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A Note on Explicit Formulas for Bernoulli and Genocchi Numbers

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Abstract

In this note, we establish an identity for the Genocchi and Bernoulli numbers in terms of Stirling numbers of the second kind. This formula generalizes some known explicit formulas for these numbers and also allows us to discover new ones.

1 Introduction and result

The Bernoulli numbers B_n are the rational numbers defined by the recurrence relation:

$$B_0 = 1 \text{ and } B_n = -\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k,\ n \geq 1.$$

Equivalently, these numbers are defined by their exponential generating series:

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}. \tag{1}$$

The Bernoulli numbers first appeared in 1712 in the book called *Katsuyo Sampo* by the Japanese mathematician Seki [24], and in 1713 in the famous book of the Swiss mathematician J. Bernoulli titled *Ars Conjectandi* [2] (see also [17]). They were introduced to express the sum of the m-th powers of the first n integers, where $n, m \geq 1$, using a polynomial in n. The following formula, where n, $m \geq 1$, is now known as Faulhaber's formula:

$$\sum_{k=1}^{n} k^m = \frac{1}{m+1}n^{m+1} + \frac{1}{2}n^m + \frac{1}{m+1}\sum_{k=1}^{\lfloor m/2 \rfloor}\binom{m+1}{2k}B_{2k}n^{m+1-2k}.$$

Genocchi numbers G_n are defined as follows:

$$G_n = 2(1-2^n)B_n, \quad n \geq 0; \tag{2}$$

or, equivalently, by their exponential generating series [9, p. 49]:

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t + 1}. \tag{3}$$

It is straightforward from (1) and (3) that

$$B_{2n+1} = G_{2n+1} = 0,\ n \geq 1.$$

In other words, we have

$$(-1)^n B_n = B_n \ \text{ and } \ (-1)^n G_n = G_n,\ n \geq 2.$$

The numbers $(-1)^n G_{2n}$ were introduced in 1755 by Euler [10, p. 479] to express the alternating sum $\sum_{k=1}^{n}(-1)^k k^m$, where $n, m \geq 1$, the following formula holds

$$\sum_{k=1}^{n}(-1)^k k^m = \frac{(-1)^n}{2}n^m + (-1)^{n+1}\sum_{k=1}^{\lfloor m/2 \rfloor}\binom{m}{2k-1}\frac{G_{2k}}{4k}n^{m+1-2k} + (1-(-1)^n)\frac{G_{m+1}}{2m+2}.$$

Genocchi numbers G_n are integers. This property was established by Genocchi [12] in 1852 (see also [19, p. 252]). The first Genocchi numbers are $0, -1, -1, 0, 1, 0, -3, 0, 17$. This sequence is listed as A036968 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [25]. After their discovery, it became clear that the Bernoulli numbers had important applications in many areas of mathematics, such as number theory and analysis [20, 21]. They appear, for example, in Weyl algebra [20, p. 184].

The search for explicit formulas for these numbers has been the subject of intense and continuous research since their discovery [14]. Among the many explicit formulas discovered, we aim in this article to list some of them, comment on them, and finally show that they are special cases of a more general explicit formula that we establish in this paper. All of these explicit formulas can be expressed in terms of Stirling numbers of the second kind $S(n, k)$, which can be defined by

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n, \tag{4}$$

The numbers $S(n, k)$ were referred to as "remarkable numbers" by d'Ocagne, who studied them meticulously in an important paper [22], published in 1887. These numbers, introduced in 1730 by Stirling [26] in his work "Methodus differentialis", had already been mentioned by many famous mathematicians (see, [20, p. 8]), among whom we can mention in particular Cesàro [8] in 1885, Worpitzky[27] in 1883, and Brinkley [3] in 1807. It is easy to deduce the following from (4) the exponential generating series of $S(n, k)$ [20, p. 52]:

$$\sum_{n=0}^{\infty} S(n,k)\frac{t^n}{n!} = \frac{1}{k!}(e^t-1)^k,\ k \geq 0. \tag{5}$$

The following well-known properties [9, p. 208] easily follow from (5):

$$S(n,k) = S(n-1,k-1) + kS(n-1,k),\ n \geq 1 \text{ and } k \geq 1, \tag{6}$$

$$S(n,0) = S(0,k) = 0,\ n \geq 1 \text{ and } k \geq 1, \tag{7}$$

$$S(0,0) = 1,\ S(n,k) = 0,\ (k > n). \tag{8}$$

The properties (6), (7), and (8) prove that the numbers $S(n, k)$ are natural numbers. This sequence of integers is listed as A008275 in the *On-Line Encyclopedia of Integer Sequences* [25]. The numbers $S(n, k)$, now known as the Stirling numbers of the second kind, play a significant role in combinatorics. The numbers $S(n, k)$ count the number of partitions of a set of n elements into k parts. They are also expressed as [9, p. 204]:

$$S(n,k) = \frac{\Delta^k 0^n}{k!},\ n \geq 0 \text{ and } k \geq 0. \tag{9}$$

Let Δ denote the difference operator defined by $\Delta(f(x)) = f(x+1) - f(x)$. The notation $\Delta^k 0^n$ means $[\Delta^k x^n]_{x=0}$. More generally, we have [9, Theorem B, p. 13]:

$$\frac{\Delta^k x^n}{k!} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^n.$$

Here, $\Delta^k 1^n$ means $[\Delta^k x^n]_{x=1}$, and we can easily observe the following property:

$$\frac{\Delta^{k+1}0^{n+1}}{(k+1)!} = S(n+1,k+1) = \frac{\Delta^k(1^n)}{k!},\ n \geq 0 \text{ and } k \geq 0. \tag{10}$$

We now list the explicit formulas for the Bernoulli and Genocchi numbers that we have generalized. The relation (2) shows that any explicit formula for G_n leads to an explicit formula for B_n. This fact has been widely used in the determination of the explicit formulas that follow.

In 1807, Brinkley [3, pp. 129–131] proved that

$$\frac{B_{m+1}}{(m+1)!} = \frac{1}{m!(2^{m+1}-1)2^{m+1}} \sum_{k=1}^{m} (-1)^{k-1} 2^{m-k} \Delta^k 0^m,\ m \text{ odd }. \tag{11}$$

According to Burstall [4, p. 251], who used Formula (11) to study congruences related to the Bernoulli numbers, Brinkley would be the first to have used the notation $\Delta^n 0^m$. By replacing m with $2n-1$, Formula (11) is equivalent to the following identity:

$$B_{2n} = \frac{2n}{2^{2n}-1} \sum_{k=1}^{2n-1} \frac{(-1)^{k-1} k!}{2^{k+1}} S(2n-1,k),\ n \geq 1.$$

In 1883, Worpitzky proved an identity [27, Relation (68), p. 224] that can be stated as follows:

$$G_n = n \sum_{k=1}^{n-1} \frac{(-1)^k k!}{2^k} S(n-1,k),\ n \geq 2. \tag{12}$$

or equivalently (by exploiting (2) and (9)):

$$B_n = \frac{n}{2^n-1} \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2^{k+1}} \Delta^k 0^{n-1},\ n \geq 2.$$

In 1885, Gomes-Teixeira [13, Formula (3), p. 290] proved an explicit formula for B_{2n}, which can be written in the following form:

$$B_{2m} = \frac{2m}{2^{2m}-1} \sum_{k=1}^{2m-1} \sum_{j=0}^{k} \frac{(-1)^{j+1}}{2^{k+1}} \binom{k}{j} j^{2m-1},\ m \geq 1, \tag{13}$$

or, equivalently,

$$B_{2m} = \frac{2m}{2^{2m}-1} \sum_{k=1}^{2m-1} \frac{(-1)^{k+1} k!}{2^{k+1}} S(2m-1,k),\ m \geq 1.$$

In other words, the formulas of Brinkley and Gomes-Teixeira are equivalent. In 1886, Cesàro [7, pp. 490–491] proved, using symbolic calculus, that

$$B_n = \frac{n}{2^n - 1}\sum_{k=1}^{n-1}\frac{(-1)^{k+1}}{2^{k+1}}\Delta^k 0^{n-1},\ n \geq 2. \tag{14}$$

$$B_n = \frac{n}{2^n - 1}\sum_{k=1}^{n-3}\frac{(-1)^k k}{2^{k+3}}\Delta^{k+1} 0^{n-2},\ n \geq 3. \tag{15}$$

$$B_n = \frac{n}{2^n - 1}\sum_{k=1}^{n-3}\frac{(-1)^{k-1}k(k-3)}{2^{k+3}}\Delta^k 0^{n-3},\ n \geq 4. \tag{16}$$

In [6, p. 316], Cesàro also proved (14) by a different method.

In 1940, Garabedian [11] proved that

$$B_{n+1} = \frac{(-1)^{n+1}(n+1)}{2^{n+1} - 1}\sum_{k=0}^{n}(-1)^k\frac{\Delta^k 1^n}{2^{k+1}},\ n \geq 0. \tag{17}$$

The identity (17), was also proved by Carlitz [5] in 1953, who provided a short proof, obtained also by Rzadkowski [23] in 2004, and by Khaldi et al. [16, p. 88] in 2021. Taking into account (10), we see that Garabedian's formula is equivalent to the following formula:

$$2(1 - 2^n)B_n = G_n = n\sum_{k=1}^{n}\frac{(-1)^{n-k}}{2^{k-1}}(k-1)!\,S(n,k),\ n \geq 1. \tag{18}$$

In his paper [5], Carlitz asserts that, apart from a notation up, the formulas of Worpitzky (12) and Garabedian (18) are identical. This assertion has been obtained by many authors (Gould [14], Rzadkowski [23] in 2004, Mazur [21] in 2008, and Khaldi et al. [16, p. 88] in 2021). However, it is easy to show that these two formulas are not identical.

The relation (18) was obtained by Liu [18, Relation (1.12)] in 2009 as a special case of an explicit formula for the generalized Genocchi numbers, and in 2015 by Guo and Qi as the main result of their paper [15, Theorem 1.1].

In 1953, Carlitz [5, Formula (6)] stated that

$$B_{n+1} = \frac{(-1)^n(n+1)}{2^{n+1} - 1}\sum_{k=0}^{n}(-1)^k 2^{-k-1}\Delta^k 0^n,\ n \geq 1. \tag{19}$$

Since $B_{n+1} = (-1)^{n+1}B_{n+1}$ for $n \geq 1$, we note that Formula (19) is nothing other than rewriting Cesàro's Formula (14) when replacing n with $n+1$.

In 2024, using composition operators, Bensaci et al. [1, Theorem 22] reproved

$$G_n = n\sum_{k=0}^{n-1}\frac{(-1)^k k!}{2^k}S(n-1,k),\ n \geq 1. \tag{20}$$

Identity (20) can easily be deduced from Worpitzky's Formula (12). Our main result is the following theorem.

Theorem 1. *For all positive integers n and r such that $n \geq r+1$, we have*

$$2(1-2^n)B_n = G_n = (-1)^n n \sum_{k=1}^{n-r} \left(\sum_{j=0}^{r} \frac{(-1)^{k+j+r}}{2^{k+j-1}} (k+j-1)!\, S(r,j) \right) S(n-r,k). \tag{21}$$

By exploiting (9) and (10), we easily deduce the following corollary from Theorem 1:

Corollary 2. *For all positive integers n and r such that $n \geq r+1$, we have*

$$(-1)^n B_n = \frac{n}{2^n-1} \sum_{k=1}^{n-r} \left(\sum_{j=0}^{r} (-1)^{k+j+r+1} 2^{r-j} \frac{(k+j-1)!}{k!} S(r,j) \right) \frac{\Delta^k 0^{n-r}}{2^{k+r}}, \tag{22}$$

$$(-1)^n B_n = \frac{n}{2^n-1} \sum_{k=0}^{n-1-r} \left(\sum_{j=0}^{r} (-1)^{k+j+r} 2^{r-j} \frac{(k+j)!}{k!} S(r,j) \right) \frac{\Delta^k 1^{n-1-r}}{2^{k+1+r}}. \tag{23}$$

Note that since $(-1)^n B_n = B_n$ for $n \geq 2$, the factor $(-1)^n$ appearing in Formulas (21), (22), and (23) can be replaced by 1 for $r \geq 1$ and for $r = 0$ with $n \geq 2$.

Theorem 1 and Corollary 2 generalize many explicit formulas for the Bernoulli numbers. Thus, Relation (21) gives

- For $r = 0$, Formula (18) obtained by Liu [18, Formula (1.12), p. 3] in 2009 and by Guo and Qi [15, Theorem 1.1, p. 35] in 2015.
- For $r = 1$, the Formula (12) established by Worpitky [27] in 1883.
- For $r = 1$ and $n = 2m$, the Formula (13) established by Gomes-Teixeira [13] in 1885.

Relation (22) provides

- For $r = 1$ and $n = m + 1$ with m odd, Formula (11) established in 1807 by Brinkley [3, p. 129 and p. 131].
- For $r \in \{1, 2, 3\}$, Formulas (14), (15), and (16), established by Cesàro in 1886 in [7, p. 490] and in [6, p. 316].

Relation (23) becomes

- For $r = 0$, Formula (17) obtained by Garabedian [11] in 1940, by Carlitz [5] in 1953, by Rzadkowski [23] in 2004, and by Khaldi et al. [16, p. 88] in 2021.

By relying on Theorem 1 and Corollary 2, we can obtain some new results. Thus, by applying (22) for $r \in \{4, 5\}$, we obtain

$$B_n = \frac{n}{2^n - 1} \sum_{k=1}^{n-4} \frac{(-1)^{k+1}(k-1)(k^2 - 5k - 2)}{2^{k+4}} \Delta^k 0^{n-4},\ n \geq 5,$$

$$B_n = \frac{n}{2^n - 1} \sum_{k=1}^{n-5} \frac{(-1)^{k+1}k(k^3 - 10k^2 + 15k + 10)}{2^{k+5}} \Delta^k 0^{n-5},\ n \geq 6.$$

Applying (23) for $r \in \{4, 5\}$, we get

$$B_n = \frac{n}{2^n - 1} \sum_{k=1}^{n-5} \frac{(-1)^k k(k+1)(k^2 - 3k - 6)}{2^{k+5}} \Delta^k 1^{n-5},\ n \geq 5,$$

$$B_n = \frac{n}{2^n - 1} \sum_{k=1}^{n-6} \frac{(-1)^k (k+1)^2(k^3 - 7k^2 - 2k + 16)}{2^{k+6}} \Delta^k 1^{n-6},\ n \geq 6.$$

The proof of Theorem 1 is based mainly on the following lemma.

2 Lemma

In what follows, we let $U(m, k)$ and $H_{n,m}$ denote the following sums:

$$U(m, k) := \sum_{j \geq 0} \frac{(-1)^{k+j+m}}{2^{k+j-1}} (k + j - 1)!\, S(m, j),\ n \geq 0,\ k \geq 1,$$

and

$$H_{n,m} := (-1)^n n \sum_{k \geq 1} U(m+1, k) S(n-m, k),\ n \geq m+1. \tag{24}$$

Lemma 3. *For all integers n, r, and $k \geq 1$, we have the following:*

$$U(r+1, k) = U(r, k+1) + kU(r, k). \tag{25}$$

$$H_{n,r} = H_{n,r-1},\ n \geq r+1. \tag{26}$$

Proof. From Property (6), we have

$$S(r+1, j) = S(r, j-1) + jS(r, j),$$

which allows us to rewrite for $r \geq 1$ and $k \geq 1$,

$$\begin{aligned} &U(r+1, k) \\ &= \sum_{j \geq 1} \frac{(-1)^{k+j+r+1}}{2^{k+j-1}} (k+j-1)!\, S(r, j-1) + \sum_{j \geq 1} \frac{(-1)^{k+j+r+1}}{2^{k+j-1}} (k+j-1)! jS(r, j) \\ &= \sum_{j \geq 0} \frac{(-1)^{k+j+r}}{2^{k+j}} (k+j)!\, S(r, j) + \sum_{j \geq 0} \frac{(-1)^{k+j+r+1}}{2^{k+j-1}} (k+j-1)! jS(r, j). \end{aligned}$$

This simplifies to

$$U(r+1,k)=\sum_{j\geq 0}\frac{(-1)^{k+j+r}}{2^{k+j}}(k+j-1)!(k-j)S(r,j).$$

Noting the identity

$$(k+j-1)!(k-j)=2k(k+j-1)!-(k+j)!,$$

we deduce that

$$\begin{aligned}
&U(r+1,k)\\
&=\sum_{j\geq 0}\frac{(-1)^{k+j+r}}{2^{k+j}}(2k(k+j-1)!-(k+j)!)S(r,j)\\
&=k\sum_{j\geq 0}\frac{(-1)^{k+j+r}}{2^{k+j-1}}(k+j-1)!\,S(r,j)+\sum_{j\geq 0}\frac{(-1)^{k+j+r+1}}{2^{k+j}}(k+j)!\,S(r,j).
\end{aligned}$$

Thus, we have

$$U(r+1,k)=kU(r,k)+U(r,k+1).$$

For $n\geq r+1$, we have

$$H_{n,r}=(-1)^n n\sum_{k\geq 1}U(r+1,k)S(n-r,k).$$

Using (25), we obtain

$$H_{n,r}=(-1)^n n\sum_{k\geq 1}(U(r,k+1)+kU(r,k))S(n-r,k).$$

This can be split into two sums as follows:

$$H_{n,r}=(-1)^n n\sum_{k\geq 2}U(r,k)S(n-r,k-1)+(-1)^n n\sum_{k\geq 1}kU(r,k)S(n-r,k).$$

Rearranging, we get

$$H_{n,r}=(-1)^n n\sum_{k\geq 1}U(r,k)(S(n-r,k-1)+kS(n-r,k)).$$

From (6), we have

$$S(n-r,k-1)+kS(n-r,k)=S(n-r+1,k).$$

We deduce that

$$H_{n,r}=(-1)^n n\sum_{k\geq 1}U(r,k)S(n-r+1,k)=H_{n,r-1}.$$

The lemma is therefore proved. $\square$

3 Proof of Theorem 1

Let us start by proving that $G_n = H_{n,0}$. For this purpose, we let $[t^n]A(t)$ denote the coefficient of t^n in the formal power series $A(t)$. According to (5), we have

$$[t^n](e^t-1)^k = \frac{k!}{n!}S(n,k). \tag{27}$$

From (3), we deduce that

$$\sum_{n=1}^{\infty}(-1)^{n+1}G_n\frac{t^{n-1}}{n!} = \frac{2e^t}{e^t+1}.$$

We can extract the coefficient

$$\begin{aligned} G_n &= (-1)^{n+1}n![t^{n-1}]\frac{2e^t}{e^t+1} \\ &= (-1)^{n+1}n![t^{n-1}]\left(\frac{1}{1+\frac{1}{2}(e^t-1)} + \frac{(e^t-1)}{1+\frac{1}{2}(e^t-1)}\right) \\ &= (-1)^{n+1}n![t^{n-1}]\left(\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1}(e^t-1)^{k-1} + \sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1}(e^t-1)^{k}\right). \end{aligned}$$

So, we get

$$G_n = (-1)^{n+1}n!\left(\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1}[t^{n-1}](e^t-1)^{k-1} + \sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1}[t^{n-1}](e^t-1)^{k}\right).$$

Thanks to (27), we obtain

$$G_n = (-1)^{n+1}n\sum_{k\geq 1}\left(-\frac{1}{2}\right)^{k-1}(k-1)!(S(n-1,k-1)+kS(n-1,k)).$$

Using (6), we deduce

$$\forall n\geq 1, \quad G_n = (-1)^n n\sum_{k=1}^{n}\frac{(-1)^k}{2^{k-1}}(k-1)!\,S(n,k).$$

From (24), we can see that

$$H_{n,0} = (-1)^n n\sum_{k=1}^{n}\frac{(-1)^k}{2^{k-1}}(k-1)!\,S(n,k).$$

So, we have

$$\forall n\geq 1, \quad G_n = H_{n,0}. \tag{28}$$

It is easy to deduce from Lemma 3 that for any integer $n \geq r+1$, we have

$$H_{n,r} = H_{n,r-1} = H_{n,r-2} = \cdots = H_{n,0}. \tag{29}$$

So, from (29) and (28), we deduce

$$\forall n \geq r+1, \quad G_n = H_{n,r}.$$

In other words

$$G_n = (-1)^n n \sum_{k=1}^{n-r} \left(\sum_{j=0}^{r} \frac{(-1)^{k+j+r}}{2^{k+j-1}} (k+j-1)!\, S(r,j) \right) S(n-r,k).$$

The proof of the theorem is then complete.

4 Conclusion

Throughout this study, we have observed that many explicit formulas have been rediscovered multiple times by different authors. The fact that these authors often do not adopt the same definition of the Bernoulli numbers, and express their results in different ways, makes this observation more difficult. Theorem 1 and Corollary 2 clarify the distinction between the known explicit formulas and also provide many others.

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