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Ramanujan and a Combinatorial Identity Involving Harmonic Numbers

Horst Alzer
Morsbacher Straße 10
51545 Waldbröl
Germany
h.alzer@gmx.de

Man Kam Kwong
Department of Applied Mathematics
The Hong Kong Polytechnic University
Hunghom, Hong Kong
mankamkwong.math@outlook.com

Abstract

We show that trigonometric series and integral formulas given by Ramanujan can be applied to deduce a new summation identity involving binomial coefficients and the classical harmonic number $H_n = 1 + 1/2 + \cdots + 1/n$. Moreover, we prove that for $n \geq 2$ the sum $H_2 + H_3 + \cdots + H_n$ is not an integer.

1 Introduction and statement of the main result

The work of Ramanujan had (and still has) tremendous influence on various fields, such as real analysis and number theory. The aim of this note is to show that certain trigonometric summation and integral formulas given by Ramanujan can be used to deduce a new combinatorial identity involving the well-known harmonic numbers

$$H_0 = 0, \quad H_n = \sum_{\nu=1}^{n} \frac{1}{\nu}, \quad n = 1, 2, 3, \ldots.$$

The properties of these numbers have been studied intensively by many authors. Ramanujan [2, p. 374] obtained a rapidly convergent series representation for the digamma function $\psi = \Gamma'/\Gamma$,

$$\psi(z+1) + \gamma = \frac{1}{2z} - \frac{1}{2\pi z^2} + \frac{\pi\cot(\pi z)}{e^{2\pi z}-1} + \sum_{\nu=1}^{\infty} \frac{z^2}{\nu(\nu^2+z^2)} + 4\sum_{\nu=1}^{\infty} \frac{\nu z^2}{(e^{2\pi\nu}-1)(\nu^4-z^4)},$$

where γ denotes Euler's constant. Let $n \geq 1$ be an integer. If $z \to n$, then we obtain

$$H_n = \frac{1}{2n} - \frac{1}{2\pi n^2} + \frac{1-(4\pi n+1)e^{2\pi n}}{2n(e^{2\pi n}-1)^2} + \sum_{\nu=1}^{\infty} \frac{n^2}{\nu(\nu^2+n^2)} + 4\sum_{\substack{\nu=1 \\ \nu\neq n}}^{\infty} \frac{\nu n^2}{(e^{2\pi\nu}-1)(\nu^4-n^4)}.$$

Lagarias [7] revealed a remarkable connection between harmonic numbers and the classical Riemann hypothesis. He proved that Riemann's hypothesis is equivalent to

$$\sigma(n) < H_n + e^{H_n}\log H_n, \quad n = 2,3,4,\ldots,$$

where $\sigma(n)$ denotes the sum of divisors of n.

In the literature, we can find numerous interesting sums and series involving H_n. We give two examples:

$$\sum_{\nu=1}^{m} (-1)^\nu \binom{m}{\nu} \frac{\nu(H_{n+\nu}-H_n)}{\binom{n+\nu}{\nu}} = \frac{m(m^2-m-n^2)}{(m+n)^2(m+n-1)^2}$$

and

$$\sum_{\nu=1}^{\infty} \frac{\binom{2\nu}{\nu}^2}{(2\nu-1)^2 16^\nu} H_\nu = \frac{4}{\pi}\big(3-4\log 2\big);$$

see Choi [3], Chu [4], Chu and Donno [5], and Sofo [12].

Here is our main result.

Theorem 1. *Let k and n be natural numbers with $k \leq n$. Then*

$$\sum_{\nu=1}^{n} \left\{ \binom{n+k-\nu-1}{n-1} - \binom{n+k-\nu-1}{k-1} \right\} H_\nu = \binom{n+k}{k}\big(H_k - H_n\big). \tag{1.1}$$

Remark 1. When $k = 1$, (1.1) reduces to a particularly simple form:

$$\sum_{\nu=2}^{n} H_\nu = (n+1)(H_n - 1). \tag{1.2}$$

Remark 2. (ii) The referee pointed out the following noteworthy representation of the difference of two harmonic numbers, see [14]:

$$\frac{1}{\binom{n}{k}} \sum_{\nu=1}^{k} \binom{n-\nu}{n-k} \frac{1}{\nu} = H_n - H_{n-k}, \quad 1 \leq k \leq n.$$

Applications of (1.1) lead to the following closely related formulas.

Corollary 2. *Let k and n be natural numbers with $2 \le k \le n$. Then*

$$\sum_{\nu=1}^{n+1} \left\{ \binom{n+k-\nu-1}{n} - \binom{n+k-\nu-1}{k-2} \right\} H_\nu = \binom{n+k}{k-1} (H_k - H_{n+1}) - \frac{1}{n+1} \binom{n+k}{k}. \tag{1.3}$$

Corollary 3. *Let k and n be natural numbers with $k \le n-1$. Then*

$$\sum_{\nu=1}^{n} \left\{ \binom{n+k-\nu-1}{n-2} - \binom{n+k-\nu-1}{k} \right\} H_\nu = \binom{n+k}{k+1} (H_k - H_n) + \frac{1}{k+1} \binom{n+k+1}{k+1}. \tag{1.4}$$

Obaid [8] presented an elegant counterpart of (1.1) which involves the famous Fibonacci numbers F_n:

$$\sum_{\nu=1}^{n} (-1)^\nu \left\{ \binom{n+k-\nu-1}{n-1} - \binom{n+k-\nu-1}{k-1} \right\} F_\nu = (-1)^{n+1} F_{n-k}, \quad 1 \le k \le n. \tag{1.5}$$

With regard to (1.1) and (1.5) it is natural to ask: do there exist similar formulas if we replace H_ν or F_ν by other special combinatorial or number theoretical sequences?

Our proofs are given in the next section. In Section 3, we offer two additional results. Among others, we show that if $n \ge 2$, then $\sum_{\nu=2}^{n} H_\nu$ is not an integer.

2 Proofs

To prove Theorem 1 we need the following four lemmas. The first two formulas were given by Ramanujan; see Berndt [1, pp. 246, 290].

Lemma 4. *Let a, N, and θ be real numbers with $N \ge 0$ and $|\theta| \le \pi/2$. Then*

$$\sum_{\nu=0}^{\infty} \binom{N}{\nu} \sin((a+2\nu)\theta) = 2^N \cos^N(\theta) \sin((a+N)\theta). \tag{2.1}$$

Lemma 5. *Let n be a positive integer. Then*

$$\int_0^{\pi/2} x \cos^n(x) \sin(nx) dx = \frac{\pi}{2^{n+2}} H_n. \tag{2.2}$$

The next result is due to Rung and Obaid [11]. It has applications in the theory of boundary value problems. See also Obaid and Rung [9].

Lemma 6. *Let k and n be integers with $1 \le k \le n$ and let θ be a real number. Then*

$$\sum_{\nu=1}^{n} 2^{\nu} \left\{ \binom{n+k-\nu-1}{k-1} - \binom{n+k-\nu-1}{n-1} \right\} \cos^{\nu}(\theta)\sin(\nu\theta) = 2^{n+k}\cos^{n+k}(\theta)\sin((n-k)\theta). \tag{2.3}$$

Moreover, we need the following combinatorial identity which is given in Gould [6, p. 6].

Lemma 7. *Let N be a nonnegative integer and let x be a real number with $x \notin \{0, 1, \ldots, N\}$. Then*

$$\sum_{\nu=0}^{N} \frac{(-1)^{\nu}}{x-\nu}\binom{N}{\nu} = \frac{(-1)^N}{(x-N)\binom{x}{N}}. \tag{2.4}$$

Proof of Theorem 1. Let $1 \le k \le n$. We apply (2.1) with $a = -2k$, $N = n+k$. Then

$$\sum_{\nu=0}^{n+k} \binom{n+k}{\nu} \sin(2(\nu-k)\theta) = 2^{n+k}\cos^{n+k}(\theta)\sin((n-k)\theta). \tag{2.5}$$

Using (2.3) and (2.5) leads to

$$\sum_{\nu=0}^{n+k}\binom{n+k}{\nu}\theta\sin(2(\nu-k)\theta) = \sum_{\nu=1}^{n}\left\{\binom{n+k-\nu-1}{k-1} - \binom{n+k-\nu-1}{n-1}\right\} 2^{\nu}\theta\cos^{\nu}(\theta)\sin(\nu\theta). \tag{2.6}$$

Next, we integrate and apply the formula

$$\int_0^{\pi/2} \theta\sin(2p\theta)d\theta = (-1)^{p+1}\frac{\pi}{4p}, \quad p \in \mathbb{Z}\setminus\{0\}.$$

This gives

$$\begin{aligned}
\int_0^{\pi/2}\sum_{\nu=0}^{n+k}\binom{n+k}{\nu}\theta\sin(2(\nu-k)\theta)d\theta &= \int_0^{\pi/2}\sum_{\substack{\nu=0\\ \nu\neq k}}^{n+k}\binom{n+k}{\nu}\theta\sin(2(\nu-k)\theta)d\theta \\
&= \sum_{\substack{\nu=0\\ \nu\neq k}}^{n+k}\binom{n+k}{\nu}\int_0^{\pi/2}\theta\sin(2(\nu-k)\theta)d\theta \\
&= -\frac{\pi}{4}\sum_{\substack{\nu=0\\ \nu\neq k}}^{n+k}\frac{(-1)^{\nu-k}}{\nu-k}\binom{n+k}{\nu}.
\end{aligned} \tag{2.7}$$

Using (2.2) yields

$$\int_0^{\pi/2}\sum_{\nu=1}^{n}\left\{\binom{n+k-\nu-1}{k-1}-\binom{n+k-\nu-1}{n-1}\right\}2^{\nu}\theta\cos^{\nu}(\theta)\sin(\nu\theta)d\theta$$

$$=\frac{\pi}{4}\sum_{\nu=1}^{n}\left\{\binom{n+k-\nu-1}{k-1}-\binom{n+k-\nu-1}{n-1}\right\}H_{\nu}. \tag{2.8}$$

From (2.6), (2.7) and (2.8) we conclude that

$$\sum_{\nu=1}^{n}\left\{\binom{n+k-\nu-1}{k-1}-\binom{n+k-\nu-1}{n-1}\right\}H_{\nu}=\sum_{\substack{\nu=0\\ \nu\neq k}}^{n+k}\frac{(-1)^{\nu-k}}{k-\nu}\binom{n+k}{\nu}. \tag{2.9}$$

We define

$$\phi_N(x)=\binom{x}{N},\quad N\in\mathbb{N},\ x\in\mathbb{R}.$$

Let $k<N$. Then

$$\phi_N'(k)=\frac{(-1)^{N-k-1}}{(k+1)\binom{N}{k+1}},\quad \phi_N''(k)=2\phi_N'(k)\sum_{\substack{\nu=0\\ \nu\neq k}}^{N-1}\frac{1}{k-\nu}.$$

From (2.4) we obtain

$$\sum_{\substack{\nu=0\\ \nu\neq k}}^{N}\frac{(-1)^{\nu}}{x-\nu}\binom{N}{\nu}=\frac{(-1)^N}{(x-N)\binom{x}{N}}-\frac{(-1)^k}{x-k}\binom{N}{k}.$$

Using l'Hospital's rule leads to

$$\begin{aligned}\sum_{\substack{\nu=0\\ \nu\neq k}}^{N}\frac{(-1)^{\nu}}{k-\nu}\binom{N}{\nu} &= \lim_{x\to k}\left(\frac{(-1)^N}{(x-N)\binom{x}{N}}-\frac{(-1)^k}{x-k}\binom{N}{k}\right)\\ &= \frac{(-1)^{k+1}}{k-N}\binom{N}{k}\left(1+\frac{(k-N)\phi_N''(k)}{2\phi_N'(k)}\right)\\ &= (-1)^{k+1}\binom{N}{k}\left(\frac{1}{k-N}+\sum_{\substack{\nu=0\\ \nu\neq k}}^{N-1}\frac{1}{k-\nu}\right).\end{aligned}$$

We set $N=n+k$ and obtain

$$\sum_{\substack{\nu=0\\ \nu\neq k}}^{n+k}\frac{(-1)^{\nu}}{k-\nu}\binom{n+k}{\nu}=(-1)^k\binom{n+k}{k}\big(H_n-H_k\big). \tag{2.10}$$

From (2.9) and (2.10) we conclude that (1.1) holds. □

Proof of Corollary 2. Let $2 \le k \le n$. We define

$$T(k,n) = \sum_{\nu=1}^{n} \left\{ \binom{n+k-\nu-1}{n-1} - \binom{n+k-\nu-1}{k-1} \right\} H_\nu. \tag{2.11}$$

Applying Pascal's formula

$$\binom{N+1}{\nu} = \binom{N}{\nu} + \binom{N}{\nu-1}$$

gives

$$\begin{aligned} T(k,n+1) &= \sum_{\nu=1}^{n+1} \left\{ \binom{n+k-\nu-1}{n} + \binom{n+k-\nu-1}{n-1} \right\} H_\nu \\ &\quad - \sum_{\nu=1}^{n+1} \left\{ \binom{n+k-\nu-1}{k-1} + \binom{n+k-\nu-1}{k-2} \right\} H_\nu \\ &= T(k,n) + \sum_{\nu=1}^{n+1} \left\{ \binom{n+k-\nu-1}{n} - \binom{n+k-\nu-1}{k-2} \right\} H_\nu. \end{aligned} \tag{2.12}$$

From (1.1) we obtain

$$T(k,n+1) - T(k,n) = \binom{n+k}{k-1}(H_k - H_{n+1}) - \frac{1}{n+1}\binom{n+k}{k}. \tag{2.13}$$

Combining (2.12) and (2.13) leads to (1.3). □

Proof of Corollary 3. Let $1 \le k \le n-1$ and let $T(k,n)$ be the sum given in (2.11). Then

$$\begin{aligned} T(k+1,n) &= \sum_{\nu=1}^{n} \left\{ \binom{n+k-\nu}{n-1} - \binom{n+k-\nu}{k} \right\} H_\nu \\ &= \sum_{\nu=1}^{n} \left\{ \binom{n+k-\nu-1}{n-1} + \binom{n+k-\nu-1}{n-2} \right\} H_\nu \\ &\quad - \sum_{\nu=1}^{n} \left\{ \binom{n+k-\nu-1}{k} + \binom{n+k-\nu-1}{k-1} \right\} H_\nu \\ &= T(k,n) + \sum_{\nu=1}^{n} \left\{ \binom{n+k-\nu-1}{n-2} - \binom{n+k-\nu-1}{k} \right\} H_\nu. \end{aligned} \tag{2.14}$$

Using (1.1) gives

$$T(k+1,n) - T(k,n) = \binom{n+k}{k+1}(H_k - H_n) + \frac{1}{k+1}\binom{n+k+1}{k+1}. \tag{2.15}$$

From (2.14) and (2.15) we conclude that (1.4) holds. □

3 Additional results

There exists a kind of converse of Theorem 1. We use (1.1) with $k = n-1$ and assume that $(K_\nu)_{\nu\geq 1}$ is a real or complex sequence. Then we obtain the following uniqueness theorem.

Theorem 8. *If*

$$\sum_{\nu=2}^{n} \left\{ \binom{2n-\nu-2}{n-1} - \binom{2n-\nu-2}{n-2} \right\} K_\nu = \binom{2n-1}{n-1} \big(K_{n-1} - K_n\big) \tag{3.1}$$

holds for integers $n \geq 2$, then there exists a constant λ such that

$$K_r = \lambda H_r, \quad r = 1, 2, \dots . \tag{3.2}$$

Proof. We consider two cases.

Case 1. $K_1 = 0$. We use induction to prove that $K_r = 0$ for $r \geq 1$. Let $K_1 = K_2 = \cdots = K_{r-1} = 0$. From (3.1) with $n = r \geq 2$ we obtain

$$-K_r = \binom{2r-1}{r-1}(-K_r).$$

Since $\binom{2r-1}{r-1} > 1$, we get $K_r = 0$.

Case 2. $K_1 \neq 0$. Since (3.1) remains valid, if we replace K_j by K_j/K_1, we may assume that $K_1 = 1$. We show that (3.2) holds with $\lambda = 1$. Let $K_j = H_j$ $(j = 1, \dots, r-1)$. Then we obtain from (3.1) with $n = r$:

$$\begin{aligned} \sum_{\nu=2}^{r-1} \left\{ \binom{2r-\nu-2}{r-1} - \binom{2r-\nu-2}{r-2} \right\} H_\nu &+ \left\{ \binom{r-2}{r-1} - \binom{r-2}{r-2} \right\} K_r \\ &= \binom{2r-1}{r-1} \big(H_{r-1} - K_r\big). \end{aligned} \tag{3.3}$$

Since the solution of a linear equation is unique, we conclude from (1.1) (with $n = r$, $k = r-1$) that (3.3) implies that $K_r = H_r$. □

In 1915, Theisinger [13] proved that if $n > 1$, then H_n is not an integer. Kürschák offered an interesting extension. He showed that if $1 \leq n < m$, then the difference $H_m - H_{n-1}$ is never an integer; see Pólya and Szegö [10, pp. 159, 381]. We apply (1.2) to prove the following related result.

Theorem 9. *Let $n \geq 2$ be a natural number. Then the sum $\sum_{\nu=2}^{n} H_\nu$ is not an integer.*

Proof. We define $J_n = (n+1)H_n$. Then

$$J_2 = \frac{9}{2} \quad \text{and} \quad J_3 = \frac{22}{3}.$$

Let $n \geq 4$. We assume that J_n is an integer. First, we show that there exists a prime number p such that $n+1 = 2p$. We have

$$J_n = \frac{1}{n!} \sum_{\nu=1}^{n} \frac{(n+1)!}{\nu}.$$

It follows that

$$n! \,\Big|\, \sum_{\nu=1}^{n} \frac{(n+1)!}{\nu}.$$

An application of Bertrand's postulate gives that there exists a prime number p such that

$$[n/2] < p < n.$$

Since $p \,|\, n!$, we obtain

$$p \,\Big|\, \sum_{\nu=1}^{n} \frac{(n+1)!}{\nu}. \tag{3.4}$$

We have

$$p \,\Big|\, \frac{(n+1)!}{\nu}, \quad \nu = 1, 2, \ldots, p-1, p+1, \ldots, n. \tag{3.5}$$

From (3.4) and (3.5) we conclude that

$$p \,\Big|\, \frac{(n+1)!}{p}.$$

This means that

$$p \,\Big|\, 1 \cdot 2 \cdots (p-1) \cdot (p+1) \cdots n \cdot (n+1).$$

It follows that there exists a number $k \in \{p+1, \ldots, n+1\}$ such that $p|k$. This implies that $2p \leq n+1$. Otherwise, if $n+1 < 2p$, then there exists a positive integer m with $pm = k$ and

$$p \leq pm = k \leq n+1 < 2p.$$

Thus, $m = 1$ and $k = p$. A contradiction. Hence $2p \leq n+1$.

We assume that n is even. Let $n = 2N$. Then

$$N + 1 = [n/2] + 1 \leq p \leq \frac{n+1}{2} = N + 1/2.$$

A contradiction. It follows that n is odd. Let $n = 2N + 1$. Then

$$N + 1 = [n/2] + 1 \leq p \leq \frac{n+1}{2} = N + 1.$$

Thus, $p = N + 1$. This gives $n + 1 = 2N + 2 = 2p$.

We suppose that there exists a natural number $n \geq 4$ such that J_n is an integer. Then $n = 2p - 1$. Here, p is an odd prime number. We have the representation

$$H_n = \frac{1}{2^{[\log_2(n)]}} \frac{a_n}{b_n},$$

where a_n and b_n are odd integers. It follows that

$$J_{2p-1} = 2pH_{2p-1} = \frac{2p}{2^{[\log_2(2p-1)]}} \frac{a_{2p-1}}{b_{2p-1}} = \frac{p}{2^{[\log_2(2p-1)]-1}} \frac{a_{2p-1}}{b_{2p-1}}.$$

From

$$\log_2(2p-1) = \frac{\log(2p-1)}{\log(2)} \geq \frac{\log(5)}{\log(2)} > 2$$

we conclude that

$$[\log_2(2p-1)] \geq 2.$$

Thus

$$J_{2p-1} = \frac{c_{2p-1}}{2^r b_{2p-1}},$$

where r is a natural number and $c_{2p-1} = pa_{2p-1}, b_{2p-1}$ are odd integers. A contradiction. This implies that $\sum_{\nu=2}^{n} H_\nu = J_n - (n+1)$ is not an integer for all $n \geq 2$. $\square$

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