Journal of Integer Sequences, Vol. 28 (2025),
Article 25.7.5

A New Criterion for the Total Positivity of Riordan Arrays

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Abstract

This paper establishes a new criterion for the total positivity of Riordan arrays with parameterized A- and Z-sequence generating functions. We demonstrate the effectiveness of this criterion by proving the total positivity of several Riordan array classes.

1 Introduction

Riordan arrays play an important unifying role in enumerative combinatorics [19, 22, 20]. Many combinatorial properties of Riordan arrays have been studied in the literature; see, for

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instance, [4, 6, 7, 10, 11, 12, 13, 16]. A *(proper) Riordan array*, denoted by $(d(x), h(x))$, is an infinite lower triangular matrix whose generating function of the kth column is $h^k(x)d(x)$ for $k = 0, 1, 2, \ldots$, where $d(0) = 1$, $h(0) = 0$ and $h'(0) \neq 0$.

A Riordan array $R = [r_{n,k}]_{n,k\geq 0}$ can also be characterized by two sequences $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ such that

$$r_{0,0} = 1, \quad r_{n+1,0} = \sum_{j\geq 0} z_j r_{n,j}, \quad r_{n+1,k+1} = \sum_{j\geq 0} a_j r_{n,k+j} \tag{1}$$

for $n, k \geq 0$. We call $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ the A- and Z-sequences of R, respectively. Let $A(x)$ and $Z(x)$ be the generating functions of A- and Z-sequences, respectively. Then it is known that [15]

$$A(x) = \frac{x}{\bar{h}(x)}, \quad Z(x) = \frac{1}{\bar{h}(x)}\left(1 - \frac{1}{d(\bar{h}(x))}\right) \tag{2}$$

where $\bar{h}(x)$ is the compositional inverse of $h(x)$, i.e., $h(\bar{h}(x)) = \bar{h}(h(x)) = x$. We call the matrix

$$J(Z(x), A(x)) = \begin{bmatrix} z_0 & a_0 & & & \\ z_1 & a_1 & a_0 & & \\ z_2 & a_2 & a_1 & a_0 & \\ z_3 & a_3 & a_2 & a_1 & a_0 \\ \vdots & & & & \ddots \end{bmatrix}$$

the coefficient matrix of the Riordan array R.

Let $M = [m_{n,k}]_{n,k\geq 0}$ be a finite or infinite matrix of real numbers. We say that M is *totally positive* (*TP* for short) if its minors of all orders are nonnegative. Total positivity has a wide variety of applications throughout the realm of applied mathematics. We refer the reader to [3, 9, 14, 17] for details.

An infinite nonnegative sequence $(a_n)_{n\geq 0}$ is a *Pólya frequency sequence* (*PF* for short) if its Toeplitz matrix

$$[a_{i-j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ a_2 & a_1 & a_0 & \\ \vdots & & & \ddots \end{bmatrix}$$

is TP. A fundamental characterization for PF sequences is due to Schoenberg and Edrei, which states that a sequence $(a_n)_{n\geq 0}$ is PF if and only if its generating function

$$\sum_{n\geq 0} a_n t^n = Ct^k e^{\gamma t} \frac{\prod_{j\geq 0}(1 + \alpha_j t)}{\prod_{j\geq 0}(1 - \beta_j t)},$$

where $C > 0$, $k \in \mathbb{N}$, α_j, β_j, $\gamma \geq 0$ and $\sum_{j\geq 0}(\alpha_j + \beta_j) < +\infty$ (see [14, p. 412] for instance). This generating function is called a *Pólya frequency formal power series*. If we identify a finite sequence $a_0, a_1, \ldots, a_n$ with the infinite sequence $a_0, a_1, \ldots, a_n, 0, \ldots$, then a finite sequence of nonnegative numbers is PF if and only if its generating function has only real zeros (see [23] for instance).

In recent years, there have been some papers concerning the total positivity of Riordan arrays. In [6], by means of A- and Z-sequences, Chen et al. gave the following criterion for the total positivity of Riordan arrays.

Proposition 1 ([6, Theorem 2.1])**.** *Let R be a Riordan array defined by (1). Then R is TP if its coefficient matrix $J(Z(x), A(x))$ is TP.*

Recently, Chen et al. [7] provided a new criterion for the total positivity of Riordan arrays in terms of formal power series $d(x)$ and $h(x)$: if both $d(x)$ and $h(x)$ are Pólya frequency formal power series, then R is totally positive. However, many Riordan arrays do not meet these conditions but are still totally positive. For example, the Riordan array (see [21, A113955] for details)

$$[r_{n,k}]_{n,k\geq 0} = \left(\frac{1}{(1-4x)c(x)}, \frac{xc(x)}{\sqrt{1-4x}}\right) = \begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 11 & 6 & 1 & & \\ 42 & 30 & 9 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},$$

where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of Catalan numbers, and its entries

$$r_{n,k} = \sum_{j=0}^{n} \binom{2n}{j}\binom{n-j}{k}.$$

The sequence consisting of sums of rows of R is $\sum_{k=0}^{n}\binom{2n}{n+k}2^k$ [21, A100192].

In this paper, we present a new criterion for the total positivity of Riordan arrays in terms of its decompositions, and give some applications.

2 Main results and applications

In this section we investigate the total positivity of a certain class of Riordan arrays $R(P, Q; a, b, r, s, t)$, whose A- and Z-sequences have the generating functions

$$A(x) = \frac{P^2(x) + sxP(x) + tx^2}{P(x) - rx} \tag{3}$$

and

$$Z(x) = \frac{(P^2(x) + sxP(x) + tx^2)Q(x) + (aP(x) + bx)(P(x) - xQ(x))}{P^2(x) - rxP(x)}, \tag{4}$$

respectively, where $P(x)$ and $Q(x)$ are formal power series, a, b, r, s and t are nonnegative numbers.

Theorem 2. *If $J(Q(x), P(x))$ is TP, $s^2 \geq 4t$, and $as + a\sqrt{s^2 - 4t} \geq 2b$, then $R(P, Q; a, b, r, s, t)$ is TP.*

Proof. Recall that if the generating functions of A- and Z-sequences of a Riordan array R satisfy

$$A(x) = A_2(x)A_1\left(\frac{x}{A_2(x)}\right) \tag{5}$$

and

$$Z(x) = A_1\left(\frac{x}{A_2(x)}\right)Z_2(x) + Z_1\left(\frac{x}{A_2(x)}\right)\left(1 - \frac{x}{A_2(x)}Z_2(x)\right), \tag{6}$$

respectively, where $A_i(x)$ and $Z_i(x)$, $i = 1, 2$, are formal power series, then R can be characterized by the product of two Riordan arrays, i.e., $R = R_1R_2$, where the two Riordan arrays R_i are defined by $A_i(x)$ and $Z_i(x)$ exactly (this fact follows easily from [13, Theorems 3.3 and 3.4]).

Note that (3) can be expressed as

$$A(x) = P(x)\frac{1 + s\frac{x}{P(x)} + t\frac{x^2}{P^2(x)}}{1 - r\frac{x}{P(x)}}$$

which is equivalent to (5) with

$$A_1(x) = \frac{1 + sx + tx^2}{1 - rx} \quad \text{and} \quad A_2(x) = P(x).$$

Similarly, (4) can be expressed as

$$Z(x) = \frac{1 + s\frac{x}{P(x)} + t\frac{x^2}{P^2(x)}}{1 - r\frac{x}{P(x)}}Q(x) + \frac{a + b\frac{x}{P(x)}}{1 - r\frac{x}{P(x)}}\left(1 - \frac{x}{P(x)}Q(x)\right)$$

which is equivalent to (6) with

$$Z_1(x) = \frac{a + bx}{1 - rx} \quad \text{and} \quad Z_2(x) = Q(x).$$

Hence $R(P, Q; a, b, r, s, t) = R_1R_2$, where R_i are defined by $A_i(x)$ and $Z_i(x)$ exactly. To prove the total positivity of $R(P, Q; a, b, r, s, t)$, it suffices to show that R_i are TP by a fact which says that the product of two TP matrices is TP.

From Proposition 1, it follows immediately that R_2 is TP, since $A_2(x) = P(x)$, $Z_2(x) = Q(x)$ and $J(Q(x), P(x))$ is TP.

We now prove that R_1 is TP. From

$$A_1(x) = \frac{1 + sx + tx^2}{1 - rx} \quad \text{and} \quad Z_1(x) = \frac{a + bx}{1 - rx},$$

it follows that

$$J(Z_1(x), A_1(x)) = \begin{bmatrix} a & 1 & & & \\ ra+b & r+s & 1 & & \\ r^2a+rb & r^2+rs+t & r+s & 1 & \\ r^3a+r^2b & r^3+r^2s+rt & r^2+rs+t & r+s & 1 \\ \vdots & & & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & \\ r & 1 & & & \\ r^2 & r & 1 & & \\ r^3 & r^2 & r & 1 & \\ \vdots & & & & \ddots \end{bmatrix} \begin{bmatrix} a & 1 & & & \\ b & s & 1 & & \\ 0 & t & s & 1 & \\ 0 & 0 & t & s & 1 \\ \vdots & & & & \ddots \end{bmatrix}.$$

It is easy to check that the first matrix in the last equation is TP since $r \geq 0$, and so is the second matrix since $s^2 \geq 4t$ and $as + a\sqrt{s^2 - 4t} \geq 2b$ (see [6, Prop. 2.6] for details). Then $J(Z_1(x), A_1(x))$ is TP, and so is R_1 by Proposition 1, as required. $\square$

Example 3. Consider the Riordan array

$$R_1 = \left(\frac{1}{(1-4x)c(x)}, \frac{xc(x)}{\sqrt{1-4x}} \right)$$

which was mentioned in the introduction. Recall that $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of Catalan numbers. By (2), we have

$$A(x) = \frac{4x^2+4x+1}{1+x} \quad \text{and} \quad Z(x) = \frac{3+8x+4x^2}{1+2x+x^2}.$$

Then it is easy to check that

$$R_1 = R(1+x, 1; 2, 2, 0, 2, 1).$$

Clearly, R_1 meets the conditions of Theorem 2, since

$$J(1, 1+x) = \begin{bmatrix} 1 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & 1 \\ \vdots & & & & \ddots \end{bmatrix}$$

is TP. Then R_1 is TP.

Example 4. Consider the Riordan array

$$R_2 = \left(\frac{1+\sqrt{5-4c(x)}}{1-3xc(x)+(1-xc(x))\sqrt{5-4c(x)}}, \frac{1-\sqrt{5-4c(x)}}{2}\right),$$

where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of Catalan numbers. Its first few entries are as follows.

$$R_2 = \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 7 & 5 & 1 & & \\ 29 & 24 & 8 & 1 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

The first column of R is the sequence of antichains in rooted plane trees on n nodes (see [21, A007852] for details). By (2), we have

$$A(x) = \frac{x^4 - 2x^3 - x^2 + 2x + 1}{1-x} \quad \text{and} \quad Z(x) = \frac{x^3 - 3x^2 + x + 2}{1-x}.$$

Then it is easy to check that

$$R_2 = R\left(\frac{1}{1-x}, \frac{1}{1-x}; 1, 1, 0, 2, 1\right).$$

Clearly, R_2 meets the conditions of Theorem 2, since

$$J\left(\frac{1}{1-x}, \frac{1}{1-x}\right) = \begin{bmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 \\ \vdots & & & & \ddots \end{bmatrix}$$

is TP. Hence R_2 is TP.

In the sequel we consider the application of Theorem 2 to a class of special interesting Riordan arrays, the Bell-type Riordan arrays. Following [11], we say that a Riordan array R is a *Bell-type* Riordan array if $z_i = a_{i+1}$ for all $i \in \mathbb{N}$, i.e., the generating functions of its A- and Z-sequences satisfy

$$A(x) - a_0 = xZ(x).$$

Many well-known triangles are actually Bell-type Riordan arrays. Examples include the Pascal triangle, the Catalan triangle [18], the Motzkin triangle [1], the ballot table [2], and the large Schröder triangle [8]. Chen and Wang gave the following criterion for the total positivity of Bell-type Riordan arrays.

Proposition 5 ([6, Theorem 2.7]). *Let R be a Bell-type Riordan array. If the A-sequence of R is PF, then R is TP.*

In the following, we consider the total positivity of Bell-type Riordan arrays of the form (3), and use $R(P;r,s,t)$ to denote such matrices for convenience.

Corollary 6. *If $P(x)$ is a Pólya frequency formal power series and $s^2 \geq 4t$, then $R(P;r,s,t)$ is TP.*

Proof. As in the proof of Theorem 2, it is straightforward to show that $R(P;r,s,t)$ can be expressed as the product of two Bell-type Riordan arrays, i.e., $R(P;r,s,t) = R_1R_2$, where R_1 and R_2 are defined by $A_1(x) = \frac{1+sx+tx^2}{1-rx}$ and $A_2(x) = P(x)$, respectively. Note that $A_1(x)$ is a Pólya frequency formal power series since $s^2 \geq 4t$. From Proposition 5, it follows immediately that both R_1 and R_2 are TP, since $A_2(x)$ is also a Pólya frequency formal power series. Thus $R(P;r,s,t)$ is TP by the Cauchy-Binet formula. □

As applications, we next give some Bell-type Riordan arrays whose A-sequences are not PF sequences, i.e., for their total positivity, Proposition 5 does not work, but Corollary 6 does.

Example 7. Consider the Bell-type Riordan array

$$R_3 = \left(\frac{1-\sqrt{1-4x}-4x}{2x(4x-1)}, \frac{1-\sqrt{1-4x}-4x}{2(4x-1)}\right).$$

Its first few entries are as follows.

$$R_3 = \begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 10 & 6 & 1 & & \\ 35 & 29 & 9 & 1 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

The first column is the sequence $\binom{2n+1}{n+1}$. By (2), we have

$$A(x) = \frac{(1+2x)^2}{1+x} = \frac{(1+x)^2 + 2x(1+x) + x^2}{1+x}.$$

Clearly,

$$R_3 = R(1+x;0,2,1),$$

and R_3 is TP.

Example 8. Consider the Bell-type Riordan array

$$R_4 = \left(\frac{-1+x+\sqrt{1-6x+5x^2}}{2x(x-1)}, \frac{-1+x+\sqrt{1-6x+5x^2}}{2(x-1)}\right)$$

(see [21, A104259] for details). Its first entries are as follows.

$$R_4 = \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 5 & 4 & 1 & & \\ 15 & 14 & 6 & 1 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

The first column is the binomial transform of Catalan numbers [21, A007317]. It is easy to check that

$$R_4 = R\left(\frac{1}{1-x}; 0, 1, 0\right),$$

since $A(x) = \frac{1+x-x^2}{1-x}$. Clearly, R_4 is TP.

Example 9. Consider the Bell-type Riordan array

$$R_5 = \left(\frac{1-\sqrt[4]{1-8x}}{2x}, \frac{1-\sqrt[4]{1-8x}}{2}\right)$$

(see [21, A202039] for details). Its first few entries are as follows.

$$R_5 = \begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 14 & 6 & 1 & & \\ 77 & 37 & 9 & 1 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

The first column is the sequence A101478 in [21]. It follows from

$$R_5 = R\left(\frac{1}{1-x}; 2, 0, 0\right)$$

that R_5 is TP, since $A(x) = \frac{1}{(1-x)(1-2x+2x^2)}$.

3 Concluding remarks

In this paper we provide a criterion for the total positivity of Riordan arrays in terms of the decomposition of Riordan arrays. In particular, we derive a criterion for the Bell-type Riordan arrays in the view of the generating function of A-sequence. Riordan arrays arise often in combinatorics and many well-known combinatorial triangles are such matrices, including the Pascal triangle, the Catalan triangle, the Motzkin triangle, the large and little Schröder triangles, the ballot table and so on. Such triangles arise often in enumeration of lattice paths, for example, the Dyck paths, the Motzkin paths, and the Schröder paths and so on [8, 15, 19]. Brenti [5] gave combinatorial proofs for the total positivity of many well-known matrices by means of lattice path techniques. It is natural to consider the combinatorial proofs for the total positivity of Riordan arrays.

4 Acknowledgments

The authors thank the anonymous referees for their careful reading and valuable suggestions. This work was supported in part by the National Natural Science Foundation of China (Grant Nos. 12271222, 12101110), the Scientific Research Fund of Liaoning Provincial Education Department (Grant No. LJ212410173071).

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2020 *Mathematics Subject Classification*: Primary 05A20; Secondary 15A45, 15B36.
Keywords: Riordan array, total positivity.

(Concerned with sequences A007317, A007852, A100192, A101478, A104259, A113955, and A202039.)

Received July 17 2025; revised versions received July 19 2025; December 5 2025; December 6 2025. Published in *Journal of Integer Sequences*, December 6 2025.

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