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Determining all Biunitary Triperfect Numbers of a Certain Form

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Abstract

A divisor d of an integer N is called a unitary divisor of N if d and N/d are relatively prime and a biunitary divisor of N if d and N/d have no common unitary divisor except 1. An integer N is called a biunitary triperfect number if the sum $\sigma^{**}(N)$ of biunitary divisors of N is equal to $3N$. We show that 2160 is the only biunitary triperfect number divisible by $27 = 3^3$.

1 Introduction

As usual, we let $\sigma(N)$ and $\omega(N)$ denote, respectively, the sum of divisors and the number of distinct prime factors of a positive integer N (throughout this paper, we consider only positive divisors). We call a positive integer N perfect if $\sigma(N) = 2N$. It is a well-known unsolved problem whether or not an odd perfect number exists. Interest to this problem has produced many analogous concepts and problems concerning divisors of an integer. For example, it is also unknown whether or not there exists an odd multiperfect number, an integer dividing the sum of its divisors. We call a positive integer N k-perfect if $\sigma(N) = kN$. Ordinary perfect numbers are 2-perfect numbers and multiperfect numbers are k-perfect numbers for some positive integer k. A 3-perfect number such as 120, 672, and 523776

is also called *triperfect*, given in sequence A005820 in the *On-Line Encyclopedia of Integer Sequences* (OEIS).

On the other hand, some special classes of divisors have also been studied in several papers. One of them is the class of unitary divisors, a divisor d of a positive integer N relatively prime to N/d. As far as we know, Vaidyanathaswamy first mentioned this class of divisors with the term "block-factor" in [10, p. 601, Example 4]. Later, Cohen [2] called such a divisor d of N a *unitary divisor*. Wall [11] introduced the notion of *biunitary divisors*, a divisor d of a positive integer N satisfying $\gcd_1(d, N/d) = 1$, where $\gcd_1(a, b)$ is the greatest common unitary divisor of two positive integers a and b.

According to Cohen [2] and Wall [11], respectively, we let $\sigma^*(N)$ and $\sigma^{**}(N)$ denote the sum of unitary and biunitary divisors of N. Moreover, we write $d \,||\, N$ if d is a unitary divisor of N. Hence, for a prime p, we have $p^e \,||\, N$ if p divides N exactly e times. Replacing σ by σ^*, Subbarao and Warren [9] introduced the notion of *unitary perfect numbers* by calling N unitary perfect if $\sigma^*(N) = 2N$. They proved that there are no odd unitary perfect numbers, and 6, 60, 90, and 87360 are the first four unitary perfect numbers. Nine years later, the fifth unitary perfect number was found by Wall [12], but no further instance has been found. Subbarao [8] conjectured that there are only finitely many unitary perfect numbers.

Similarly, a positive integers N is called *biunitary perfect* if $\sigma^{**}(N) = 2N$. Wall [11] showed that 6, 60, and 90, the first three unitary perfect numbers, are the only biunitary perfect numbers.

Combining the notion of multiperfect numbers and biunitary perfect numbers, Hagis [4] introduced the notion of biunitary multiperfect numbers, integers N such that $\sigma^{**}(N) = kN$ for some positive integer k and proved that there is no odd biunitary multiperfect number. Smallest instances for $\sigma^{**}(N) = kN$ with $k \geq 3$ are $N = 120, 672, 2160, \ldots$ with $k = 3$ and $N = 30240, 1028160, 6168960, \ldots$ with $k = 4$. All biunitary multiperfect numbers below 4.66×10^{12} as well as many larger ones are given in A189000.

Now we can call an integer N satisfying $\sigma^{**}(N) = kN$ biunitary k-perfect and biunitary 3-perfect numbers biunitary triperfect. Haukkanen and Sitaramaiah [5] determined all biunitary triperfect numbers N such that $2^a \,||\, N$ with $1 \leq a \leq 6$ or $a = 8$, and such ones with $a = 7$ under several conditions. In this paper, we determine all biunitary triperfect numbers divisible by $27 = 3^3$.

Theorem 1. *There exists the only one biunitary triperfect number $N = 2160$ divisible by 3^3.*

Comparison of results on ordinary, unitary, and biunitary divisors such as Wall's result mentioned above leads us to expect that certain biunitary problems are more accessible than corresponding ordinary and unitary problems. Indeed, we proved [14] that $M = 2$ and 9 are the only biunitary superperfect numbers, integers satisfying $\sigma^{**}(\sigma^{**}(N)) = 2N$, while the corresponding equation $\sigma^*(\sigma^*(N)) = 2N$ has many solutions 2, 9, 165, 238, ... as given in A038843 and we only proved [13] that 9 and 165 are the only *odd* solutions.

Indeed, our proof is completely elementary, based on the idea used in [14]. If $\sigma^{**}(N) = 3N$ with a factorization $N = \prod_i p_i^{e_i}$, then a prime $p \neq 3$ dividing $\sigma^{**}(p_i^{e_i})$ must divide N since $\sigma^{**}(p_i^{e_i}) \mid \sigma^{**}(N) = 3N$. We see that $\sigma^{**}(2^e 3^f)/(2^e 3^f)$ tends to 3 as e and f grows and for

e and f large, $\sigma^{**}(2^e)$ and $\sigma^{**}(3^f)$ produce new prime factors of N. In many cases, this causes $\sigma^{**}(N)/N > 3$, which is a contradiction (in [14], we focused on only one prime 2 to arrive at the contradiction $\sigma^{**}(\sigma^{**}(N))/N > 2$ in many cases). In other cases, we are led to a contradiction that $p^{f+1} \mid N$ under the assumption that p divides N exactly f times or $\sigma^{**}(N)/N < 3$.

2 Preliminary lemmas

In this section, we give several preliminary lemmas concerning the sum of biunitary divisors used to prove our main theorems.

Before all, we recall two basic facts from [11]. The sum of biunitary divisors function σ^{**} is multiplicative. Moreover, if p is a prime and e is a positive integer, then

$$\sigma^{**}(p^e) = \begin{cases} p^e + p^{e-1} + \cdots + 1 = \frac{p^{e+1}-1}{p-1}, & \text{if } e \text{ is odd;} \\ \frac{p^{e+1}-1}{p-1} - p^{e/2} = \frac{(p^{e/2}-1)(p^{e/2+1}+1)}{p-1}, & \text{if } e \text{ is even.} \end{cases} \tag{1}$$

We note that, putting $e = 2s - 1 - \delta$ with $\delta \in \{0, 1\}$, this can be represented by the single formula

$$\sigma^{**}(p^e) = \frac{(p^{s-\delta} - 1)(p^s + 1)}{p - 1}, \tag{2}$$

provided that $e > 0$.

From these facts, we can deduce the following lemmas almost immediately. We note that these lemmas were already proved in our previous paper [14]. However, we would like to give proofs for self-containedness.

Lemma 2. *Let n be a positive integer. Then, $\sigma^{**}(n)$ is odd if and only if n is a power of 2 (including 1). More exactly, $\sigma^{**}(n)$ is divisible by 2 at least $\omega(n)$ times if n is odd and at least $\omega(n) - 1$ times if n is even.*

Proof. Whether e is even or odd, $\sigma^{**}(p^e)$ is odd if and only if $p = 2$ by (2). Factoring $n = 2^e \prod_{i=1}^{r} p_i^{e_i}$ into distinct odd primes $p_1, p_2, \ldots, p_r$ with $e \geq 0$ and $e_1, e_2, \ldots, e_r > 0$, each $\sigma^{**}(p_i^{e_i})$ is even. Hence, $\sigma^{**}(n) = \sigma^{**}(2^e) \prod_{i=1}^{r} \sigma^{**}(p_i^{e_i})$ is divisible by 2 at least r times, where $r = \omega(n)$ if n is odd and $\omega(n) - 1$ if n is even. □

Lemma 3. *For any prime p and any positive integer e, $\sigma^{**}(p^e)/p^e \geq 1 + 1/p^2$. Moreover, $\sigma^{**}(p^e)/p^e \geq 1 + 1/p$ unless $e = 2$ and $\sigma^{**}(p^e)/p^e \geq (1 + 1/p)(1 + 1/p^3)$ if $e \geq 3$. More generally, for any positive integers m and $e \geq 2m - 1$, we have $\sigma^{**}(p^e)/p^e \geq \sigma^{**}(p^{2m})/p^{2m}$ and, unless $e = 2m$, $\sigma^{**}(p^e)/p^e \geq 1 + 1/p + \cdots + 1/p^m$.*

Proof. If $e \geq 2m - 1$ and e is odd, then $p^e, p^{e-1}, \ldots, p, 1$ are biunitary divisors of p^e. If $e > 2m$ and e is even, then $p^e, p^{e-1}, \ldots, p^{e-m}$ are biunitary divisors of p^e since $e - m > e/2$. Hence, if $e \geq 2m - 1$ and $e \neq 2m$, then $\sigma^{**}(p^e) = p^e + p^{e-1} + \cdots + 1 > p^e + \cdots + p^{e-m} = p^e(1+1/p+\cdots+1/p^m)$. Since $\sigma^{**}(p^{2m})/p^{2m} < 1+1/p+\cdots+1/p^m$, $\sigma^{**}(p^e)/p^e$ with $e \geq 2m-1$ takes its minimum value at $e = 2m$. □

Now we quote the following lemma of Bang [1], which has been rediscovered (and extended to numbers of the form $a^n - b^n$) by many authors such as Zsigmondy [15], Dickson [3] and Kanold [6]. Also see Shapiro [7, Theorem 6.4A.1].

Lemma 4. *If $a \geq 2$ and $n \geq 2$ are integers, then $a^n - 1$ has a prime factor which does not divide $a^m - 1$ for any $m < n$, unless $(a, n) = (2, 1), (2, 6)$ or $n = 2$ and $a + 1$ is a power of 2. Furthermore, such a prime factor must be congruent to 1 modulo n.*

As a corollary, we obtain the following lemma:

Lemma 5. *If $a \geq 2$ and $n \geq 1$ are integers, then $a^n + 1$ has a prime factor which does not divide $a^m + 1$ for any $m < n$ unless $(a, n) = (2, 3)$. Furthermore, such a prime factor must be congruent to 1 modulo $2n$.*

Proof. By Lemma 4, $a^{2n} - 1 = (a^n - 1)(a^n + 1)$ has a prime factor p which does not divide $a^m - 1$ for any $m < 2n$. Since p does not divide $a^n - 1$, the prime p must divide $a^n + 1$. On the other hand, for $m < n$, p cannot divide $a^m + 1$ since p does not divide $a^{2m} - 1$. Finally, such a prime factor p must be congruent to 1 modulo $2n$. □

Now we prove that $\sigma^{**}(2^e)$ and $\sigma^{**}(3^f)$ must produce a new prime factor which is not very large.

Lemma 6. *Let f be a positive integer and write $f = 2t - 1 - \eta$ with $\eta \in \{0, 1\}$ and t a positive integer. If $f \geq 5$, then at least one of the following statements holds.*

(A) $\eta = 0$ and $\sigma^{**}(3^f)$ has a prime factor p_1 with $5 < p_1 \leq (3^t - 1)/2$,

(B) $p_1 = 5 \mid \sigma^{**}(3^f)$ and $f \equiv 2 \pmod 4$ or $f = 7, 8$,

(C) $\sigma^{**}(3^f)$ has a prime factor p_1 with $5 < p_1 \leq \sqrt{(3^{t-\eta} - 1)/2}$, or

(D) 4 divides t, $\eta = 1$, and $p_1 = (3^{t-1} - 1)/2$ is prime.

Proof. We begin by observing that $\sigma^{**}(3^f) = (3^t + 1)(3^{t-\eta} - 1)/2$ from (2). We put $t = 2^h v$ with v odd.

If $\eta = 0$ and $f \neq 7$, then, we see that $t = 3$ or $t \geq 5$. Hence, Lemma 4 implies that $(3^t - 1)/2$ has a prime factor greater than 5. Defining p_1 to be the smallest among such prime factors, we have $5 < p_1 \leq (3^t - 1)/2$ and (A) holds. If $f = 7$, then $\sigma^{**}(3^f) = (3^8 - 1)/2 = 2^4 \times 5 \times 41$ and (B) holds.

Henceforth we assume that $\eta = 1$. If t is odd and $t \neq 5, 9$, then, $(t - 1)/2 = 3$ or $(t - 1)/2 \geq 5$ and, like above, $(3^{(t-1)/2} - 1)/2$ has a prime factor greater than 5. Take the smallest p_1 among such prime factors. Then, we have $p_1 \leq (3^{(t-1)/2} - 1)/2 < \sqrt{(3^{t-1} - 1)/2}$ and $p_1 \mid (3^{(t-1)/2} - 1)/2 \mid (3^{t-1} - 1)/2 \mid \sigma^{**}(3^f)$. Hence, (C) holds.

If $t \equiv 2 \pmod 4$, then $5 \mid (3^t + 1) \mid \sigma^{**}(3^f)$ and (B) holds.

If $4 \mid t$, then, defining p_1 to be the smallest prime factor of $(3^{t-1} - 1)/2$, clearly $p_1 = (3^{t-1} - 1)/2$ is prime or $p_1 \leq \sqrt{(3^{t-1} - 1)/2}$. Since $t - 1$ is odd, we can never have $p_1 = 2$

or $p_1 = 5$. Thus we see that $p_1 = (3^{t-1}-1)/2$ is prime or $5 < p_1 \le \sqrt{(3^{t-1}-1)/2}$, implying (D) and (C), respectively.

If $t = 9$, then $7 \mid (3^3+1) \mid (3^9+1) \mid \sigma^{**}(3^f)$ and (C) holds. Finally, if $t = 5$, then $\sigma^{**}(3^f) = \sigma^{**}(3^8) = 2^5 \times 5 \times 61$ and (B) holds. □

Lemma 7. *Let $e \ge 6$ be an integer and write $e = 2s-1-\delta$ with $\delta \in \{0,1\}$ and s a positive integer. If $e \ne 8, 12$, then at least one of the following statements holds.*

(a) $\delta = 0$ and $\sigma^{**}(2^e)$ has at least two prime factors q_1 and q_2 with $5 < q_1, q_2 \le 2^s - 1$,

(b) $p_2 = 5 \mid \sigma^{**}(2^e)$.

(c) $\sigma^{**}(2^e)$ has at least two prime factors q_1 and q_2 each of which satisfies either $q_i = 2^{s-1}-1$, $q_i = 2^s+1$, or $5 < q_i \le \sqrt{2^s-3}$. Moreover, if $q_i = 2^{s-1}-1$ or $q_i = 2^s+1$ for $i = 1$ or 2, then 4 divides s and $\delta = 1$.

Proof. We begin by observing that $\sigma^{**}(2^e) = (2^s+1)(2^{s-\eta}-1)$ from (2). We put $s = 2^g m$ with m odd.

If $\delta = 0$, $m > 1$, and $e \ne 11, 23$, then $s \ne 6, 12$ and Lemma 4 yields that 2^s-1 has a prime factor $q_1 \equiv 1 \pmod{s}$ and $2^{2s}-1$ has a prime factor $q_2 \equiv 1 \pmod{2s}$ not dividing 2^s-1 or $2^{2^{g+1}}-1$. Clearly, we see that $q_1 \ne q_2$ and q_2 divides $(2^s+1)/(2^{2^g}+1)$. Moreover, since $s \ge 3$ is odd, we see that $q_1 \equiv q_2 \equiv 1 \pmod{2s}$ and q_2 divides $(2^s+1)/(2^{2^g}+1) < 2^s-1$. Then, we have $5 < q_1, q_2 \le 2^s-1$ and (a) holds. If $e = 11$ or $e = 23$, then $7 \times 13 \mid (2^{12}-1) \mid \sigma^{**}(2^f)$ and therefore we can take $(q_1, q_2) = (7, 13)$, which yields (a) again.

If $\delta = 0$, $m = 1$ and $s = 2^g \ge 16$, then we work as above but with $2^{s/2}-1$ and 2^s-1 instead of 2^s-1 and $2^{2s}-1$, respectively. Now we can take two prime factors q_1, q_2 of 2^s-1 with $5 < q_1, q_2 \le 2^s-1$, which yields (a). If $e = 7$ or $e = 15$, then $5 \mid (2^8-1) \mid \sigma^{**}(2^e)$ and (b) holds.

Henceforth, assume that $\delta = 1$. If s is odd and $e \ne 8, 12, 16, 24$, then $s \ne 5, 7, 9, 13$ and we see from Lemma 4 that we can take a prime factor q_1 of $2^{(s-1)/2}-1$ such that $q_1 \equiv 1 \pmod{(s-1)/2}$ and a prime factor q_2 of $2^{(s-1)/2}+1$ such that $q_2 \equiv 1 \pmod{(s-1)/2}$. Since $2^{(s-1)/2}-1$ and $2^{(s-1)/2}+1$ are relatively prime, we have $q_1 \ne q_2$. Now $5 < q_1, q_2 \le 2^{(s-1)/2}+1 < \sqrt{2^s-3}$ and (c) holds.

If $g = 1$, then $5 = (2^2+1) \mid (2^s+1) \mid \sigma(2^e)$.

Assume that $g \ge 2$. Let q_1 and q_2 be the smallest prime factors of $2^{s-1}-1$ and 2^s+1, respectively. Since $s-1$ is odd, we cannot have $q_1 = 3$ or $q_1 = 5$. Similarly, since 4 divides s, we cannot have $q_2 = 3$ or $q_2 = 5$.

Since 2^s is square and $2^{s-1}-1 \equiv 3 \pmod{4}$, neither of 2^s+1, 2^s-1, nor $2^{s-1}-1$ can be square. Hence, we see that either $q_1 = 2^{s-1}-1$ is prime or $q_1 \le \sqrt{2^s-3}$. Similarly, $q_2 = 2^s+1$ is prime or $q_2 \le \sqrt{2^s-3}$. Hence, we see that (c) holds.

If $e = 24$, then we can take $(q_1, q_2) = (7, 13)$ since $7 \times 13 \mid (2^{12}-1) \mid \sigma^{**}(2^{24})$ to obtain (c). Finally, if $e = 16$, then we can take $(q_1, q_2) = (17, 19)$ since $17 \times 19 \mid (2^8-1)(2^9+1) = \sigma^{**}(2^{16})$ to obtain (c) observing that $p_2 = 19 < \sqrt{2^9-3}$. □

We also use the following miscellaneous divisibility results.

Lemma 8.

(I) For any prime p and $g \geq 2$, $\sigma^{**}(p^g)$ has a prime factor ≥ 5.

(II) For any odd prime p and $g \geq 4$, $\sigma^{**}(p^g)$ has a prime factor ≥ 7.

(III) If $f \geq 18$ or $f = 13, 14$, then $\sigma^{**}(3^f)$ has at least two distinct prime factors ≥ 127.

(IV) If $g \geq 2$, then $\sigma^{**}(13^g)$ has at least two distinct prime factors in addition to 2, 3, 41, and 547.

(V) For any $g \geq 1$, $\sigma^{**}(41^g)$ has a prime factor in addition to 2, 3, 5, and 13.

Proof.

(I) We write $g = 2u - 1 - \gamma$ with $\gamma \in \{0, 1\}$. For $g \geq 2$, we have $u \geq 2$ and, by Lemma 5, $p^u + 1$ has a one prime factor ≥ 5.

(II) We write $g = 2u - 1 - \gamma$ with $\gamma \in \{0, 1\}$. For $g \geq 4$, we have $u \geq 3$ and, by Lemma 5, $p^u + 1$ has a one prime factor ≥ 7.

(III) By Lemma 4, $(3^s - 1)/2$ has at least one prime factor ≥ 127 for $s \geq 126$ and, by Lemma 5, $3^t + 1$ has at least one prime factor ≥ 127 for $t \geq 126$. Now, we can confirm that $(3^s - 1)/2$ has at least one prime factor ≥ 127 for $s \geq 7$ and $3^t + 1$ has at least one prime factor ≥ 127 for $t \geq 10$ (see, for example, https://homes.cerias.purdue.edu/~ssw/cun/pmain125.txt). Hence, putting $f = 2t - 1 - \eta$ with $\eta \in \{0, 1\}$, $\sigma^{**}(3^f) = (3^t + 1)(3^{t-\eta} - 1)/2$ has at least two distinct prime factors ≥ 127 for $f \geq 18$. For $f = 13, 14$, (I) can be confirmed by noting that $\sigma^{**}(3^{13}) = 2^2 \times 547 \times 1093$ and $\sigma^{**}(3^{14}) = 2 \times 17 \times 193 \times 547$.

(IV) With the aid of Lemmas 4 and 5, we see that $(13^s - 1)/12$ has at least one prime factor ≥ 11 for $s \geq 3$ and $13^t + 1$ has at least one prime factor ≥ 11 for $t \geq 2$ (these can be confirmed by direct calculation for $3 \leq s \leq 6$ and $2 \leq t \leq 5$). If 41 divides $13^t + 1$, then $t \equiv 20 \pmod{40}$ and 14281 also divides $13^t + 1$. Similarly, if 41 divides $(13^s - 1)/12$, then 14281 also divides $(13^s - 1)/12$. If 547 divides $(13^u - 1)/12$, then $21 \mid s$ and 61 also divides $(13^s - 1)/12$. Since $13^{21} \equiv 1 \pmod{547}$, 547 can never divide $13^t + 1$. Hence, for $g \geq 5$, putting $g = 2u - 1 - \gamma$ with $\gamma \in \{0, 1\}$, each of $13^u + 1$ and $(13^{u-\gamma} - 1)/12$ has at least one prime factor p_1 and p_2, respectively, both other than 2, 3, 7, 41, or 547. If $p_1 = p_2$, then p_1 must divide $13^\gamma + 1$ and therefore $p_1 = 2$ or $p_1 = 7$, which is a contradiction. Hence, $\sigma^{**}(13^g) = (13^u + 1)(13^{u-\gamma-1} - 1)/12$ has at least two prime factors besides 2, 3, 41, and 547 for $g \geq 5$. For $2 \leq g \leq 4$, (IV) can be easily confirmed.

(V) For $p = 41$, from Lemma 5, we see that $41^u + 1$ has a prime factor greater than 5 for $u \geq 3$. If 13 divides $41^u + 1$, then $u \equiv 6 \pmod{12}$ and 29 also divides $41^u + 1$. Hence, for $g \geq 2$, putting $g = 2u - 1 - \gamma$ with $\gamma \in \{0, 1\}$, $41^u + 1$ has a prime factor in addition to 2, 3, 5, and 13 and so does $\sigma^{**}(41^g)$. For $g = 1$, (V) is clear since $\sigma^{**}(41) = 41 + 1 = 2 \times 3 \times 7$.

□

3 Proof of Theorem 1

Assume that N is a biunitary triperfect number with $3^3 \mid N$. We put integers e and f by $2^e \mid\mid N$ and $3^f \mid\mid N$ and write $e = 2s - 1 - \delta$ with $\delta \in \{0, 1\}$ and $f = 2t - 1 - \eta$ with $\eta \in \{0, 1\}$.

In this section, once we write p_i for a prime factor of N with an index i, e_i denotes the exponent of p_i dividing N. Clearly, $\sigma^{**}(p_i^{e_i})$ divides $3N$ for each i. Thus, if p is a prime other than 3 and p^g divides $\sigma^{**}(p_i^{e_i})$ for some i, then p^g divides N.

We begin with small e and f. Although Haukkanen and Sitaramaiah [5] proved that $e \geq 7$ whether $3^3 \mid N$ or not, we give a proof for small e in the view of self-containedness.

Lemma 9. *We must have $e \geq 4$.*

Proof. We cannot have $e = 0$ as Hagis [4] has shown. Indeed, if $N > 1$ is odd, then, by Lemma 2, $\sigma^{**}(N)$ must be even and $\sigma^{**}(N) \neq 3N$. If $e = 1$, then N can have no odd prime factor other than 3 and therefore we must have $\sigma^{**}(3^f) = 2^e$. By Lemma 4, we must have $f = 1$, contrary to the assumption that $f \geq 3$.

If $e = 2$, then $p_1 = 5 = \sigma^{**}(2^2) \mid \sigma^{**}(N) = 3N$ and therefore 5 must divide N. From Lemma 2, we see that N has no further prime factor. Hence, $\sigma^{**}(3^f)$ can have no prime factor other than 2 and 5. By Lemma 8 (II), we must have $f = 3$. Now $e_1 \geq 2$ since $5^2 \mid \sigma^{**}(2^2 3^3) \mid 3N$. However, from Lemma 8 (I), we see that $p_2 \mid \sigma^{**}(5^{e_1}) \mid \sigma^{**}(N) = 3N$ for some $p_2 > 5$ and $2^3 \mid N$ by Lemma 2, which is a contradiction.

If $e = 3$, then $p_1 = 5$ must divide N. If $e_1 \neq 2$, then, with the aid of Lemma 3, we have

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^3 3^4 5)}{2^3 3^4 5} = \frac{28}{9} > 3, \tag{3}$$

contrary to the assumption that $\sigma^{**}(N) = 3N$. Hence, we must have $e_1 = 2$ and $p_2 = 13$ must divide N since $13 \mid \sigma^{**}(5^2) \mid \sigma^{**}(N) = 3N$. With the aid of Lemma 8 (II), we see that $\sigma^{**}(13^{e_2})$ has an odd prime factor p_3 in addition to 3, 5, and 13. By Lemma 2, we must have $2^4 \mid \sigma^{**}(3^f 5^2 13^{e_2} p_3^{e_3}) \mid 3N$ and $2^4 \mid N$, which is a contradiction. Thus, we cannot have $e = 3$. □

Lemma 10. *We cannot have $e = 5$.*

Proof. Assume that $e = 5$. Then, $\sigma^{**}(2^5) = 3^2 \times 7$ and $p_1 = 7$ must divide N. If $e_1 \neq 2$, then

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^5 3^4 7)}{2^5 3^4 7} = \frac{28}{9} > 3 \tag{4}$$

and if $f \neq 4$ and $e_1 = 2$, then $5 \mid \sigma^{**}(p_1^{e_1}) = 7^2 + 1$ and

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^5 3^6 7^2 5^2)}{2^5 3^6 7^2 5^2} = \frac{6929}{2268} > 3, \tag{5}$$

which are both contradictions. If $f = 4$ and $e_1 = 2$, then $p_2 = 5$ divides N and $2^6 \mid \sigma^{**}(3^4 7^2 5^{e_2}) \mid 3N$, which is a contradiction. □

Lemma 11. *We cannot have $e = 6$.*

Proof. Assume that $e = 6$. We observe that $p_1 = 7$ and $p_2 = 17$ must divide N since $7 \times 17 = \sigma^{**}(2^6) \mid \sigma^{**}(N) = 3N$. If $e_1 \neq 2$ and $e_2 \neq 2$, then

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^6 3^4 7 \times 17)}{2^6 3^4 7 \times 17} = \frac{28}{9} > 3, \tag{6}$$

which is a contradiction. If $e_1 \neq 2$ and $e_2 = 2$, then, since 5 divides $17^2 + 1$, $p_3 = 5$ must divide N and

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^6 3^4 5^2 7)}{2^6 3^4 5^2 7} = \frac{6188}{2025} > 3, \tag{7}$$

which is a contradiction again.

Now we must have $e_1 = 2$. We see that $p_3 = 5$ and $e_3 \geq 2$ since $5^2 \mid (7^2 + 1) = \sigma^{**}(7^2)$. We observe that we cannot have $f = 4$ since $2^7 \mid \sigma^{**}(3^4 7^2 17^{e_2} 5^{e_3})$. If $e_3 > 2$, then

$$\frac{\sigma^{**}(N)}{N} > \frac{\sigma^{**}(2^6 3^6 7^2 5)}{2^6 3^6 7^2 5} = \frac{45305}{13608} > 3, \tag{8}$$

which is a contradiction. Hence, $e_3 = 2$ and $p_4 = 13$ must divide N. However, this is impossible. Indeed, if $e_4 = 2$, then $5^3 \mid \sigma^{**}(7^2 13^2) \mid 3N$, contrary to $e_3 = 2$, and if $e_4 \neq 2$, then

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^6 3^6 7^2 5^2 13)}{2^6 3^6 7^2 5^2 13} = \frac{9061}{2916} > 3, \tag{9}$$

which is a contradiction. □

Lemma 12. *We cannot have $e \geq 7$ and $f = 3$.*

Proof. Assume that $e \geq 7$ and $f = 3$. Clearly $p_1 = 5$ must divide N since $5 \mid \sigma^{**}(3^3) \mid 3N$. If $e_1 \neq 2$, then

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^8 3^3 5)}{2^8 3^3 5} = \frac{55}{16} > 3, \tag{10}$$

which is a contradiction.

If $e_1 = 2$, then, since $\sigma^{**}(5^{e_1}) = 2 \times 13$, $p_2 = 13$ must divide N. We must have $e_2 = 2$, $p_3 = 17$ (we note that $17 \mid \sigma^{**}(13^{e_2}) = \sigma^{**}(13^2)$), and $e_3 = 2$ since otherwise

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^8 3^3 5^2 13)}{2^8 3^3 5^2 13} = \frac{77}{24} > 3 \tag{11}$$

or

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^8 3^3 5^2 13^2 17)}{2^8 3^3 5^2 13^2 17} = \frac{165}{52} > 3, \tag{12}$$

which is a contradiction. But then, we must have $5^3 \mid \sigma^{**}(3^3 13^2 17^2) \mid 3N$, contrary to $e_1 = 2$. □

Lemma 13. *We cannot have $e \geq 7$ and $f = 4$.*

Proof. Assume that $e \geq 7$ and $f = 4$. Clearly $p_1 = 7$ must divide N since $7 \mid \sigma^{**}(3^4) \mid 3N$. If $e_1 \neq 2$, then

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^8 3^4 7)}{2^8 3^4 7} = \frac{55}{18} > 3, \tag{13}$$

which is a contradiction.

If $e_1 = 2$, then, since $\sigma^{**}(7^{e_1}) = 2 \times 5^2$, $p_2 = 5$ must divide N. We must have $e_2 = 2$, $p_3 = 13$, and $e_3 = 2$, since otherwise

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^8 3^4 7^2 5)}{2^8 3^4 7^2 5} = \frac{275}{84} > 3 \tag{14}$$

or

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^8 3^4 7^2 5^2 13)}{2^8 3^4 7^2 5^2 13} = \frac{55}{18} > 3, \tag{15}$$

which is a contradiction. Hence, we must have $5^3 \mid N$ since $5^3 \mid \sigma^{**}(7^2 13^2)$, which is also a contradiction. □

Now we prove remaining cases. Assume that $f \geq 5$, $e \geq 7$ and $e \neq 8, 12$. Then, a prime p_1 taken from Lemma 6 must divide N since $p_1 \mid \sigma^{**}(3^f) \mid 3N$ and $p_1 \neq 3$. If (b) in Lemma 7 holds, then $p_2 = 5$ must divide N and we have $p_1 > 5 = p_2$. If (a) or (c) in Lemma 7 holds, then we can choose a prime $p_2 \in \{q_1, q_2\}$ not equal to p_1. Like above, p_2 must divide N. We see that if (c) in Lemma 7 holds, then we have either (c′) $5 < p_2 \leq \sqrt{2^s - 3}$ or (d) $4 \mid s$, $\delta = 1$, and either of $p_2 = 2^{s-1} - 1$ or $p_2 = 2^s + 1$ is prime. We put $N_1 = 2^e p_2^{e_2}$ if (a), (b), or (c′) holds or (d) holds with $e_2 \neq 2$. It is clear that $N_1 \mid\mid N$. If (d) holds with $e_2 = 2$, then, we observe that $p_i \equiv 2 \pmod 5$ and $5 \mid (p_i^2 + 1) \mid \sigma^{**}(N) = 3N$ and thus we see that $N_1 = 2^e 5^{e_3} \mid\mid N$. Similarly, we put $N_2 = 3^f p_1^{e_1}$ if (A), (B), or (C) holds or (D) holds with $e_2 \neq 2$ and $N_2 = 3^f 5^{e_3}$ if (D) holds with $e_2 = 2$ to see that $N_2 \mid\mid N$.

We observe that $\sigma^{**}(N_1)/N_1 > 2$ and $\sigma^{**}(N_2)/N_2 > 3/2$. Indeed, if (a) holds, then

$$\frac{\sigma^{**}(N_1)}{N_1} \geq \frac{(2^{e+1} - 1)(p_2^2 + 1)}{2^e p_2^2} \geq \frac{(2^{2s} - 1)((2^s - 1)^2 + 1)}{2^{2s-1}(2^s - 1)^2} > 2. \tag{16}$$

If (c′) holds, then

$$\frac{\sigma^{**}(N_1)}{N_1} \geq \frac{(2^{s-1}-1)(2^s+1)(p_2^2+1)}{2^{2s-2}p_2^2} \geq \frac{(2^{s-1}-1)^2(2^s-2)}{2^{2s-3}(2^s-3)} > 2. \tag{17}$$

If (d) holds with $e_2 \neq 2$, then

$$\frac{\sigma^{**}(N_1)}{N_1} \geq \frac{(2^{s-1}-1)(2^s+1)(p_2+1)}{2^{2s-2}p_2} \geq \frac{(2^{s-1}-1)(2^s+2)}{2^{2s-2}} > 2. \tag{18}$$

If (b) holds, then

$$\frac{\sigma^{**}(N_1)}{N_1} \geq \frac{(2^{s-\eta}-1)(2^s+1)(5^{e_2}+1)}{2^{2s-1-\eta}5^{e_2}} \geq \frac{26(2^{s-\eta}-1)(2^s+1)}{25\times 2^{2s-1-\eta}} > 2, \tag{19}$$

which also holds with e_2 replaced by e_3 in the case (d) with $e_2 = 2$. Thus, we see that $\sigma^{**}(N_1)/N_1 > 2$ in any case.

Similarly, if (A) holds, then

$$\frac{\sigma^{**}(N_2)}{N_2} \geq \frac{(3^{f+1}-1)(p_1^2+1)}{2\times 3^f p_1^2} \geq \frac{(3^{2t}-1)((3^t-1)^2+4)}{2\times 3^{2t-1}(3^t-1)^2} > \frac{3}{2}. \tag{20}$$

If (C) holds, then

$$\frac{\sigma^{**}(N_2)}{N_2} \geq \frac{(3^{t-\eta}-1)(3^t+1)(p_1^2+1)}{2\times 3^{2t-1-\eta}p_1^2} \geq \frac{(3^{t-\eta}-1)(3^t+1)(3^{t-\eta}+1)}{2\times 3^{2t-1-\eta}(3^{t-\eta}-1)} > \frac{3}{2}. \tag{21}$$

If (D) holds with $e_1 \neq 2$, then

$$\frac{\sigma^{**}(N_2)}{N_2} \geq \frac{(3^{t-1}-1)(3^t+1)(p_1+1)}{2\times 3^{2t-2}p_1} \geq \frac{(3^{t-1}-1)(3^t+1)(3^{t-1}+1)}{2\times 3^{2t-2}(3^{t-1}-1)} > \frac{3}{2}. \tag{22}$$

If (B) holds, then

$$\frac{\sigma^{**}(N_2)}{N_2} \geq \frac{(3^{t-\eta}-1)(3^t+1)(5^{e_1}+1)}{2\times 3^{2t-1-\eta}5^{e_1}} \geq \frac{26(3^{t-1}-1)(3^t+1)}{2\times 25\times 3^{2t-1-\eta}} > \frac{3}{2}, \tag{23}$$

which also holds with e_1 replaced by e_3 in the case (D) with $e_1 = 2$. Thus, we see that $\sigma^{**}(N_2)/N_2 > 3/2$ in any case.

If (a) or (c′) holds, then $\gcd(N_1, N_2) = 1$ and

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(N_1)}{N_1} \times \frac{\sigma^{**}(N_2)}{N_2} > 3, \tag{24}$$

which is a contradiction. If (b) or (d) holds and (A) or (C) also holds, then, we have $\gcd(N_1, N_2) = 1$ and a contradiction (24) again.

Now we settle two cases (i) (b) or (d) holds and (B) or (D) also holds and (ii) $e = 8$. In both cases, 5 must divide N since 5 divides $\sigma^{**}(2^8) = 3^2 \times 5 \times 11$. We rewrite $p_1 = 5$. If $e_1 \neq 2$, then

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^8 3^6 5)}{2^8 3^6 5} = \frac{5863}{1728} > 3, \tag{25}$$

which is a contradiction. Hence, $e_1 = 2$.

Now, rewriting $p_2 = 13$, p_2 must divide N. Hence, if $e \neq 8$, then

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^{10} 3^6 5^2 13^2)}{2^{10} 3^6 5^2 13^2} > 3 \tag{26}$$

and, if $e = 8$ and $f \neq 6$, then

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^8 3^8 5^2 13^2)}{2^8 3^8 5^2 13^2} > 3 \tag{27}$$

and we have a contradiction. Thus, we must have $e = 8$ and $f = 6$. Since $f = 6$, $p_3 = 7$ must divide N. If $e_3 = 2$, then $5^3 \mid \sigma^{**}(2^8 7^2) \mid 3N$, which is incompatible with $e_1 = 2$. If $e_3 \neq 2$, then

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^8 3^6 7)}{2^8 3^6 7} = \frac{29315}{9072} > 3, \tag{28}$$

a contradiction again.

Now we assume that $e = 12$. Then, $p_1 = 7$ must divide N since $7 \mid (2^6 - 1) \mid \sigma^{**}(2^{12})$. If $e_1 \neq 2$, then

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^{12} 3^6 7)}{2^{12} 3^6 7} = \frac{22919}{6912} > 3, \tag{29}$$

and, if $e_1 = 2$, then 5 divides N and

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^{12} 3^6 7^2 5^2)}{2^{12} 3^6 7^2 5^2} = \frac{297947}{96768} > 3. \tag{30}$$

Thus, we have a contradiction.

Now the only remaining case is the case $e = 4$. We prove that if $e = 4$, then $N = 2160$ to complete the proof of Theorem 1. We begin by showing that if $N \neq 2160$, then $f \geq 5$ and 5 cannot divide N.

Lemma 14. *If $e = 4$ and $f \leq 4$, then $N = 2160$.*

Proof. If $f = 4$, then $\sigma^{**}(3^f) = 2^4 7$ and therefore $p_1 = 7$ must divide N and $2^5 \mid \sigma^{**}(3^f 7^{e_1}) \mid 3N$, contrary to the assumption that $e = 4$.

If $f = 3$, then $\sigma^{**}(3^f) = 2^3 5$ and $p_1 = 5$ must divide N. If $e_1 = 1$, then $\sigma^{**}(2^4 3^3 5) = 2^4 3^4 5$ and therefore $N = 2^4 3^3 5$. If $e_1 = 2$, then $p_1^2 + 1 = 2 \times 13$ and $p_2 = 13$ must divide N, which is impossible since $2^5 \mid \sigma^{**}(3^3 5^2 13^{e_2})$. If $e_1 \geq 3$, then

$$\frac{\sigma^{**}(N)}{N} > \frac{\sigma^{**}(2^4 3^3 5)}{2^4 3^3 5} = 3, \tag{31}$$

which is a contradiction. Hence, if $f = 3$, then we have $N = 2^4 3^3 5$. □

Now we must have $f \geq 5$ if $N \neq 2160$.

Lemma 15. *If $e = 4$ and $f \geq 5$, then 5 cannot divide N.*

Proof. Assume that $p_1 = 5$ divides N. If $e_1 \neq 2$, then we must have $f = 6$ since otherwise

$$\frac{\sigma^{**}(N)}{N} > \frac{\sigma^{**}(2^4 3^3 5)}{2^4 3^3 5} = 3, \tag{32}$$

which is a contradiction. Noting that $13 \mid \sigma^{**}(3^6) \mid 3N$, $p_2 = 13$ must divide N, $e_2 = 2$, $p_3 = 17$ must divide N and $e_3 = 2$ since otherwise

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^4 3^6 5 \times 13)}{2^4 3^6 5 \times 13} = \frac{287}{90} > 3 \tag{33}$$

or

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^4 3^6 5 \times 13^2 17)}{2^4 3^6 5 \times 13^2 17} = \frac{41}{13} > 3, \tag{34}$$

a contradiction again. Since $5^2 \mid \sigma^{**}(13^2 17^2)$ and we assumed that $e_1 \neq 2$, we must have $e_1 \geq 3$ and

$$\frac{\sigma^{**}(N)}{N} \geq \frac{\sigma^{**}(2^4 3^6 5^4 13^2)}{2^4 3^6 5^4 13^2} = \frac{4879}{1625} > 3, \tag{35}$$

which is impossible. Thus, we must have $e_1 = 2$ and $p_2 = 13$ must divide N since $13 \mid \sigma^{**}(5^2) \mid 3N$.

If $f \geq 9$, then, with the aid of Lemma 8 (III), we see that $\sigma^{**}(3^f)$ must have two odd prime factors p_3 and p_4 in addition to 5 and 13, which is incompatible with $e = 4$ by Lemma 2.

If $f = 8$, then $61 \mid \sigma^{**}(3^f)$ and $p_3 = 61$ must divide N. Noting that $2^2 \mid (3^5 + 1) \mid \sigma^{**}(3^8)$, we must have $2^5 \mid \sigma^{**}(3^8 5^2 13^{e_2} 61^{e_3})$, which is a contradiction. If $f = 7$, then we have a similar contradiction from $p_3 = 41 \mid (3^4 + 1) \mid \sigma^{**}(3^f)$. If $f = 6$, then $p_3 = 41$ must divide N again. By Lemma 8 (V), $\sigma^{**}(41^{e_3})$ has an odd prime factor p_4 other than 3, 5, p_3 and, we must have $2^5 \mid \sigma^{**}(3^6 5^2 13^{e_2} 41^{e_3} p_4^{e_4})$, which is a contradiction.

If $f = 5$, then $7 \mid \sigma^{**}(3^5)$ and $p_3 = 7$ must divide N. However, we must have $2^5 \mid \sigma^{**}(3^5 5^2 13^{e_2} 5^{e_3}) \mid 3N$ from Lemma 2, contrary to $e = 4$. Thus we conclude that 5 cannot divide N. □

Now we prove that we can never have $\sigma^{**}(N) = 3N$ if $N \neq 2160$ and $e = 4$. By Lemmas 14 and 15, we must have $f \geq 5$ and 5 can never divide N.

If $f \geq 18$ or $f = 13, 14$, then, By Lemma 8 (IV), $\sigma^{**}(3^f)$ has at least two odd prime factors $p_1, p_2 \geq 127$. By Lemma 2, we see that N can have at most one more prime factor p_3. Thus, we must have

$$\frac{\sigma^{**}(N)}{N} < \frac{27 \times 3 \times 7 \times 127 \times 131}{16 \times 2 \times 6 \times 126 \times 130} < 3, \tag{36}$$

which is a contradiction.

If $15 \le f \le 17$, then $\sigma^{**}(3^f)$ has at least three odd prime factors not less than 31 and

$$\frac{\sigma^{**}(N)}{N} < \frac{27 \times 3 \times 31 \times 37 \times 41}{16 \times 2 \times 30 \times 36 \times 40} < 3, \tag{37}$$

which is a contradiction. If $f = 7, 8, 10, 11$, then $5 \mid \sigma^{**}(3^f)$ and 5 must divide N, which is impossible by Lemma 15.

If $f = 6, 12$, then $p_1 = 13$ and $p_2 = 41, 547$, respectively, must divide N. By Lemma 4, we see that $\sigma^{**}(13^{e_1})$ can have at most one prime factor other than 2, 3, or p_2, which is impossible for $f = 6$ and $f = 12$ by Lemma 8 (IV). If $f = 9$, then, 11×61 must divide N and, noting that $2^2 \mid \sigma^{**}(3^9)$, N can have no prime factor except 2, 3, 11, or 61. Hence, $\sigma^{**}(61^{e_2})$ has no prime factor except 2, 3, or 11, which is impossible since $61^5 + 1 = 2 \times 11 \times 31 \times 1238411$.

Finally, if $f = 5$, then $\sigma^{**}(3^5) = 2^2 \times 7 \times 13$ and therefore $p_2 = 7$ and $p_3 = 13$ must divide N. If $e_3 \ge 2$, then $\sigma^{**}(13^{e_3})$ has another prime factor p_4 in addition to 2, 3, and 7 from Lemma 8 (IV) and $2^5 \mid \sigma^{**}(3^5 7^{e_2} 13^{e_3} p_4^{e_4}) \mid 3N$, which is a contradiction. If $e_3 = 1$, then we have a similar contradiction that $2^5 \mid \sigma^{**}(3^5 7) \mid 3N$.

Hence, we can never have $\sigma^{**}(N) = 3N$ if $N \ne 2160$ and $e = 4$. This completes the proof of Theorem 1.

4 Concluding remarks

Our arguments in this paper and [14] lead us to strongly believe that the equations $\sigma^{**}(\sigma^{**}(N)) = 3N$ and $\sigma^{**}(\sigma^{**}(N)) = 4N$ have no solutions other than the known twelve solutions given in A318175. Moreover, based on our theorem and known biunitary multiperfect numbers, we can pose the following problems, which are so far unsolved:

- For each integer $k \ge 3$, are there infinitely or only finitely many integers N for which $\sigma^{**}(N) = kN$?
- For a prime power $p^g \ne 1$, does there exist at least one N for which $\sigma^{**}(N) = kN$ and $p^g \,||\, N$?
- For an integer $d > 2$, does there exist at least one integer N for which $\sigma^{**}(N) = kN$ and $\gcd(d, N) = 1$?

Problems involving $\sigma^{**}(\sigma^{**}(N))$ are posed in [14].

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