254, 26642, 27324, 27402, 27754, 27956, 27974, 28073, \\
& & 28218, 28720, 28845, 28890, 28956, 28974\} \pmod{29040}.
\end{array}$$

Observe that the intersection of the $\mathcal{A}$-sets above is nonempty; in fact, there are 216 distinct residues modulo 163843680 with the smallest being 3817. The result now follows from Theorem 1. $\square$

4 Padovan

Let $(P_k)_{k\geq 0}$ be the sequence of Padovan numbers defined by $P_0 = 1$, $P_1 = 1$, $P_2 = 1$, and $P_k = P_{k-2} + P_{k-3}$ for all $k \geq 3$. We proceed with a similar theorem for the Padovan numbers.

Theorem 3. *There are infinitely many Padovan numbers P_k that cannot be represented as $p^a + q^b$ for some primes p, q and nonnegative integers a, b. In fact, this occurs for every sufficiently large $k \equiv 45750 \pmod{967206240}$.*

Proof. First consider the following table.

$$\begin{array}{lcl}
n \equiv 1 \pmod{2} & \& & P_k \equiv 2^1 \pmod{3} \\
n \equiv 2 \pmod{4} & \& & P_k \equiv 2^2 \pmod{5} \\
n \equiv 4 \pmod{8} & \& & P_k \equiv 2^4 \pmod{17} \\
n \equiv 2 \pmod{3} & \& & P_k \equiv 2^2 \pmod{7} \\
n \equiv 0 \pmod{12} & \& & P_k \equiv 2^0 \pmod{13} \\
n \equiv 16 \pmod{24} & \& & P_k \equiv 2^{16} \pmod{241}
\end{array}$$

Table 2: Padovan congruences.

Note assumptions (i) and (ii) in Theorem 1 hold. The solutions for k such that P_k is odd and for each of the congruences involving P_k in Table 2 are

$$\begin{array}{lcl}
\mathcal{A}(1,2) & = & \{0,1,2,5\} \pmod{7} \\
\mathcal{A}(2^1,3) & = & \{3,4,7\} \pmod{13} \\
\mathcal{A}(2^2,5) & = & \{6,9,15,18\} \pmod{24} \\
\mathcal{A}(2^4,17) & = & \{11, 37, 49, 53, 61, 92, 112, 115, 124, 135, 150, 173, 176, 186, 187, 227, \\
 & & 246, 257, 279, 282\} \pmod{288} \\
\mathcal{A}(2^2,7) & = & \{6,19,20,25,27,30,32,33,34\} \pmod{48} \\
\mathcal{A}(2^0,13) & = & \{0, 1, 2, 30, 66, 72, 92, 100, 131, 157, 170, 171, 176, 178, 181\} \\
 & & \pmod{183}
\end{array}$$

$$\mathcal{A}(2^{16}, 241) = \{78, 285, 495, 545, 560, 801, 983, 1042, 1065, 1160, 1346, 1892, 1996, 2294, 2398, 2697, 3580, 3698, 4578, 4687, 4912, 5133, 5592, 6027, 6751, 7462, 7559, 7681, 7690, 7808, 8536, 8584, 9027, 9080, 9140, 9180, 9718, 10135, 10470, 11252, 11393, 11500, 11607, 11617, 11635, 11644, 11721, 12154, 12290, 12481, 12524, 13076, 13082, 13355, 13972, 14650, 15012, 15051, 15620, 16272, 17019, 17038, 17424, 17705, 17782, 18044, 18438, 18532, 18569, 18643, 19437, 19530, 19698, 19730, 19794, 20020, 20131, 20305, 20427, 20538, 20671, 20806, 21086, 21279, 21553, 21726, 22170, 22461, 22488, 22889, 23014, 23055, 23095, 23409, 23716, 23918, 24100, 24156, 24503, 25260, 25664, 25683, 26107, 26123, 26135, 26302, 27037, 27714, 27729, 27857, 27982, 28686, 29228, 29289, 29367, 29964, 30474, 30525, 30877, 31017, 31581, 31637, 31752, 31777, 32185, 32647, 32778, 33167, 33322, 33392, 33516, 34123, 34238, 34705, 34776, 34916, 35105, 35482, 35873, 35957, 36115, 36188, 36230, 36240, 36316, 36362, 36706, 36852, 37119, 37293, 37344, 37416, 37496, 37906, 38088, 38284, 38694, 38707, 38708, 38713, 38715, 38718, 38720, 38721, 38722, 38758, 39067, 39132, 39286, 39324, 39350, 39670, 39859, 39938, 40142, 40723, 41159, 41571, 41760, 41772, 42112, 42739, 42923, 43088, 43636, 43764, 43983, 44075, 44265, 44296, 45034, 45073, 45750, 45815, 45967, 46074, 46251, 46428, 46467, 46663, 46985, 47047, 48464, 48686, 48694, 48771, 49263, 49398, 49790, 50054, 50112, 50851, 50889, 51865, 52352, 52374, 52862, 53043, 53651, 53906, 54056, 54257, 54283, 54564, 54830, 55270, 55507, 55748, 55864, 55899, 56051, 56207, 56263, 56791, 56841, 56953, 57435, 57708, 57751, 57837\} \pmod{58080}.$$

The intersection of the $\mathcal{A}$-sets above is nonempty, and there are 360 distinct residues modulo 967206240 with the smallest being 45750. The result follows from Theorem 1. $\square$

5 Van der Laan

Let $(V_k)_{k\geq 0}$ be the sequence of Van der Laan numbers defined by $V_0 = 1$, $V_1 = 0$, $V_2 = 1$, and $V_k = V_{k-2} + V_{k-3}$ for all $k \geq 3$.

Theorem 4. *There are infinitely many Van der Laan numbers V_k that cannot be represented as $p^a + q^b$ for some primes p, q and nonnegative integers a, b. In fact, this occurs for every sufficiently large $k \equiv 45752 \pmod{967206240}$.*

The proof of the theorem follows from the observation that $V_k = P_{k-2}$. From this fact, we are able to use the same covering as Table 2 and deduce that the intersection of the $\mathcal{A}$-sets is nonempty; in fact, there are 360 total intersections with the smallest solution coming from $k \equiv 45752 \pmod{967206240}$.

6 Perrin

Let $(E_k)_{k\geq 0}$ be the sequence of Perrin numbers defined by $E_0 = 3$, $E_1 = 0$, $E_2 = 2$, and $E_k = E_{k-2} + E_{k-3}$ for all $k \geq 3$.

Theorem 5. *There are infinitely many Perrin numbers E_k that cannot be represented as $p^a + q^b$ for some primes p, q and nonnegative integers a, b. In fact, this occurs for every sufficiently large $k \equiv 472190 \pmod{967206240}$.*

Proof. First consider the following table.

$n \equiv 1 \pmod{2}$	&	$E_k \equiv 2^1 \pmod{3}$
$n \equiv 0 \pmod{4}$	&	$E_k \equiv 2^0 \pmod{5}$
$n \equiv 6 \pmod{8}$	&	$E_k \equiv 2^6 \pmod{17}$
$n \equiv 1 \pmod{3}$	&	$E_k \equiv 2^1 \pmod{7}$
$n \equiv 2 \pmod{12}$	&	$E_k \equiv 2^2 \pmod{13}$
$n \equiv 18 \pmod{24}$	&	$E_k \equiv 2^{18} \pmod{241}$

Table 4: Perrin congruences.

Note assumptions (i) and (ii) in Theorem 1 hold. The solutions for k such that E_k is odd and for each of the congruences involving E_k in Table 4 are

$$\begin{aligned}
\mathcal{A}(1,2) &= \{0,3,5,6\} \pmod{7}\\
\mathcal{A}(2^1,3) &= \{2,4,5,6,10,12\} \pmod{13}\\
\mathcal{A}(2^0,5) &= \{14,22\} \pmod{24}\\
\mathcal{A}(2^6,17) &= \{41, 60, 94, 115, 121, 151, 156, 158, 198, 227, 263\} \pmod{288}\\
\mathcal{A}(2^1,7) &= \{2,4,14,27,28,45\} \pmod{48}\\
\mathcal{A}(2^2,13) &= \{10, 20, 32, 43, 48, 50, 54, 60, 70, 75, 77, 86, 101, 118, 130, 153,\\
&\quad 159, 178\} \pmod{183}\\
\mathcal{A}(2^{18},241) &= \{162, 224, 867, 983, 1020, 1240, 1799, 1901, 1906, 1908, 2184,\\
&\quad 2446, 2977, 3401, 3454, 3475, 3520, 3580, 3624, 3816, 3879, 4583,\\
&\quad 4923, 5212, 5331, 5559, 5591, 6521, 7011, 7021, 7338, 7417, 7550,\\
&\quad 7741, 7864, 8338, 8378, 8440, 8492, 8647, 8686, 8760, 8776, 9122,\\
&\quad 9270, 9460, 9534, 9611, 10026, 10274, 10279, 10281, 11175, 11185,\\
&\quad 11375, 11591, 11615, 11859, 11917, 12099, 12357, 12647, 13085,\\
&\quad 13500, 13586, 13772, 14186, 14191, 14193, 14389, 14552, 14636,\\
&\quad 14740, 15174, 15295, 15537, 15648, 15769, 15957, 16078, 16867,\\
&\quad 17165, 17716, 19070, 19294, 19870, 20280, 20375, 20421, 20497,\\
&\quad 21364, 21495, 21591, 21746, 22114, 22232, 22532, 23133, 23280,\\
&\quad 23528, 23905, 24032, 24055, 24136, 24355, 24843, 25129, 25179,\\
&\quad 25281, 25485, 25686, 26058, 26077, 26110, 26196, 26417, 26705,
\end{aligned}$$

26999, 27030, 27055, 27297, 27767, 27819, 28079, 28329, 28772, 29586, 29716, 29759, 29798, 30537, 31046, 31063, 31318, 31655, 31929, 32507, 32574, 33204, 33846, 34154, 34311, 34397, 34707, 34738, 34800, 34849, 34986, 35089, 35200, 35777, 36109, 36144, 36373, 36412, 36488, 36664, 36674, 37478, 37515, 37684, 37879, 38361, 38715, 38742, 39004, 39033, 39042, 40211, 40450, 40596, 41029, 41337, 41518, 41792, 41834, 41921, 42317, 42476, 42741, 43097, 43371, 43485, 44022, 44194, 44378, 44466, 45061, 45097, 45204, 45980, 47105, 47335, 47846, 48137, 48349, 48456, 48907, 49084, 49090, 49145, 49442, 49571, 49660, 49882, 50186, 51127, 51131, 51391, 51467, 51581, 51873, 51943, 52401, 52786, 53268, 53705, 53893, 53984, 54048, 54427, 55121, 55318, 55974, 56073, 56091, 56784, 56821, 57082, 57427, 57453} (mod 58080).

The intersection of the $\mathcal{A}$-sets above is nonempty, and there are 288 distinct residues modulo 967206240 with the smallest being 472190. The result follows from Theorem 1. $\square$

7 Leonardo

Let $(L_k)_{k\geq 0}$ be the sequence of Leonardo numbers defined by $L_0 = 1$, $L_1 = 1$ and $L_k = L_{k-1} + L_{k-2} + 1$ for $k \geq 2$. It is easy to show that $L_k = 2L_{k-1} - L_{k-3}$ for all $k \geq 3$; that is, the Leonardo sequence can be converted into a third-order linear homogeneous recurrence sequence.

Theorem 6. *There are infinitely many Leonardo numbers L_k that cannot be represented as $p^a + q^b$ for some primes p, q and nonnegative integers a, b. In fact, this occurs for every sufficiently large $k \equiv 539073 \pmod{3543120}$.*

Proof. First consider the following table.

$$\begin{array}{lcl}
n \equiv 0 \pmod{2} & \& & L_k \equiv 2^0 \pmod{3} \\
n \equiv 3 \pmod{4} & \& & L_k \equiv 2^3 \pmod{5} \\
n \equiv 0 \pmod{3} & \& & L_k \equiv 2^0 \pmod{7} \\
n \equiv 5 \pmod{12} & \& & L_k \equiv 2^5 \pmod{13} \\
n \equiv 7 \pmod{18} & \& & L_k \equiv 2^7 \pmod{19} \\
n \equiv 13 \pmod{36} & \& & L_k \equiv 2^{13} \pmod{37} \\
n \equiv 1 \pmod{9} & \& & L_k \equiv 2^1 \pmod{73}
\end{array}$$

Table 7: Leonardo congruences.

Note assumptions (i) and (ii) in Theorem 1 hold. Also, note L_k is always odd. The solutions for k in each of the congruences involving L_k in Table 7 are

$$\begin{array}{lcl}
\mathcal{A}(2^0,3) & = & \{0,1,6\} \pmod 8 \\
\mathcal{A}(2^3,5) & = & \{2,13,15,16\} \pmod{20} \\
\mathcal{A}(2^0,7) & = & \{0,1,5,14\} \pmod{16} \\
\mathcal{A}(2^5,13) & = & \{17,23\} \pmod{28} \\
\mathcal{A}(2^7,19) & = & \{9\} \pmod{18} \\
\mathcal{A}(2^{13},37) & = & \{5,31\} \pmod{76} \\
\mathcal{A}(2^1,73) & = & \{15,57\} \pmod{148}.
\end{array}$$

Observe that the intersection of the $\mathcal{A}$-sets above is nonempty; in fact, every integer $k \equiv 539073 \pmod{3543120}$ is in the intersection of the $\mathcal{A}$-sets above. The result follows from Theorem 1. □

8 Narayana's cows

Let $(N_k)_{k\geq 0}$ denote Narayana's cows sequence defined by $N_0 = 0$, $N_1 = 1$, $N_2 = 1$, and $N_k = N_{k-1} + N_{k-3}$ for all $k \geq 3$. We refer to these as NC numbers.

Theorem 7. *There are infinitely many NC numbers N_k that cannot be represented as $p^a + q^b$ for some primes p, q and nonnegative integers a, b. In fact, this occurs for every sufficiently large $k \equiv 137042 \pmod{718391520}$.*

Proof. First consider the following table.

$$\begin{array}{lcl}
n \equiv 0 \pmod 2 & \& & N_k \equiv 2^0 \pmod 3 \\
n \equiv 3 \pmod 4 & \& & N_k \equiv 2^3 \pmod 5 \\
n \equiv 5 \pmod 8 & \& & N_k \equiv 2^5 \pmod{17} \\
n \equiv 2 \pmod 3 & \& & N_k \equiv 2^2 \pmod 7 \\
n \equiv 9 \pmod{12} & \& & N_k \equiv 2^9 \pmod{13} \\
n \equiv 1 \pmod{24} & \& & N_k \equiv 2^1 \pmod{241}
\end{array}$$

Table 8: Narayana congruences.

Note assumptions (i) and (ii) in Theorem 1 hold. The solutions for k such that N_k is odd and for each of the congruences involving N_k in Table 8 are

$$\begin{array}{lcl}
\mathcal{A}(1,2) & = & \{1,2,3,5\} \pmod 7 \\
\mathcal{A}(2^0,3) & = & \{1,2,3,6\} \pmod 8 \\
\mathcal{A}(2^3,5) & = & \{5,9,11,14,21,22,24\} \pmod{31} \\
\mathcal{A}(2^5,17) & = & \{18,\ 49,\ 54,\ 58,\ 68,\ 78,\ 105,\ 110,\ 115,\ 152,\ 163,\ 193, \\
& & 216,\ 228,\ 242,\ 278,\ 281\} \pmod{288} \\
\mathcal{A}(2^2,7) & = & \{6,13,14,17,20,21,22,42,46\} \pmod{57} \\
\mathcal{A}(2^9,13) & = & \{29,\ 34,\ 60,\ 67,\ 68,\ 79,\ 83,\ 89,\ 92,\ 119,\ 122,\ 136,\ 154, \\
& & 156\} \pmod{168}
\end{array}$$

$$\mathcal{A}(2^1, 241) = \{4, 190, 241, 667, 707, 944, 1117, 1199, 1522, 1767, 1910, 2043, 2302, 2610, 2677, 2771, 2777, 3079, 3203, 3232, 3252, 3482, 3546, 4145, 4172, 4276, 4330, 4340, 4411, 4447, 4518, 4521, 4628, 4680, 5020, 5484, 5857, 5925, 5993, 6425, 6513, 6646, 6668, 7266, 7542, 8956, 9006, 9032, 9198, 9384, 9485, 9520\} \pmod{9680}.$$

The intersection of the $\mathcal{A}$-sets above is nonempty, and there are 147 distinct residues modulo 718391520 with the smallest being 137042. The result follows from Theorem 1. $\square$

9 Concluding remarks

Given a third-order integer valued linear homogeneous recurrence sequence $\mathbf{r} = (r_k)_{k\geq 0}$, consider the sequence $\overline{\mathbf{r}} = (\overline{r_k})_{k\geq 0}$ defined with the same recurrence relation as $\mathbf{r}$ but with initial conditions $\overline{r_j} = r_j + 2m_j p_1 p_2 \cdots p_t$ for each $j \in \{0, 1, 2\}$, where m_0, m_1, m_2 are integers and $p_1, p_2, \ldots, p_t$ are primes. Then $\overline{r_k} \equiv r_k \pmod{p_i}$ for every $1 \leq i \leq t$ and $k \geq 0$, which implies that $\mathcal{A}_{\mathbf{r}}(x, p_i) \subseteq \mathcal{A}_{\overline{\mathbf{r}}}(x, p_i)$ for each $1 \leq i \leq t$. Moreover, $\mathcal{A}_{\mathbf{r}}(1, 2) \subseteq \mathcal{A}_{\overline{\mathbf{r}}}(1, 2)$. Thus, for each of the named sequences described within this work, one can define another sequence containing infinitely many terms that cannot be represented as a sum of two prime powers. For example, consider the Tribonacci numbers. Given integers m_0, m_1, and m_2, define the sequence $\overline{\mathbf{r}}$ by

$$\overline{r_k} = T_k + 2 \cdot 3 \cdot 5 \cdot 17 \cdot 7 \cdot 13 \cdot 241 \cdot m_k$$

for $k \in \{0, 1, 2\}$ and $\overline{r_k} = \overline{r_{k-1}} + \overline{r_{k-2}} + \overline{r_{k-3}}$ for all $k \geq 3$. From Theorem 2 we deduce for every choice of integers m_0, m_1, and m_2 that $\overline{\mathbf{r}}$ has infinitely many k such that $\overline{r_k}$ cannot be represented as $p^a + q^b$ for some primes p, q and nonnegative integers a, b. In fact, this occurs for every sufficiently large $k \equiv 3817 \pmod{163843680}$. Therefore, our results can be used to produce classes of degree three linear homogeneous recurrence sequences such that each sequence has infinitely many terms that cannot be represented as the sum of two prime powers. The following is an immediate corollary summarizing this.

Corollary 1. *There are infinitely many degree three linear homogeneous recurrence sequences such that each sequence has infinitely many terms that cannot be represented as $p^a + q^b$ for some primes p, q and nonnegative integers a, b.*

Lastly, we wonder for each of the named sequences described within this work if there exist infinitely many terms that are not the sum and not the difference of two prime powers. More precisely, let $(r_k)_{k\geq 0}$ denote one of the sequences studied in this work. Does there exist infinitely many k such that r_k cannot be represented as $p^a \pm q^b$ for some primes p, q and nonnegative integers a, b (i.e., cannot be represented as the sum or difference of two prime powers)? The current techniques to prove such results are related to proving the existence of infinitely many terms of the sequence that are simultaneously Sierpiński and Riesel numbers.

For the Fibonacci numbers, such a question appears to be related to the open problem of whether there exist infinitely many Fibonacci numbers that are both Sierpiński and Riesel numbers [8].

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