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Some New Results on the Minuscule Polynomials of Type A

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Abstract

We prove two recent conjectures of Bourn and Erickson (2023) regarding a certain family of polynomials $N_n(x)$. The first conjecture says they have only real zeros, and the second concerns the sum of their coefficients. These polynomials arise as the numerators of generating functions in the context of the discrete one-dimensional earth mover's distance (EMD), and also have a connection to the Wiener index of minuscule lattices. Additionally, we prove that the coefficients of $N_n(x)$ are asymptotically normal, the coefficient matrix of $N_n(x)$ is totally positive, and the polynomial sequence $(N_n(x))_{n\geq 1}$ is x-log-concave.

1 Introduction

For $n \in \mathbb{N}^+$ define the polynomials $N_n(x) = \sum_{k=1}^{n-1} N_{n,k} x^k$, where

$$N_{n,k} := \frac{k(n-k)}{4n+2}\binom{2n+2}{2k+1}. \tag{1.1}$$

As the coefficients $N_{n,k}$ are the Wiener indices of a minuscule lattice of type A [7], we refer to $N_n(x)$ as the *minuscule polynomial of type A*. These polynomials also appeared as the numerators of generating functions in the context of the discrete one-dimensional earth mover's distance (EMD) [2, 3]. Below are the first few values of $N_n(x)$:

$$\begin{aligned}
N_1(x) &= 0,\\
N_2(x) &= 2x,\\
N_3(x) &= 8x + 8x^2,\\
N_4(x) &= 20x + 56x^2 + 20x^3,\\
N_5(x) &= 40x + 216x^2 + 216x^3 + 40x^4,\\
N_6(x) &= 70x + 616x^2 + 1188x^3 + 616x^4 + 70x^5.
\end{aligned}$$

We note that some coefficient sequences of the polynomials $N_n(x)$ are in the OEIS [15]. For example, the sequence $2, 8, 20, 40, 70, 112, \ldots$, the leading terms with the explicit expression $2\binom{n+1}{3}$, is entry A007290; the sequence $1, 7, 27, 77, 182, 378, \ldots$, the coefficients of $8x^{n-2}$ with the explicit expression $\frac{2n-1}{5}\binom{n+1}{4}$, is entry A005585; and the coefficient sequence $2, 8, 8, 20, 56, 20, \ldots$ is entry A375853. Recently, Bourn and Erickson [3] proved a conjecture of Bourn and Willenbring [2] regarding the palindromicity and unimodality of $N_n(x)$, and proposed the following conjectures.

Conjecture 1. *The polynomial $N_n(x)$ has only real zeros for every $n \in \mathbb{N}^+$.*

Conjecture 2. *For all positive integers n, we have $N_n(1) = (n-1)2^{2n-3}$.*

This paper was originally motivated by the above conjectures. Actually, we prove more results about the coefficients (1.1). More precisely, in Section 2, we prove Conjecture 1, which implies that the polynomial $N_n(x)$ is log-concave and γ-positive; in Section 3, we establish a polynomial identity for $N_n(x^2)$, which reduces directly to Conjecture 2 for $x = 1$, and by applying the latter identity we show that the coefficients of $N_n(x)$ are asymptotically normal. In Section 4, we further prove that the polynomial sequence $(N_n(x))_{n\geq 1}$ is x-log-concave, i.e., $N_n^2(x) - N_{n+1}(x)N_{n-1}(x) \in \mathbb{N}[x]$ for all $n \geq 2$. Lastly, we prove that the coefficient matrix $[N_{n,k}]_{n,k\geq 0}$ is totally positive in Section 5.

For the reader's convenience we recall some terminology for later use. Let $f(x) = \sum_{k=0}^{n} f_k x^k$ be a polynomial with real coefficients and of degree n. We say that the polynomial $f(x)$ is

- *palindromic* (with respect to n) if $f_k = f_{n-k}$ for all $0 \leq k \leq n$,

- *unimodal* if there exists an index m such that

$$f_0 \le f_1 \le \cdots \le f_m \ge \cdots \ge f_{n-1} \ge f_n,$$

- *log-concave* if $f_k^2 \ge f_{k-1}f_{k+1}$ for all $1 \le k \le n-1$.

It is known that the log-concavity of $f(x)$ implies its unimodality [4].

Let $f(x)$ (resp., $g(x)$) be a polynomial of real coefficients with $\deg(f(x)) = n$ (resp., $\deg(g(x)) = m$) and having only real zeros $r_n \le \cdots \le r_2 \le r_1$ (resp., $s_m \le \cdots \le s_2 \le s_1$). We say that $g(x)$ *interlaces* $f(x)$ if $m = n-1$ and

$$r_n \le s_{n-1} \le \cdots \le s_2 \le r_2 \le s_1 \le r_1, \tag{1.2}$$

and that $g(x)$ *alternates left of* $f(x)$ if $m = n$ and

$$s_n \le r_n \le \cdots \le s_2 \le r_2 \le s_1 \le r_1. \tag{1.3}$$

We write $g(x) \preceq f(x)$ if $g(x)$ *interlaces* $f(x)$ or $g(x)$ *alternates left of* $f(x)$.

2 The polynomials $N_n(x)$ have only real zeros

A finite sequence $\mathbf{\Lambda} = (\lambda_k)_{k=0}^n$ of real numbers is called a *multiplier n-sequence* if a polynomial $a_0+a_1x+\cdots+a_nx^n$ of degree at most n has only real zeros, then so does the new polynomial $a_0\lambda_0+a_1\lambda_1x+\cdots+a_n\lambda_nx^n$. The following is a classical algebraic characterization of multiplier n-sequence.

Lemma 3 ([6, Theorem 3.7])**.** *A real sequence $(\lambda_k)_{k=0}^n$ is a multiplier n-sequence if and only if the polynomial*

$$\sum_{k=0}^{n} \binom{n}{k} \lambda_k x^k$$

has only real zeros with same sign.

Example 4. From the binomial formula $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$ we derive successively

$$nx(1+x)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} x^k, \tag{2.1a}$$

$$nx(1+nx)(1+x)^{n-2} = \sum_{k=0}^{n} k^2 \binom{n}{k} x^k, \tag{2.1b}$$

and

$$n(n-1)x(1+x)^{n-2} = \sum_{k=0}^{n} k(n-k) \binom{n}{k} x^k. \tag{2.1c}$$

By the above lemma, the sequence $(k(n-k))_{k=0}^n$ is a multiplier n-sequence.

The Chebyshev polynomials of the second kind $U_n(x)$ can be defined by the generating function

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1-2xt+t^2}.$$

They also have the following explicit formulae [16, p. 696]:

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} x^{n-2k}(x^2-1)^k \tag{2.2a}$$

$$= 2^n \prod_{k=1}^{n} \left(x - \cos\left(\frac{k\pi}{n+1} \right) \right). \tag{2.2b}$$

Theorem 5. *The polynomial $N_n(x)$ has only real zeros for every positive integer n. Moreover, we have $N_n(x) \preceq N_{n+1}(x)$ for every $n \in \mathbb{N}^+$.*

Proof. By formula (2.2a), we have

$$U_{2n+1}(x) = x^{2n+1} \sum_{k=0}^{n} \binom{2n+2}{2k+1} \left(\frac{x^2-1}{x^2} \right)^k. \tag{2.3a}$$

For $n+2 \le k \le 2n+1$, let $k = 2n+2-\ell$, then $\ell \in [n]$ and formula (2.2b) implies

$$U_{2n+1}(x) = \prod_{k=1}^{n} \left(x - \cos\left(\frac{k\pi}{2n+2} \right) \right) \cdot x \cdot 2^{2n+1} \cdot \prod_{\ell=1}^{n} \left(x - \cos\left(\frac{(2n+2-\ell)\pi}{2n+2} \right) \right)$$

$$= x \cdot 2^{2n+1} \prod_{k=1}^{n} \left(x^2 - \cos^2\left(\frac{k\pi}{2n+2} \right) \right). \tag{2.3b}$$

Substituting $x^2 \to 1/(1-x)$ in $x^{-2n-1}U_{2n+1}(x)$ and combining (2.3a) and (2.3b), we have

$$f_n(x) := \sum_{k=0}^{n} \binom{2n+2}{2k+1} x^k = 2^{2n+1} \prod_{k=1}^{n} \left(1 - (1-x)\cos^2\left(\frac{k\pi}{2n+2} \right) \right), \tag{2.3c}$$

which proves that all zeros of the polynomial $f_n(x)$ are real. By the above example, the polynomial $N_n(x)$ has only real zeros.

By (2.3c), it is not difficult to see that $f_n(x) \preceq f_{n+1}(x)$ since the inequalities

$$1 - \frac{1}{\cos^2\left(\frac{(k+1)\pi}{2n+4}\right)} \le 1 - \frac{1}{\cos^2\left(\frac{k\pi}{2n+2}\right)} \le 1 - \frac{1}{\cos^2\left(\frac{k\pi}{2n+4}\right)}$$

hold for each $k \in [n]$. Given two polynomials $f(x)$ and $g(x)$ with non-negative coefficients and non-zero constant. Note the fact [13, Theorem 6.3.8] that if $f(x) \preceq g(x)$, then $xf'(x) \preceq xg'(x)$. Additionally, the relation $f(x) \preceq g(x)$ with $\deg(f(x)) = \deg(g(x)) - 1 = n$ implies $x^n f(1/x) \preceq x^{n+1} g(1/x)$. Due to $f_n(x) \preceq f_{n+1}(x)$, it is routine to verify that $N_n(x) \preceq N_{n+1}(x)$ by the facts which are mentioned before. $\square$

Remark 6. In view of [13, Theorem 6.3.4], one can also prove the property that the polynomial $f_n(x)$ has only real zero by the Hurwitz stability of $(1+x)^{2n+2}$.

It is known [4, Lemma 7.1.1] that if all zeros of a polynomial with non-negative coefficients are real, then it is log-concave; and if a positive sequence is log-concave, then it is unimodal. Thus, Theorem 5 implies the following result, which generalizes the unimodality of $N_n(x)$ [3, Corollary 4.2].

Corollary 7. *The polynomial $N_n(x)$ is log-concave for each $n \in \mathbb{N}^+$.*

By (1.1) it is obvious that $N_n(x)$ is palindromic. Combining this with Theorem 5, we derive the following γ-positivity from a known result [4, Remark 7.3.1], which provides another generalization of both the palindromicity and unimodality of $N_n(x)$ as stated in [3, Corollaries 3.2 and 4.2].

Corollary 8. *There are non-negative integers $\gamma_{n,k}$ such that*

$$N_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n,k} x^k (1+x)^{n-2k}. \tag{2.4}$$

Below we list the above formula for the first few values of n:

$$\begin{aligned}
N_1(x) &= 0,\\
N_2(x) &= 2x,\\
N_3(x) &= 8x(1+x),\\
N_4(x) &= 20x(1+x)^2 + 16x^2,\\
N_5(x) &= 40x(1+x)^3 + 96x^2(1+x),\\
N_6(x) &= 70x(1+x)^4 + 336x^2(1+x)^2 + 96x^3.
\end{aligned}$$

It would be interesting to find a combinatorial interpretation of the coefficients $\gamma_{n,k}$. The sequence of the half-sums of the gamma coefficients $\sum_k \gamma_{n,k}/2$, which starts with $1, 4, 18, 68, 251$, is entry A376072 in the OEIS [15].

3 Asymptotic normality of the coefficients of $N_n(x)$

In this section, we first establish an identity about $N_n(x)$, which can be viewed as a polynomial version of Conjecture 2. As an application, we show that the coefficients of $N_n(x)$ are asymptotically normal.

3.1 An identity related to minuscule polynomial of type A

Proposition 9. *We have*

$$N_n(x^2) = \frac{n+1}{8}\Big((1+x)^{2n} + (1-x)^{2n}\Big) - \frac{1}{16x}\Big((1+x)^{2n+2} - (1-x)^{2n+2}\Big). \tag{3.1}$$

Proof. Although identity (3.1) can be verified by Maple using a mechanical procedure [11], we provide an alternative proof by manipulating binomial formulas. From the binomial identity $\sum_{k=0}^{2n+2}\binom{2n+2}{k}x^k = (1+x)^{2n+2}$, we derive

$$\sum_{k=0}^{n+1}\binom{2n+2}{2k}x^{2k} = \frac{1}{2}\Big((1+x)^{2n+2}+(1-x)^{2n+2}\Big), \tag{3.2}$$

$$\sum_{k=0}^{n}\binom{2n+2}{2k+1}x^{2k+1} = \frac{1}{2}\Big((1+x)^{2n+2}-(1-x)^{2n+2}\Big). \tag{3.3}$$

Differentiating (3.3) we get

$$\sum_{k=0}^{n}(2k+1)\binom{2n+2}{2k+1}x^{2k} = (n+1)\Big((1+x)^{2n+1}+(1-x)^{2n+1}\Big). \tag{3.4}$$

Subtracting (3.3) divided by x from (3.4) we get

$$\begin{aligned}\sum_{k=1}^{n}2k\binom{2n+2}{2k+1}x^{2k} =& (n+1)\Big((1+x)^{2n+1}+(1-x)^{2n+1}\Big)\\ &-\frac{1}{2x}\Big((1+x)^{2n+2}-(1-x)^{2n+2}\Big).\end{aligned} \tag{3.5}$$

Differentiating (3.4) yields

$$\sum_{k=1}^{n}(2k+1)(2k)\binom{2n+2}{2k+1}x^{2k-1} = (n+1)(2n+1)\Big((1+x)^{2n}-(1-x)^{2n}\Big). \tag{3.6}$$

It follows from the last two identities that

$$\begin{aligned}\sum_{k=1}^{n}k^2\binom{2n+2}{2k+1}x^{2k-1} &= \frac{(n+1)(2n+1)}{4}\Big((1+x)^{2n}-(1-x)^{2n}\Big)\\ &-\frac{(n+1)}{4x}\Big((1+x)^{2n+1}+(1-x)^{2n+1}\Big)+\frac{1}{8x^2}\Big((1+x)^{2n+2}-(1-x)^{2n+2}\Big).\end{aligned} \tag{3.7}$$

Combining (3.5) and (3.7) gives (3.1). □

By Proposition 9, the following result is immediate and the proof is left to the reader.

Proposition 10. *We have*

$$\sum_{n=1}^{\infty}N_n(x^2)\frac{z^n}{n!} = \frac{2x(1+x)^2z-1-x^2}{16x}e^{(1+x)^2z}+\frac{2x(1-x)^2z+1+x^2}{16x}e^{(1-x)^2z}.$$

Setting $x=1$ in (3.1) yields the following result, which is Conjecture 2.

Corollary 11. *For all positive integers n, we have*

$$N_n(1) = (n-1)2^{2n-3}. \tag{3.8}$$

The sequence $(n \cdot 2^{2n-1})_{n\geq 0}$ is entry A002699 in the OEIS [15]. Let $2^{[n]}$ denote the *power set* of $[n]$. Consider the expansion formula

$$(a+b+c+d)^n = \sum_{f:[n]\to[4]} a^{|f^{-1}(1)|}b^{|f^{-1}(2)|}c^{|f^{-1}(3)|}d^{|f^{-1}(4)|}. \tag{3.9}$$

Encoding a mapping $f:[n]\to[4]$ by a pair of subsets $(X,Y)\in 2^{[n]}\times 2^{[n]}$ with $f^{-1}(1)=X\backslash Y$, $f^{-1}(2)=Y\setminus X$, $f^{-1}(3)=X\cap Y$, and $f^{-1}(4)=\overline{X\cup Y}$, we can rewrite identity (3.9) as

$$\sum_{(X,Y)\in 2^{[n]}\times 2^{[n]}} a^{|X\backslash Y|}b^{|Y\backslash X|}c^{|X\cap Y|}d^{|\overline{X\cup Y}|} = (a+b+c+d)^n. \tag{3.10a}$$

Setting $a=b=z$ and $c=d=1$ we find that

$$\sum_{(X,Y)\in 2^{[n]}\times 2^{[n]}} z^{|X\Theta Y|} = 2^n(1+z)^n, \tag{3.10b}$$

where $X\Theta Y := (X\cup Y)\setminus(X\cap Y)$ is the symmetric difference. Differentiating (3.10b) and setting $z=1$ yields a known result, i.e., [9, Eq. (26)],

$$S(n) := \sum_{(X,Y)\in 2^{[n]}\times 2^{[n]}} |X\Theta Y| = n\cdot 2^{2n-1}. \tag{3.10c}$$

Bourn and Erickson [3] asked for a combinatorial proof of (3.8), i.e., $N_n(1) = S(n-1)$ using (3.10c). In this regard, it would be interesting to find a combinatorial proof of (3.1) in terms of $2^{[n]}\times 2^{[n]}$.

3.2 Asymptotic normality

A random variable X is said to be *standard normal distributed* when

$$\text{Prob}\{X\leq x\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt, \tag{3.11}$$

and denoted $X\sim\mathcal{N}(0,1)$. Let $\mathcal{N}(x)$ be the function on the right of (3.11).

Let $a(n,k)$ be a double-indexed sequence of non-negative numbers. Recall [5] that the numbers $a(n,k)$ are *asymptotically normal with mean* μ_n and *variance* σ_n^2 provided that for the normalized probabilities

$$p(n,k) := \frac{a(n,k)}{\sum_{k=0}^{n} a(n,k)} \qquad (0\leq k\leq n)$$

we have, for each fixed $x \in \mathbb{R}$,

$$\sum_{k \leq \mu_n + x\sigma_n} p(n,k) \to \mathcal{N}(x), \quad \text{as } n \to +\infty.$$

The following classical result was initially published by Bender [1, Theorem 2], although Harper [8] already used this technique to show the asymptotic normality of the Stirling numbers of the second kind. We refer the reader to the survey article [5] and references therein for background on asymptotic normality in enumeration.

Lemma 12. *Let $A_n(x) = \sum_{k=0}^{n} a(n,k)x^k$ be a monic polynomial whose zeros are all real and non-positive. If $\lim_{n\to+\infty} \sigma_n = +\infty$, then the numbers $a(n,k)$ are asymptotically normal with mean μ_n and variance σ_n^2. Moreover,*

$$\mu_n = \sum_{k=1}^{n} \frac{1}{1+r_k} \quad \textit{and} \quad \sigma_n^2 = \sum_{k=1}^{n} \frac{r_k}{(1+r_k)^2},$$

where $-r_k \leq 0$ for $1 \leq k \leq n$ are the zeros of $A_n(x)$.

Theorem 13. *The distribution of the coefficients of $N_n(x)$ defined in (1.1) is asymptotically normal as $n \to +\infty$ with mean $\mu_n = n/2$ and variance*

$$\sigma_n^2 = \frac{n^2-n-2}{8(n-1)}. \tag{3.12}$$

Proof. Let $p(n,k) = N_{n,k}/N_n(1)$ for $1 \leq k \leq n-1$. Then $\sum_{k=1}^{n-1} p(n,k) = 1$. By the palindromicity of $N_{n,k}$, we have $p(n,k) = p(n,n-k)$. By the definition of mean, it is easy to see that

$$\mu_n = \sum_{k=1}^{n-1} k \cdot p(n,k) = \frac{1}{2}\sum_{k=1}^{n-1}(k+n-k)\cdot p(n,k) = \frac{n}{2}. \tag{3.13}$$

On the other hand, we have

$$\mu_n = \sum_{k=1}^{n-1} k \cdot p(n,k) = \frac{\sum_{k=1}^{n-1} kN_{n,k}}{N_n(1)} = \frac{N_n'(1)}{N_n(1)}.$$

Combining the above formula with (3.13) results in that

$$\left.\frac{(N_n(x^2))'}{N_n(x^2)}\right|_{x=1} = 2 \cdot \frac{N_n'(1)}{N_n(1)} = n. \tag{3.14}$$

If r_k are the negatives of the (real) zeros of $N_n(x)$ with $k \in [n-1]$, then $N_n(x) = 2\binom{n+1}{3}\prod_{k=1}^{n-1}(x+r_k)$. By Lemma 12, the variance σ_n^2 is given by

$$\sigma_n^2 = \sum_{k=1}^{n-1}(k-\mu_n)^2 \cdot p(n,k) = \sum_{k=1}^{n-1} \frac{r_k}{(1+r_k)^2}. \tag{3.15}$$

It is easily verified that

$$\sum_{k=1}^{n-1}\frac{4xr_k}{(x^2+r_k)^2}=\frac{(N_n(x^2))'}{N_n(x^2)}+x\left(\frac{(N_n(x^2))'}{N_n(x^2)}\right)'$$
$$=\frac{(N_n(x^2))'}{N_n(x^2)}-x\left(\frac{(N_n(x^2))'}{N_n(x^2)}\right)^2+x\frac{(N_n(x^2))''}{N_n(x^2)}. \tag{3.16}$$

Using the formula of $N_n(x^2)$ in Proposition 9, for $n\ge 2$, we have (by Maple)

$$(N_n(x^2))''\big|_{x=1}=\big(2n^3-3n^2+n-2\big)2^{2n-4}. \tag{3.17}$$

Combining (3.14)–(3.17) and Corollary 11, we have

$$\begin{aligned}4\sigma_n^2&=\left(\frac{(N_n(x^2))'}{N_n(x^2)}-x\left(\frac{(N_n(x^2))'}{N_n(x^2)}\right)^2+x\frac{(N_n(x^2))''}{N_n(x^2)}\right)\Bigg|_{x=1}\\&=n-n^2+\frac{2^{2n-4}(2n^3-3n^2+n-2)}{2^{2n-3}(n-1)}\\&=\frac{n^2-n-2}{2(n-1)},\end{aligned}$$

which implies (3.12) and that $\lim_{n\to+\infty}\sigma_n=+\infty$. The proof is complete by Lemma 12. □

4 x-log-concavity of polynomials

A polynomial sequence $(f(x))_{n\ge 0}$ is said to be *x-log-concave* if the polynomial

$$f_n^2(x)-f_{n+1}(x)f_{n-1}(x)$$

has only non-negative coefficients for each $n\in\mathbb{N}^+$. The definition of x-log-concavity is introduced by Stanley [14]. The corresponding definition of x-log-convexity of a polynomial sequence is that all coefficients of the polynomial $f_{n+1}(x)f_{n-1}(x)-f_n^2(x)$ are non-negative for each n. The properties x-log-concavity and x-log-convexity have received much attention in combinatorics.

The first few values of $D_n(x):=N_n^2(x)-N_{n+1}(x)N_{n-1}(x)$ $(n\ge 2)$ are listed as follows:

$$\begin{aligned}D_2(x)&=4x^2,\\D_3(x)&=24x^2+16x^3+24x^4,\\D_4(x)&=80x^2+192x^3+480x^4+192x^5+80x^6,\\D_5(x)&=200x^2+1040x^3+4280x^4+5344x^5+4280x^6+1040x^7+200x^8.\end{aligned}$$

Then we have the following result.

Theorem 14. *The polynomial sequence $(N_n(x))_{n\geq 1}$ is x-log-concave.*

Proof. By Proposition 9 it follows that

$$\begin{aligned}
&N_{n+1}^2(x^2) - N_{n+2}(x^2)N_n(x^2) \\
&= \frac{1}{64}\Big((1+x)^{4n+4} + (1-x)^{4n+4} - 2\Big(1+(8n^2+16n+6)x^2+x^4\Big)(1-x^2)^{2n}\Big),
\end{aligned}$$

whose verification is left to the reader. Therefore, we have

$$D_{n+1}(x) = \frac{1}{32}\sum_{k=0}^{2n+2}\binom{4n+4}{2k}x^k - \frac{1}{32}\Big(1+(8n^2+16n+6)x+x^2\Big)(1-x)^{2n}.$$

Note that the degree of $D_{n+1}(x)$ is $2n$ and $D_{n+1}(0)=0$. Let

$$D_{n+1}(x) = \frac{1}{32}\sum_{k=1}^{2n} D_{n,k}x^k.$$

Then the above formula of $D_n(x)$ gives

$$D_{n,k} = \binom{4n+4}{2k} - (-1)^k\binom{2n}{k} - (-1)^k\binom{2n}{k-2} + (-1)^k(8n^2+16n+6)\binom{2n}{k-1}.$$

We now proceed to show that $D_{n,k}\geq 0$ by the parity of k. Let $(a)_n = a(a+1)\cdots(a+n-1)$ for $n\geq 1$ and $(a)_0=1$.

- For $1\leq k\leq n$ we have

$$\begin{aligned}
\frac{(2k)!(2n-2k+2)!}{(2n)!}D_{n,2k} &= \frac{(2n+1)_{2n+4}}{(2k+1)_{2k}(2n-2k+3)_{2n-2k+2}} + E(n,k) \\
&\geq E(n,k), \qquad (4.1)
\end{aligned}$$

  where $E(n,k) = 2k(2n-2k+2)(8n^2+16n+6) - 2k(2k-1) - (2n-2k+2)(2n-2k+1)$. We notice that

$$\begin{aligned}
E(n+k,k) &= 32k^3n + 64k^2n^2 + 32kn^3 + 32k^3 + 128k^2n \\
&\quad + 96kn^2 + 60k^2 + 88kn + 26k - 4n^2 - 6n - 2,
\end{aligned}$$

  which is clearly non-negative for $n\geq 0$ and $k\geq 1$.

- For $1\leq k\leq n-1$ we have

$$\frac{(2k+1)!(2n-2k+1)!}{(2n)!}D_{n,2k+1} = \frac{(2n+1)_{2n+4}}{(2k+2)_{2k+1}(2n-2k+2)_{2n-2k+1}} - F(n,k), \quad (4.2)$$

where $F(n,k) = 32(n+1)^2(n-k)k + 2(n+1)(2n+1)(4n+3)$. We now proceed to prove the non-negativity of the right-hand side of (4.2), i.e.,

$$(2k+2)_{2k+1} \cdot (2n-2k+2)_{2n-2k+1} F(n,k) \leq (2n+1)_{2n+4}. \tag{4.3}$$

Let $g(k) := (2k+2)_{2k+1} \cdot (2n-2k+2)_{2n-2k+1}$ and

$$h(k) := F(n,k) \cdot g(k). \tag{4.4}$$

We now show that the function $h(k)$ is bounded by the right-hand side of (4.3) for $k \in [1, n-1]$. Firstly, for $k=1$ we have

$$(2n+1)_{2n+4} - h(1) = 1024(n-1)n(n+1)^2(n+2)(4n-3)(2n+1)_{2n-2}, \tag{4.5}$$

which is non-negative when $n \geq 1$. As $h(k) = h(n-k)$ for $k \in [1, n-1]$, it suffices to prove that $h(k)$ is decreasing on the interval $[1, n/2]$, i.e., $h(k) - h(k+1) \geq 0$ for $1 \leq k \leq n/2 - 1$.

By the definition of $h(k)$, it is not difficult to get the formula

$$h(k) - h(k+1) = \frac{g(k+1)}{(4k+3)(4k+5)} \cdot 32\big(p_1(k) + (n+1)^2 p_2(k)\big), \tag{4.6}$$

where

$$\begin{aligned}
p_1(k) &:= (n+1)(n-2k-1)(n+1)(2n+1)(4n+3),\\
p_2(k) &:= 32(n+2)k^3 - 48(n+2)(n-1)k^2 + (16n^3 - 48n + 62)k - 15n + 15.
\end{aligned}$$

The function $p_1(k)$ is obviously non-negative if $1 \leq k \leq n/2 - 1$. For $p_2(k)$, we have

$$p_2'(k) = 96(n+2)\big(k^2 - (n-1)k\big) + 16n^3 - 48n + 62.$$

Thus the function $p_2'(k)$ in k goes upwards and the symmetric axis is $(n-1)/2$. Note that the function $p_2'(k)$ takes the values

$$p_2'(1) = 16n^3 - 96n^2 - 48n + 446 \geq 0$$

when $n \geq 6$, and

$$p_2'(n/2 - 1) = -8n^3 + 48n + 62 \leq 0$$

when $n \geq 3$. Following the discussion above, the function $p_2(k)$ first increases and then decreases when $1 \leq k \leq n/2 - 1$. Moreover, we get that

$$p_2(1) = (n-3)(16n^2 - 79) \quad \text{and} \quad p_2(n/2 - 1) = 4n^3 - 16n - 15,$$

which are non-negative when $n \geq 3$. The function $p_2(k)$ is therefore non-negative when $1 \leq k \leq n/2 - 1$.

In conclusion, the proof is complete. $\square$

Recall that a polynomial $f(x)$ is *weakly Hurwitz stable* if $f(x)$ is non-vanishing or identically zero when $\Re(x) > 0$, namely, all zeros of $f(x)$ lie on the closed left half-plane. It is easy to see that the weak Hurwitz stability of polynomial $f(x) \in \mathbb{R}[x]$ implies that all coefficients of $f(x)$ have same sign. That is, the weak Hurwitz stability is stronger than x-log-concavity in a sense. Using mathematical software Matlab, we have computed that the polynomial $N_{n+1}^2(x) - N_{n+2}(x)N_n(x)$ is weakly Hurwitz stable for $n \leq 50$. Based on this, it is reasonable to propose the following conjecture.

Conjecture 15. *The polynomial $N_{n+1}^2(x) - N_{n+2}(x)N_n(x)$ is weakly Hurwitz stable for each positive integer n.*

5 Total positivity of the coefficient matrix

Let M be a (finite or infinite) matrix of real numbers. The matrix M is called *totally positive* (TP) if all minors of M are non-negative. Let $(a_n)_{n\geq 0}$ be an infinite sequence of real numbers, and define the Toeplitz matrix of $(a_n)_{n\geq 0}$ as

$$[a_{n-k}]_{n,k\geqslant 0} = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5.1}$$

Recall that $(a_n)_{n\geq 0}$ is said to be a *Pólya frequency* (PF) sequence if the Toeplitz matrix of $(a_n)_{n\geq 0}$ is TP. The following result is a classical representation theorem for PF sequences.

Lemma 16 ([10, Schoenberg-Edrei Theorem, p. 154])**.** *A non-negative sequence $(a_n)_{n\geq 0}$ ($a_0 \neq 0$) is a* PF *sequence if and only if the generating function of $(a_n)_{n\geq 0}$ has the form*

$$\sum_{n=0}^{+\infty} a_n x^n = a_0 e^{\gamma x} \frac{\prod_{i=1}^{+\infty}(1+\alpha_i x)}{\prod_{i=1}^{+\infty}(1-\beta_i x)},$$

where $\alpha_i, \beta_i, \gamma \geq 0$ and $\sum_{i=1}^{+\infty}(\alpha_i + \beta_i) < +\infty$.

A (infinite) sequence $\mathbf{\Lambda} = (\lambda_k)_{k\geq 0}$ of real numbers is called a *multiplier sequence (of the first kind)* if whenever a polynomial $a_0 + a_1x + \cdots + a_nx^n$ has only real zeros, then so does the polynomial $a_0\lambda_0 + a_1\lambda_1 x + \cdots + a_n\lambda_n x^n$. The following result is a transcendental characterization of multiplier sequence.

Lemma 17 ([12, p. 93]). *A non-negative sequence* $(\lambda_n)_{n\geq 0}$, $(\lambda_0 \neq 0)$, *is a multiplier sequence if and only if the generating function of* $(\lambda_n)_{n\geq 0}$ *is a real entire function which has the form*

$$\sum_{n=0}^{+\infty} \frac{\lambda_n}{n!} x^n = \lambda_0 e^{\gamma x} \prod_{i=1}^{+\infty} (1 + \alpha_i x),$$

where $\gamma \geq 0, \alpha_i \geq 0$, *and* $\sum_{i=1}^{+\infty} \alpha_i < +\infty$.

In addition, there is a classical algebraic characterization of multiplier sequence as in the following.

Lemma 18 ([12, p. 100]). *A non-negative sequence* $(\lambda_k)_{k\geq 0}$ *is a multiplier sequence if and only if the polynomial*

$$\sum_{k=0}^{n} \binom{n}{k} \lambda_k x^k$$

has only real zeros with same sign for each $n \geq 1$.

Theorem 19. *The matrix* $N = [N_{n,k}]_{n\geq k\geq 0}$ *is* TP.

Proof. By (1.1), we have

$$N_{n,k} = \frac{k(n-k)}{4n+2} \binom{2n+2}{2k+1} = \frac{k(2n+2)!}{(4n+2)(2k+1)!} \times \frac{n-k}{(2n-2k+1)!}.$$

Obviously, the total positivity of the matrix N is equivalent to that of the matrix

$$T := \left[\frac{n-k}{(2n-2k+1)!} \right]_{n\geq k\geq 0}.$$

Therefore, it suffices to show that the sequence $\left(\frac{k}{(2k+1)!}\right)_{k\geq 0}$ is PF.

Combining Lemmas 16 and 17, a non-negative sequence $\left(\frac{a_k}{k!}\right)_{k\geq 0}$ is PF if the sequence $(a_k)_{k\geq 0}$ is a multiplier sequence. Then we need to prove that the sequence $\left(\frac{k\times k!}{(2k+1)!}\right)_{k\geq 0}$ is a multiplier sequence. Invoking Lemma 18 it is sufficient to show that the polynomial

$$g_n(x) := \sum_{k=0}^{n} \binom{n}{k} \frac{k \times k!}{(2k+1)!} x^k$$

has only real zeros for each $n \in \mathbb{N}$.

Define the polynomial

$$h_n(x) := \sum_{k=0}^{n} \frac{1}{(2k+1)!(n-k)!} x^k.$$

Note that the polynomial $h_n(x)$ satisfies the following recurrence relation:

$$(2n+2)(2n+3)h_{n+1}(x) = (4n+6+x)h_n(x) + 4xh_n'(x).$$

Applying the above recurrence relation, it is not difficult to verify that

$$\exp\Big(\frac{x}{4} + \frac{2n+1}{2}\ln(x)\Big)(2n+2)(2n+3)h_{n+1}(x) = \Big(\exp\Big(\frac{x}{4} + \frac{2n+1}{2}\ln(x)\Big)(4xh_n(x))\Big)'.$$

By induction on n and Rolle's theorem, it is easy to see that $h_n(x)$ has only real zeros for each positive integer n. As $g_n(x) = n!xh_n'(x)$, the polynomial $g_n(x)$ also has only real zeros, which implies the desired result. □

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